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RESEARCH ARTICLE

Time delay in the Einstein-Straus solution

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Abstract We compute the time delay of strong lensing in the framework of the Einstein-Straus solution in presence of a cosmological constant and restrict ourselves to the case of a spatially flat universe. The calculations are done to first order in the ratio of Schwarzschild radius to peri-lens. We apply our results to the lensed quasar SDSS J1004+4112. Our predictions for the time delay between the images C and D of the quasar range from 6 to 13 years and are compatible with the experimental lower bound.

Keywords Cosmological parameters · Lensing

1 Introduction

Before the work of Rindler and Ishak [1], the general believe was that the deflexion angle of light passing near an isolated static and spherically symmetric mass is independent of the cosmological constant. This believe was based on the argument that the cosmological constant disappears from the geodesic equation for massless particles.

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In september 2007, Rindler and Ishak [1] corrected this believe. They pointed out that it is not sufficient to consider the geodesic equation but also the metric itself must be considered. In the work of Rindler and Ishak the source emitting the light and the Earth were supposed to be at rest with respect to the lens. Also all the masses, including those of the Earth and of the source, were neglected, except the mass of the lens. Since then, there is a rich controversy about whether or not a cosmological constant modifies the bending of light near an isolated spherical mass. Sereno [2,3], Schücker [4,5], Miraghaei and Nouri-Zonoz [6], Kantowski, Chen and Dai [7], confirm Rindler and Ishak's result, while Khriplovich and Pomeransky [8], Park [9], Gibbons, Warnick and Werner [10], Simpson, Peacock and Heavens [11] contradict Rindler and Ishak's findings.

Recently, Schücker [12] redid the calculations of the bending of light by a spherically symmetric mass distribution, which is taken to be a cluster of galaxies by relaxing all the previous hypotheses except that of sphericity: the observer and the source are allowed to move with respect to the cluster, the masses of the other clusters are included in the form of a homogeneous isotropic dust and the observer as well as the source are taken comoving with respect to the dust. The appropriate framework of this computation is the Einstein-Straus solution [13,14] that matches the Kottler solution at the inside of the Schücking radius with the Friedmann solution at the outside. Einstein and Straus' first motivation was to explain why the cosmic expansion does not affect smaller length scales such as planetary and atomic systems. Schücker's calculations confirm previous calculations by Ishak, Rindler, Dossett, Moldenhauer and Allison [15]: Taking into account realistic cosmic velocities attenuates the effect of the cosmological constant on the bending of light without however cancelling it. Although there has never been a claim that the time delay was independent of a cosmological constant, it is also interesting to do the computation of the time delay in the framework of the Einstein-Straus solution. This computation is interesting at least on the experimental side since our results may then be compared to available experimental lower bounds on time delay.

The paper is organized as follows: Section 2 is devoted to a brief presentation of the Einstein-Straus solution with a cosmological constant in the case of a spatially flat universe. Then, in Sect. 3, the geodesics of light are integrated and a general expression for the time delay is derived and subsequently applied to the quasar SDSS J1004+4112. Finally, our findings are summarized in Sect. 4.

We will use the same units as in reference [12]: astrometers (am), astroseconds (as), astrograms (ag)

$$\begin{aligned} \mathrm{am} &= 1.30\times 10^{26}\ \mathrm{m} = 4221\ \mathrm{Mpc}, \qquad \mathrm{as} = 4.34\times 10^{17}\ \mathrm{s} = 13.8\ \mathrm{Gyr}, \\ \mathrm{ag} &= 6.99\times 10^{51}\ \mathrm{kg} = 3.52\times 10^{21}\ \mathrm{M}_\odot, \end{aligned} \tag{1}$$

where $\mathrm{M}_\odot$ denotes one solar mass. In these units

$$\begin{aligned} &\mathrm{c} = 1 \mathrm{am\ as}^{-1},\ 8\pi G = 1\ \mathrm{am}^3\mathrm{as}^{-2}\mathrm{ag}^{-1},\ H_0 = 1\ \mathrm{as}^{-1}, \\ &\hbar = 3.86\times 10^{-121}\mathrm{am}^2\ \mathrm{as}^{-1}\mathrm{ag}. \end{aligned} \tag{2}$$

For spatially flat universes, to which we will restrict ourselves in the following, we may set the scale factor today $a_0 := a(t=0) = 1$ am.

2 The Einstein-Straus solution with a cosmological constant

We will consider hereafter the Einstein-Straus solution [13,14] generalized to include the cosmological constant [16] but restrict ourselves to spatially flat universes. We will need the Jacobian of the transformation passing between the Friedmann and Schwarzschild coordinates to calculate the geodesics of photons. Let us quote the results obtained by Schücker [12]. Let (T, r, θ, φ) and $(t, \chi, \theta, \varphi)$ stand respectively for Kottler and Friedmann coordinates. The Kottler metric

$$ds^2 = B(r)dT^2 - B(r)^{-1}dr^2 - r^2 d\Omega^2, \tag{3}$$

with

$$B(r) = 1 - \frac{2GM}{r} - \frac{\Lambda}{3}r^2, \tag{4}$$

prevails inside a vacuole of radius $r_{Sch\ddot{u}}(T)$, $r < r_{Sch\ddot{u}}$. The Friedmann spatially flat metric is given by:

$$ds^2 = dt^2 - a(t)^2 \left(d\chi^2 + \chi^2 d\Omega^2\right), \tag{5}$$

with the scale factor $a(t)$ determined by the first order Friedmann equation

$$\frac{da}{dt} = \sqrt{A/a + \Lambda a^2/3}, \tag{6}$$

where

$$A = a_0^3 \rho_{dust0}/3 = 1 - \frac{\Lambda}{3}, \tag{7}$$

prevails outside the vacuole $\chi \geq \chi_{Sch\ddot{u}}$. It is worthwhile to notice that due to (6), the scale factor is strictly monotonic. The two solutions are glued together at the constant Schücking radius $\chi_{Sch\ddot{u}}$

$$r_{Sch\ddot{u}}(T) := a(t)\chi_{Sch\ddot{u}}. \tag{8}$$

By taking into account the fact that the central mass must be equal to the dust density times the volume of the ball with Schücking radius $r_{Sch\ddot{u}}$

$$A := \frac{2M}{8\pi \chi^3_{Sch\ddot{u}}} = \frac{2GM}{\chi^3_{Sch\ddot{u}}}. \tag{9}$$

In the following, it is useful to introduce

$$B(r_{Sch\ddot{u}}) =: B_{Sch\ddot{u}} = 1 - \frac{A}{a}\chi^2_{Sch\ddot{u}} - \frac{\Lambda}{3}a^2\chi^2_{Sch\ddot{u}} \tag{10}$$

and

$$C_{Sch\ddot{u}} := \sqrt{1 - B_{Sch\ddot{u}}}. \tag{11}$$

Schücker [12] computed the Jacobian of the coordinate transformation $(T, r) \to (t, \chi)$ at the Schücking radius and also the inverse of the Jacobian (corresponding to the Jacobian of the inverse coordinate transformation $(t, \chi) \to (T, r)$) with the results

$$\left.\frac{\partial t}{\partial T}\right|_{Sch\ddot{u}} = 1, \quad \left.\frac{\partial t}{\partial r}\right|_{Sch\ddot{u}} = -\frac{C_{Sch\ddot{u}}}{B_{Sch\ddot{u}}}, \quad \left.\frac{\partial \chi}{\partial T}\right|_{Sch\ddot{u}} = -\frac{C_{Sch\ddot{u}}}{a}, \quad \left.\frac{\partial \chi}{\partial r}\right|_{Sch\ddot{u}} = \frac{1}{a B_{Sch\ddot{u}}}, \tag{12}$$

and

$$\left.\frac{\partial T}{\partial t}\right|_{Sch\ddot{u}} = \frac{1}{B_{Sch\ddot{u}}}, \quad \left.\frac{\partial T}{\partial \chi}\right|_{Sch\ddot{u}} = a\frac{C_{Sch\ddot{u}}}{B_{Sch\ddot{u}}}, \quad \left.\frac{\partial r}{\partial t}\right|_{Sch\ddot{u}} = C_{Sch\ddot{u}}, \quad \left.\frac{\partial r}{\partial \chi}\right|_{Sch\ddot{u}} = a. \tag{13}$$

In the following we also need to pass between Kottler time T and Friedmann time t at the Schücking radius. To this end we will use the result also obtained by Schücker [12]

$$\left.\frac{dt}{dT}\right|_{Sch\ddot{u}} = B_{Sch\ddot{u}}. \tag{14}$$

3 Integrating the geodesics of light

We have the following situation, Fig. 1: a first photon is emitted by the source, a quasar, at a time t'_S and follows an upper straight line trajectory until its arrival at a time $t'_{Sch\ddot{u}S}$ on the Schücking sphere in the half space containing the source. It is then bent inside the Schücking radius, until it emerges from the Schücking sphere in the half space containing the Earth at time $t'_{Sch\ddot{u}E}$, then follows a straight line until its arrival on Earth at time $t'_0 = 0$. A second photon is emitted by the quasar at a time

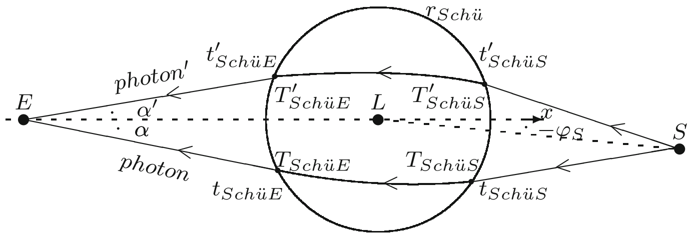

Fig. 1 Two light rays emitted by a source S, bent inside the Schücking sphere and finally received at Earth E. The travel times of the two photons differ, giving rise to a time delay

t_S follows a lower straight line trajectory, arrives on the Schücking sphere in the half space containing the source at time $t_{Sch\ddot{u}S}$, it is then bent inside the Schücking sphere and emerges from the Schücking sphere in the half space containing the Earth at time $t_{Sch\ddot{u}E}$, follows again a straight line until its receipt on Earth at the same time as the first photon $t_0 = t'_0 = 0$. We will here be interested in the computation of the time delay $t_S - t'_S$.

Let us first integrate the first order Friedmann solution (6) for the scale factor $a(t)$ in the spatially flat case with cosmological constant Λ and dust density $\rho_{dust0} = (3 - \Lambda)$ (we will take the experimentally favored value $\Lambda = 0.77 \times 3\ \mathrm{am}^{-2} \pm 20\%$) with final condition $a_0 = 1$. One may show that (6) is equivalent to the Friedmann second order equation for $a(t)$

$$\frac{2}{a}\frac{d^2a}{dt^2} + \frac{1}{a^2}\left(\frac{da}{dt}\right)^2 = \Lambda, \tag{15}$$

with final conditions

$$\left.\frac{da}{dt}\right|_{t=0} = 1, \qquad a_0 = 1. \tag{16}$$

The integration of (6) or (15) may be done analytically with the result

$$a(t) = \left(\frac{1 - \frac{\Lambda}{3}}{\frac{\Lambda}{3}}\right)^{1/3} \left[\sinh^2\left(\frac{3}{2}\sqrt{\frac{\Lambda}{3}}t + \operatorname{arcsinh}\left(\sqrt{\frac{\frac{\Lambda}{3}}{1 - \frac{\Lambda}{3}}}\right)\right)\right]^{1/3}. \tag{17}$$

We will also need to solve the equation

$$\frac{d\chi}{dt} = -\frac{1}{a}, \tag{18}$$

for various final conditions, with χ having the meaning of a geodesic distance, not to be confused with a luminosity distance. Since these are the final conditions, at the arrival on Earth, which are known, we will proceed backwards in time in three steps: we will determinate $t'_{Sch\ddot{u}E}$ and $t_{Sch\ddot{u}E}$, then $t'_{Sch\ddot{u}S}$ and $t_{Sch\ddot{u}S}$, and finally t'_S and t_S.

Step 1: Determination of $t'_{Sch\ddot{u}E}$ and $t_{Sch\ddot{u}E}$.

Here we will be interested in the photon trajectory between the Earth and the Schücking sphere with the Friedmann metric. The non vanishing Christoffel symbols of the Friedmann metric in the plane $\theta = \pi/2$ are given by

$$\Gamma^t_{\chi\chi} = aa_t, \quad \Gamma^t_{\varphi\varphi} = aa_t\chi^2, \quad \Gamma^\chi_{\varphi\varphi} = -\chi,\ \Gamma^\chi_{t\chi} = \frac{a_t}{a}, \quad \Gamma^\varphi_{t\varphi} = \frac{a_t}{a}, \quad \Gamma^\varphi_{\chi\varphi} = \frac{1}{\chi}, \tag{19}$$

and the geodesic equation reads

$$\ddot{t} + aa_t\dot{\chi}^2 + aa_t\chi^2\dot{\varphi}^2 = 0, \quad \ddot{\chi} + 2\frac{a_t}{a}\dot{t}\dot{\chi} - \chi\dot{\varphi}^2 = 0, \quad \ddot{\varphi} + 2\frac{a_t}{a}\dot{t}\dot{\varphi} + \frac{2}{\chi}\dot{\chi}\dot{\varphi} = 0, \tag{20}$$

with final conditions at $p = 0$

$$t = 0, \quad \chi = \chi_E, \quad \varphi = \pi, \quad \dot{t} = 1, \quad \dot{\chi} = \cos\alpha', \quad \dot{\varphi} = \frac{\sin\alpha'}{\chi_E}, \tag{21}$$

for the upper trajectory photon, where we use the fact that the physical angle α' coincides with the coordinate angle $\arctan(|\chi\dot{\varphi}/\dot{\chi}|)$.

For the lower trajectory photon, the final conditions differ

$$t = 0, \quad \chi = \chi_E, \quad \varphi = -\pi, \quad \dot{t} = 1, \quad \dot{\chi} = \cos\alpha, \quad \dot{\varphi} = -\frac{\sin\alpha}{\chi_E}. \tag{22}$$

The solution of the geodesic equation is

$$\dot{t} = \frac{1}{a}, \qquad \frac{\chi'_p}{\chi} = \sin(\varphi - \alpha'), \qquad \dot{\varphi} = \frac{\chi'_p}{a^2\chi^2}, \tag{23}$$

where χ'_p, the would be peri-lens, is given by

$$\chi'_p = \chi_E \sin\alpha', \tag{24}$$

for the upper trajectory photon and

$$\dot{t} = \frac{1}{a}, \qquad \frac{\chi_p}{\chi} = -\sin(\varphi + \alpha), \qquad \dot{\varphi} = -\frac{\chi_p}{a^2\chi^2}, \tag{25}$$

where χ_p, the would be peri-lens, is given by

$$\chi_p = \chi_E \sin\alpha, \tag{26}$$

for the lower trajectory photon.

The polar angles $\varphi'_{Sch\ddot{u}E}$ and $\varphi_{Sch\ddot{u}E}$ at which the lower and the upper trajectory photons emerge from the Schücking sphere are given respectively by

$$\varphi'_{Sch\ddot{u}E} = \pi - \arcsin\left(\frac{\chi'_p}{\chi_{Sch\ddot{u}}}\right) + \alpha', \tag{27}$$

$$\varphi_{Sch\ddot{u}E} = -\pi + \arcsin\left(\frac{\chi_p}{\chi_{Sch\ddot{u}}}\right) - \alpha, \tag{28}$$

where $\chi_{Sch\ddot{u}}$, the Schücking radius, is given by

$$\chi_{Sch\ddot{u}} = \left(\frac{M}{4\pi(1-\Lambda/3)}\right)^{1/3}, \tag{29}$$

where we make use of (7) and (9). It is easy to see using elementary Euclidean geometry that the geodesic distance $\chi'_{Sch\ddot{u}E,E}$ between the upper trajectory photon when it emerges from the Schücking sphere and the Earth is given by

$$\chi'_{Sch\ddot{u}E,E} = \sqrt{\chi_E^2 + \chi_{Sch\ddot{u}}^2 + 2\,\chi_E\,\chi_{Sch\ddot{u}}\cos\varphi'_{Sch\ddot{u}E}}, \tag{30}$$

where χ_E is the Earth-lens geodesic distance.

In the same way the geodesic distance $\chi_{Sch\ddot{u}E,E}$, between the lower trajectory photon when it emerges from the Schücking sphere and the Earth is given by an analogous expression

$$\chi_{Sch\ddot{u}E,E} = \sqrt{\chi_E^2 + \chi_{Sch\ddot{u}}^2 + 2\,\chi_E\,\chi_{Sch\ddot{u}}\cos\varphi_{Sch\ddot{u}E}}, \tag{31}$$

The Earth-Lens geodesic distance χ_E may be deduced from the knowledge of the redshift z_E according to the scheme

$$z_E \;\rightarrow\; a_E = \frac{1}{1+z_E} \;\rightarrow\; \tilde{t}_E := \tilde{t}(a_E) \;\rightarrow\; \chi(\tilde{t}_E) =: \chi_E, \tag{32}$$

where $\tilde{t}(a)$ denotes the inverse of the scale factor $a(t)$ and where χ is the solution of the first order differential Eq. (18), subject to the initial condition $\chi(0) = 0$, meaning that the photon reaches the Earth today at $t_0 = 0$. Moreover, since the scale factor a is strictly positif, χ is a strictly decreasing function and thus injective. Therefore, $\chi(t)$ can be inverted to give t in term of χ.

The knowledge of z_E allows to determine a_E, then one deduces the corresponding time $\tilde{t}_E$. Injecting in $\chi(t)$, one finally deduces χ_E. This is the same procedure that will be used to deduce χ_S

$$z_S \;\rightarrow\; a_S = \frac{1}{1+z_S} \;\rightarrow\; \tilde{t}_S := \tilde{t}(a_S) \;\rightarrow\; \chi(\tilde{t}_S) =: \chi_S. \tag{33}$$

On the other hand, the geodesic distance between the Earth and the upper trajectory photon in the time interval from its crossing of the Schücking sphere in the half space containing the Earth $t'_{Sch\ddot{u}E}$ until its arrival at Earth $t'_0 = 0$ is governed by Eq. (18) with the final condition

$$\chi(0) = 0, \tag{34}$$

meaning that at $t'_0 = 0$ the photon reaches the Earth. From (18) and (34), one deduces that

$$\int_{0}^{\chi'_{Sch\ddot{u}E,E}} d\chi = -\int_{0}^{t'_{Sch\ddot{u}E}} \frac{1}{a(t)} dt \tag{35}$$

i.e.,

$$\chi'_{Sch\ddot{u}E,E} = -\int_{0}^{t'_{Sch\ddot{u}E}} \frac{1}{a(t)} dt = \int_{t'_{Sch\ddot{u}E}}^{0} \frac{1}{a(t)} dt. \tag{36}$$

Then, comparing with Eq. (30), one gets

$$\sqrt{\chi_E^2 + \chi_{Sch\ddot{u}}^2 + 2\,\chi_E\,\chi_{Sch\ddot{u}} \cos\varphi'_{Sch\ddot{u}E}} = \int_{t'_{Sch\ddot{u}E}}^{0} \frac{1}{a(t)} dt. \tag{37}$$

Equation (37) may then be used to deduce $t'_{Sch\ddot{u}E}$.

The geodesic distance between the Earth and the lower trajectory photon in the time interval from its crossing of the Schücking sphere in the half space containing Earth, $t_{Sch\ddot{u}E}$, until its arrival at Earth ($t_0 = t'_0 = 0$), is also governed by (18) with the final condition (34). In an analogous way, one gets

$$\chi_{Sch\ddot{u}E,E} = \int_{t_{Sch\ddot{u}E}}^{0} \frac{1}{a(t)} dt. \tag{38}$$

Then, comparison with Eq. (31) gives

$$\sqrt{\chi_E^2 + \chi_{Sch\ddot{u}}^2 + 2\,\chi_E\,\chi_{Sch\ddot{u}} \cos\varphi_{Sch\ddot{u}E}} = \int_{t_{Sch\ddot{u}E}}^{0} \frac{1}{a(t)} dt, \tag{39}$$

which may serve to deduce $t_{Sch\ddot{u}E}$.

However, we find it more reliable to proceed in a different way: we compute $t_{Sch\ddot{u}E}$ by difference with $t'_{Sch\ddot{u}E}$. Combining (37) and (39), one gets

$$\chi_{Sch\ddot{u}E,E} - \chi'_{Sch\ddot{u}E,E} = \int_{t_{Sch\ddot{u}E}}^{0} \frac{1}{a(t)} dt - \int_{t'_{Sch\ddot{u}E}}^{0} \frac{1}{a(t)} dt = \int_{t_{Sch\ddot{u}E}}^{t'_{Sch\ddot{u}E}} \frac{1}{a(t)} dt. \tag{40}$$

Since $a(t)$ does vary significantly only on cosmological time scales and since $|t'_{SchüE} - t_{SchüE}|$ is very much smaller than cosmological time scales then

$$\int_{t_{SchüE}}^{t'_{SchüE}} \frac{1}{a(t)} dt \simeq \frac{t'_{SchüE} - t_{SchüE}}{a(t'_{SchüE})}. \tag{41}$$

On the other hand, combining (30) and (31)

$$\begin{aligned} \chi_{SchüE,E} - \chi'_{SchüE,E} = &\sqrt{\chi_E^2 + \chi_{Schü}^2 + 2\chi_E \chi_{Schü} \cos \varphi_{SchüE}} \\ &- \sqrt{\chi_E^2 + \chi_{Schü}^2 + 2\chi_E \chi_{Schü} \cos \varphi'_{SchüE}}. \end{aligned} \tag{42}$$

But,

$$\begin{aligned} \cos \varphi'_{SchüE} &= -\cos \left(-\arcsin \left(\frac{\chi_E \sin \alpha'}{\chi_{Schü}} \right) + \alpha' \right) \\ &\simeq -\cos \left(-\frac{\chi_E \, \alpha'}{\chi_{Schü}} + \alpha' \right) \simeq -1 + \frac{1}{2} \left(\frac{\chi_E - \chi_{Schü}}{\chi_{Schü}} \right)^2 \alpha'^2, \end{aligned} \tag{43}$$

and similarly

$$\cos \varphi_{SchüE} \simeq -1 + \frac{1}{2} \left(\frac{\chi_E - \chi_{Schü}}{\chi_{Schü}} \right)^2 \alpha^2, \tag{44}$$

and then one gets up to second order in the physical angles α and α'

$$\chi_{SchüE,E} - \chi'_{SchüE,E} \simeq (\chi_E - \chi_{Schü}) \frac{\chi_E}{2\chi_{Schü}} \left(\alpha^2 - \alpha'^2 \right). \tag{45}$$

Combining (41) and (45), one gets an approximate expression for $t'_{SchüE} - t_{SchüE}$

$$t'_{SchüE} - t_{SchüE} \simeq (\chi_E - \chi_{Schü}) \frac{a(t'_{SchüE}) \, \chi_E}{2\chi_{Schü}} \left(\alpha^2 - \alpha'^2 \right), \tag{46}$$

which may be used to deduce $t_{SchüE}$ if $t'_{SchüE}$ has been determined from (37). Since $\alpha > \alpha'$, the lower trajectory photon emerges from the Schücking sphere before the upper trajectory photon.

The upper trajectory photon emerges from the Schücking sphere with 4-velocity

$$\dot{t}'_{SchüE} = \frac{1}{a'_{SchüE}}, \quad \dot{\chi}'_{SchüE} = -\frac{\cos(\varphi'_{SchüE} - \alpha')}{a'^2_{SchüE}}, \quad \dot{\varphi}'_{SchüE} = \frac{\chi'_p}{a'^2_{SchüE} \chi^2_{Schü}}, \tag{47}$$

where

$$a'_{Sch\ddot{u}E} := a(t'_{Sch\ddot{u}E}), \tag{48}$$

and the lower trajectory photon emerges from the Schücking sphere with 4-velocity

$$\dot{t}_{Sch\ddot{u}E} = \frac{1}{a_{Sch\ddot{u}E}}, \quad \dot{\chi}_{Sch\ddot{u}E} = -\frac{\cos(\varphi_{Sch\ddot{u}E} + \alpha)}{a^2_{Sch\ddot{u}E}}, \quad \dot{\varphi}_{Sch\ddot{u}E} = \frac{-\chi_p}{a^2_{Sch\ddot{u}E}\chi^2_{Sch\ddot{u}}}, \tag{49}$$

where

$$a_{Sch\ddot{u}E} := a(t_{Sch\ddot{u}E}). \tag{50}$$

Let γ'_F, γ_F be the smaller physical angles between the un-oriented direction of the upper trajectory photon and the direction towards the lens, and between the un-oriented direction of the lower trajectory photon and the direction towards the lens. We have

$$\gamma'_F = \arctan\left(\left|\chi_{Sch\ddot{u}}\frac{\dot{\varphi}'_{Sch\ddot{u}E}}{\dot{\chi}'_{Sch\ddot{u}E}}\right|\right) = \pi - \left(\varphi'_{Sch\ddot{u}E} - \alpha'\right) = \arcsin\left(\frac{\chi'_p}{\chi_{Sch\ddot{u}}}\right), \tag{51}$$

and

$$\gamma_F = \arctan\left(\left|\chi_{Sch\ddot{u}}\frac{\dot{\varphi}_{Sch\ddot{u}E}}{\dot{\chi}_{Sch\ddot{u}E}}\right|\right) = \pi + (\varphi_{Sch\ddot{u}E} + \alpha) = \arcsin\left(\frac{\chi_p}{\chi_{Sch\ddot{u}}}\right). \tag{52}$$

Step 2: Determination of $t'_{Sch\ddot{u}S}$ and $t_{Sch\ddot{u}S}$.

We have first to translate the 4-velocities of the upper and lower trajectories photons into the coordinates (T, r, φ). They will serve as final conditions for the geodesic equation inside the Schücking sphere where prevails Kottler metric. Using the inverse Jacobian (13), one gets

$$\dot{r}'_{Sch\ddot{u}E} = \frac{C'_{Sch\ddot{u}E} - \cos\left(\varphi'_{Sch\ddot{u}E} - \alpha'\right)}{a'_{Sch\ddot{u}E}} \tag{53}$$

and

$$\dot{r}_{Sch\ddot{u}E} = \frac{C_{Sch\ddot{u}E} - \cos(\varphi_{Sch\ddot{u}E} + \alpha)}{a_{Sch\ddot{u}E}}, \tag{54}$$

with

$$C'_{Sch\ddot{u}E} = C_{Sch\ddot{u}}(t'_{Sch\ddot{u}E}) \tag{55}$$

and

$$C_{Sch\ddot{u}E} = C_{Sch\ddot{u}}(t_{Sch\ddot{u}E}), \tag{56}$$

where $C_{Sch\ddot{u}}(t)$ is given by

$$\begin{aligned} C_{Sch\ddot{u}}(t) = \sqrt{1 - B_{Sch\ddot{u}}(t)} &= \sqrt{\frac{A}{a(t)}\chi^2_{Sch\ddot{u}} + \frac{\Lambda}{3}a^2(t)\chi^2_{Sch\ddot{u}}} \\ &= \chi_{Sch\ddot{u}}\sqrt{\frac{A}{a(t)} + \frac{\Lambda}{3}a^2(t)}. \end{aligned} \tag{57}$$

Let γ'_K and γ_K denote respectively the smaller coordinate angles between the un-oriented direction of the upper trajectory photon and the direction towards the lens and between the un-oriented direction of the lower trajectory photon and the direction towards the lens. We have

$$\gamma'_K =: \arctan\left(\left|r'_{Sch\ddot{u}E}\frac{\dot{\varphi}'_{Sch\ddot{u}E}}{\dot{r}'_{Sch\ddot{u}E}}\right|\right) = \arctan\left(\frac{\sin\gamma'_F}{C'_{Sch\ddot{u}E} + \cos\gamma'_F}\right), \tag{58}$$

and

$$\gamma_K =: \arctan\left(\left|r_{Sch\ddot{u}E}\frac{\dot{\varphi}_{Sch\ddot{u}E}}{\dot{r}_{Sch\ddot{u}E}}\right|\right) = \arctan\left(\frac{\sin\gamma_F}{C_{Sch\ddot{u}E} + \cos\gamma_F}\right). \tag{59}$$

Moreover, we have at our disposal an initial condition, [12], which we may use to set $T'_{Sch\ddot{u}E} = t'_{Sch\ddot{u}E}$. We have thus specified the final conditions of the geodesic equation inside the Schücking sphere.

Making use of the Christoffel symbols of the Kottler metric in the equatorial plane $\theta = \pi/2$

$$\Gamma^T_{Tr} = \frac{B'}{2B}, \quad \Gamma^r_{TT} = \frac{BB'}{2}, \quad \Gamma^r_{rr} = -\frac{B'}{2B}, \quad \Gamma^r_{\varphi\varphi} = -rB, \quad \Gamma^{\varphi}_{r\varphi} = \frac{1}{r}, \tag{60}$$

the geodesic equations then read

$$\ddot{T} + \frac{B'(r)}{B(r)}\dot{T}\dot{r} = 0, \tag{61}$$

$$\ddot{r} + \frac{1}{2}B(r)B'(r)\dot{T}^2 - \frac{1}{2}\frac{B'(r)}{B(r)}\dot{r}^2 - rB(r)\,\dot{\varphi}^2 = 0, \tag{62}$$

$$\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0, \tag{63}$$

from which we deduce three first integrals

$$\dot{T} = 1/B(r), \tag{64}$$

$$\dot{\varphi}r^2 = J, \tag{65}$$

$$\frac{\dot{r}^2}{B(r)} + \frac{J^2}{r^2} - \frac{1}{B(r)} = -E. \tag{66}$$

Equation (65) comes from invariance of the metric under rotations and J has the meaning of an angular momentum per unit mass, (66) comes from invariance of the metric under time translations and E has the meaning of energy per unit mass. For the photon $E = 0$.

Eliminating the affine parameter between (65) and (66), one gets in the case of a photon

$$\frac{dr}{d\varphi} = \pm r\sqrt{\frac{r^2}{J^2} - B}. \tag{67}$$

At the peri-lens r'_p for the upper trajectory photon $\left.\frac{dr}{d\varphi}\right|_{r'_p} = 0$, from which one deduces the expression of J in terms of r'_p

$$J = \frac{r'_p}{\sqrt{B(r'_p)}}. \tag{68}$$

Similarly at the peri-lens r_p for the lower trajectory photon $\left.\frac{dr}{d\varphi}\right|_{r_p} = 0$ and one deduces the expression for J in terms of r_p

$$J = \frac{r_p}{\sqrt{B(r_p)}}. \tag{69}$$

Replacing J by the appropriate expression (68) or (69), one gets

$$\frac{d\varphi}{dr} = \pm \frac{1}{r\sqrt{r^2/r'^2_p - 1}} \left(1 - \frac{s}{r} - \frac{s}{r'_p}\frac{r}{r + r'_p}\right)^{-1/2}, \tag{70}$$

valid for the upper trajectory photon and

$$\frac{d\varphi}{dr} = \pm \frac{1}{r\sqrt{r^2/r^2_p - 1}} \left(1 - \frac{s}{r} - \frac{s}{r_p}\frac{r}{r + r_p}\right)^{-1/2}. \tag{71}$$

valid for the lower trajectory photon, where s denotes the Schwarzschild radius $s = 2GM$.

It is worthwhile to notice that the cosmological constant has disappeared from (70) and also from (71).

Since $s/r'_p \ll 1$ and $s/r_p \ll 1$, we will hereafter only retain terms up to linear order in s/r'_p or in s/r_p. In this approximation

$$r'_p \simeq a(t'_{Schü E})\chi_{Schü} \sin\gamma'_K - GM \tag{72}$$

and

$$r_p \simeq a(t_{Schü E})\chi_{Schü} \sin\gamma_K - GM. \tag{73}$$

Eliminating now the affine parameter between (64) and (66) and taking into account (68), one gets

$$\frac{dT}{dr} = \pm\frac{1}{B(r)\sqrt{1 - \dfrac{r'^2_p}{r^2}\dfrac{B(r)}{B(r'_p)}}}, \tag{74}$$

for the upper trajectory photon and

$$\frac{dT}{dr} = \pm\frac{1}{B(r)\sqrt{1 - \dfrac{r^2_p}{r^2}\dfrac{B(r)}{B(r_p)}}}, \tag{75}$$

for the lower trajectory photon.

Let us denote $T'_{Schü S}$ and $T'_{Schü E}$ the Kottler times at which the upper trajectory photon penetrates inside and emerges from the vacuole respectively and $t'_{Schü S}$ and $t'_{Schü E}$ the corresponding Friedmann times. From now on, we will denote the expression of $d\varphi/dr$ and dT/dr valid for the upper trajectory photon by $d\varphi'/dr$ and dT'/dr respectively and will continue to denote the expressions valid for the lower trajectory by $d\varphi/dr$ and dT/dr. We have

$$T'_{Schü E} - T'_{Schü S} = \int_{r'_P}^{r(t'_{Schü E})} \left|\frac{dT'}{dr}\right| dr + \int_{r'_p}^{r(t'_{Schü S})} \left|\frac{dT'}{dr}\right| dr, \tag{76}$$

with $r(t) = a(t)\chi_{Schü}$ and where dT'/dr is given by (74). To obtain (76), we have used the fact that T' increases when r decreases from $r(t'_{Schü S})$ to r'_p as well as when r increases from r'_p to $r(t'_{Schü E})$.

Using the relation (14) relating the Kottler time T and the Friedmann time t, one gets

$$T'_{Sch\ddot{u}S} = T'_{Sch\ddot{u}E} - \int_{t'_{Sch\ddot{u}S}}^{t'_{Sch\ddot{u}E}} \frac{dt}{B_{Sch\ddot{u}}(t)}. \tag{77}$$

Moreover, we have at our disposal a free initial condition [12] which we may use to set $t'_{Sch\ddot{u}E} = T'_{Sch\ddot{u}E}$ for instance. Combining (76) and (77), one gets

$$\int_{r'_P}^{r(t'_{Sch\ddot{u}E})} \left|\frac{dT'}{dr}\right| dr + \int_{r'_P}^{r(t'_{Sch\ddot{u}S})} \left|\frac{dT'}{dr}\right| dr - \int_{t'_{Sch\ddot{u}S}}^{t'_{Sch\ddot{u}E}} \frac{dt}{B_{Sch\ddot{u}}(t)} = 0, \tag{78}$$

from which one can deduce $t'_{Sch\ddot{u}S}$. If one is interested in $T'_{Sch\ddot{u}S}$ one can use (77) to obtain it.

Let us denote $T_{Sch\ddot{u}S}$ and $T_{Sch\ddot{u}E}$ the Kottler times at which the lower trajectory photon penetrates into and leaves the Schücking sphere respectively and by $t_{Sch\ddot{u}S}$ and $t_{Sch\ddot{u}E}$ the corresponding Friedmann times.

To compute $t_{Sch\ddot{u}S}$, one can proceed in a similar manner as for $t'_{Sch\ddot{u}S}$. We have the analogous of (76), (77) and (78)

$$T_{Sch\ddot{u}E} - T_{Sch\ddot{u}S} = \int_{r_P}^{r(t_{Sch\ddot{u}E})} \left|\frac{dT}{dr}\right| dr + \int_{r'_P}^{r(t_{Sch\ddot{u}S})} \left|\frac{dT}{dr}\right| dr, \tag{79}$$

$$T_{Sch\ddot{u}S} = T_{Sch\ddot{u}E} - \int_{t_{Sch\ddot{u}S}}^{t_{Sch\ddot{u}E}} \frac{dt}{B_{Sch\ddot{u}}(t)} \tag{80}$$

and

$$\int_{r_P}^{r(t_{Sch\ddot{u}E})} \left|\frac{dT}{dr}\right| dr + \int_{r_P}^{r(t_{Sch\ddot{u}S})} \left|\frac{dT}{dr}\right| dr - \int_{t_{Sch\ddot{u}S}}^{t_{Sch\ddot{u}E}} \frac{dt}{B_{Sch\ddot{u}}(t)} = 0, \tag{81}$$

from which one can deduce $t_{Sch\ddot{u}S}$.

But as in the case of $t_{SchüE}$, we prefer to proceed in a different manner: We determine $t_{SchüS}$ by difference with $t'_{SchüS}$. Combining (76) and (79), one obtains

$$\begin{aligned}
&(T'_{SchüE} - T'_{SchüS}) - (T_{SchüE} - T_{SchüS}) \\
&= \int_{r'_p}^{r(t'_{SchüE})} \left|\frac{dT'}{dr}\right| dr + \int_{r'_p}^{r(t'_{SchüS})} \left|\frac{dT'}{dr}\right| dr \\
&- \int_{r_p}^{r(t_{SchüE})} \left|\frac{dT}{dr}\right| dr - \int_{r_p}^{r(t_{SchüS})} \left|\frac{dT}{dr}\right| dr.
\end{aligned} \tag{82}$$

On the other hand, using (14), we have

$$T'_{SchüE} - T_{SchüE} = \int_{t_{SchüE}}^{t'_{SchüE}} \frac{dt}{B_{Schü}(t)} \simeq \frac{t'_{SchüE} - t_{SchüE}}{B_{Schü}(t'_{SchüE})}, \tag{83}$$

and

$$T'_{SchüS} - T_{SchüS} = \int_{t_{SchüS}}^{t'_{SchüS}} \frac{dt}{B_{Schü}(t)} \simeq \frac{t'_{SchüS} - t_{SchüS}}{B_{Schü}(t'_{SchüS})}, \tag{84}$$

where we have used the fact that $B_{Schü}$ does not vary appreciably on the time intervals $[t_{SchüE}, t'_{SchüE}]$ and $[t'_{SchüS}, t_{SchüS}]$ since these are smaller than cosmological scales.

Substituting (83) and (84) into (82) one gets

$$\begin{aligned}
\frac{t'_{SchüE} - t_{SchüE}}{B_{Schü}(t'_{SchüE})} - \frac{t'_{SchüS} - t_{SchüS}}{B_{Schü}(t'_{SchüS})} &\simeq \int_{r'_p}^{r(t'_{SchüE})} \left|\frac{dT'}{dr}\right| dr + \int_{r'_p}^{r(t'_{SchüS})} \left|\frac{dT'}{dr}\right| dr \\
&\quad - \int_{r_p}^{r(t_{SchüE})} \left|\frac{dT}{dr}\right| dr - \int_{r_p}^{r(t_{SchüS})} \left|\frac{dT}{dr}\right| dr \\
&\simeq \int_{r'_p}^{r(t'_{SchüE})} \left|\frac{dT'}{dr}\right| dr - \int_{r_p}^{r(t'_{SchüE})} \left|\frac{dT}{dr}\right| dr
\end{aligned}$$

$$+\int_{r'_p}^{r(t'_{SchüS})}\left|\frac{dT'}{dr}\right|dr-\int_{r_p}^{r(t'_{SchüS})}\left|\frac{dT}{dr}\right|dr$$
$$-\int_{r(t'_{SchüE})}^{r(t_{SchüE})}\left|\frac{dT}{dr}\right|dr-\int_{r(t'_{SchüS})}^{r(t_{SchüS})}\left|\frac{dT}{dr}\right|dr. \tag{85}$$

But since we deal with smaller length and time scales than cosmological ones:

$$\int_{r(t'_{SchüE})}^{r(t_{SchüE})}\left|\frac{dT}{dr}\right|dr\simeq\left|\frac{dT}{dr}\right|_{r(t'_{SchüE})}\left[a(t_{SchüE})-a(t'_{SchüE})\right]\chi_{Schü},$$
$$\int_{r(t'_{SchüS})}^{r(t_{SchüS})}\left|\frac{dT}{dr}\right|dr\simeq\left|\frac{dT}{dr}\right|_{r(t'_{SchüS})}\left[a(t_{SchüS})-a(t'_{SchüS})\right]\chi_{Schü}.$$

Using Eq. (6), one deduces

$$a(t_{SchüE})-a(t'_{SchüE})\simeq\sqrt{\frac{A}{a(t'_{SchüE})}+\frac{\Lambda}{3}a^2(t'_{SchüE})}\left(t_{SchüE}-t'_{SchüE}\right),$$
$$a(t_{SchüS})-a(t'_{SchüS})\simeq\sqrt{\frac{A}{a(t'_{SchüS})}+\frac{\Lambda}{3}a^2(t'_{SchüS})}\left(t_{SchüS}-t'_{SchüS}\right),$$

with

$$A=\frac{1}{3}\rho_{dust0}\quad a_0^3=\frac{1}{3}\rho_{dust0}. \tag{86}$$

On the other hand, using the results of reference [12]

$$\int_{r'_p}^{r(t'_{SchüE})}\left|\frac{dT'}{dr}\right|dr-\int_{r_p}^{r(t'_{SchüE})}\left|\frac{dT}{dr}\right|dr$$
$$\simeq\frac{M}{8\pi}\left[\frac{1}{2}\left(1-\frac{r'^2_p}{r^2_p}\right)\frac{8\pi r_p^2}{Mr(t'_{SchüE})}\right.$$

$$-\frac{3}{2}\left(\frac{r_p^2}{r_p'^2}-1\right)\frac{M}{8\pi r_p^2\sqrt{\frac{\Lambda}{3}}}\ \mathrm{arctanh}\left(\sqrt{\frac{\Lambda}{3}}r(t'_{Sch\ddot{u}E})\right)$$
$$-2\ln\left(\frac{r'_p}{r_p}\right)\Bigg].$$

It suffices to make the following replacements

$$x\rightarrow\frac{r'_p}{r_p},\qquad \epsilon_T\rightarrow\frac{r_p}{r(t'_{Sch\ddot{u}E})},\qquad \delta\rightarrow\frac{M}{8\pi r_p},\qquad \lambda\rightarrow r_p\sqrt{\frac{\Lambda}{3}} \tag{87}$$

in Eq. (23) of reference [21].

In the same manner

$$\int_{r'_p}^{r(t'_{Sch\ddot{u}S})}\left|\frac{dT'}{dr}\right|dr-\int_{r_p}^{r(t'_{Sch\ddot{u}S})}\left|\frac{dT}{dr}\right|dr$$
$$\simeq\frac{M}{8\pi}\left[\frac{1}{2}\left(1-\frac{r_p'^2}{r_p^2}\right)\frac{8\pi r_p^2}{Mr(t'_{Sch\ddot{u}S})}\right.$$
$$-\frac{3}{2}\left(\frac{r_p^2}{r_p'^2}-1\right)\frac{M}{8\pi r_p^2\sqrt{\frac{\Lambda}{3}}}\ \mathrm{arctanh}\left(\sqrt{\frac{\Lambda}{3}}r(t'_{Sch\ddot{u}S})\right)$$
$$\left.-2\ln\left(\frac{r'_p}{r_p}\right)\right].$$

Collecting together the previous results, one gets an analytical approximation for $t_{Sch\ddot{u}S}-t'_{Sch\ddot{u}S}$

$$t_{Sch\ddot{u}S}-t'_{Sch\ddot{u}S}\simeq\left\{\frac{M}{8\pi}\left[\frac{1}{2}\left(1-\frac{r_p'^2}{r_p^2}\right)\frac{8\pi r_p^2}{M\chi_{Sch\ddot{u}}}\left(\frac{1}{a(t'_{Sch\ddot{u}E})}+\frac{1}{a(t'_{Sch\ddot{u}S})}\right)\right.\right.$$
$$-\frac{3}{2}\left(\frac{r_p^2}{r_p'^2}-1\right)\frac{M}{8\pi r_p^2\sqrt{\Lambda/3}}\left(\mathrm{arctanh}\left(\sqrt{\frac{\Lambda}{3}}r(t'_{Sch\ddot{u}E})\right)\right.$$
$$\left.\left.+\ \mathrm{arctanh}\left(\sqrt{\frac{\Lambda}{3}}r(t'_{Sch\ddot{u}S})\right)\right)-4\ln\left(\frac{r'_p}{r_p}\right)\right]$$
$$+\left[\chi_{Sch\ddot{u}}\sqrt{\frac{A}{a(t'_{Sch\ddot{u}E})}+\frac{\Lambda}{3}a^2(t'_{Sch\ddot{u}E})}\left|\frac{dT}{dr}\right|_{r(t'_{Sch\ddot{u}E})}\right.$$
$$\left.-\frac{1}{B_{Sch\ddot{u}}(t'_{Sch\ddot{u}E})}\right]$$

$$\times a(t'_{Sch\ddot{u}E})\,(\chi_E-\chi_{Sch\ddot{u}})\,\frac{\chi_E}{2\chi_{Sch\ddot{u}}}\left(\alpha^2-\alpha'^2\right)\Bigg\}$$
$$\times\left(\frac{1}{B_{Sch\ddot{u}}(t'_{Sch\ddot{u}S})}\right.$$
$$\left.+\chi_{Sch\ddot{u}}\sqrt{\frac{A}{a(t'_{Sch\ddot{u}S})}+\frac{\Lambda}{3}a^2(t'_{Sch\ddot{u}S})}\left|\frac{dT}{dr}\right|_{r(t'_{Sch\ddot{u}S})}\right)^{-1}, \quad (88)$$

where we have also used the expression (46) for $(t'_{Sch\ddot{u}E}-t_{Sch\ddot{u}E})$. Then the knowledge of $t'_{Sch\ddot{u}S}$ allows one to deduce $t_{Sch\ddot{u}S}$.

Let us now determine in turn the polar angles $\varphi'_{Sch\ddot{u}S}$ and $\varphi_{Sch\ddot{u}S}$ at which the upper and lower trajectories photons penetrate inside the Schücking sphere. Since the angle φ' increases all the way from $r'_{Sch\ddot{u}S}$ to r'_p and from r'_p to $r'_{Sch\ddot{u}E}$, one gets

$$\varphi'_{Sch\ddot{u}S}=\varphi'_{Sch\ddot{u}E}-\int\limits_{r'_p}^{r(t'_{Sch\ddot{u}S})}\left|\frac{d\varphi'}{dr}\right|dr-\int\limits_{r'_p}^{r(t'_{Sch\ddot{u}E})}\left|\frac{d\varphi'}{dr}\right|dr. \quad (89)$$

To the linear order in the ratio s/r'_p, Schwarzschild radius $s = 2GM$ divided by peri-lens r'_p, one gets [12]

$$\varphi'_{Sch\ddot{u}S}\simeq\varphi'_{Sch\ddot{u}E}-\pi+\arcsin\left(\frac{r'_p}{r'_{Sch\ddot{u}E}}\right)+\arcsin\left(\frac{r'_p}{r'_{Sch\ddot{u}S}}\right)$$
$$-\frac{1}{2}\frac{s}{r'_{Sch\ddot{u}E}}\sqrt{\frac{r'^2_{Sch\ddot{u}E}}{r'^2_p}-1}-\frac{1}{2}\frac{s}{r'_{Sch\ddot{u}S}}\sqrt{\frac{r'^2_{Sch\ddot{u}S}}{r'^2_p}-1}$$
$$-\frac{1}{2}\frac{s}{r'_p}\sqrt{\frac{r'_{Sch\ddot{u}E}-r'_p}{r'_{Sch\ddot{u}E}+r'_p}}-\frac{1}{2}\frac{s}{r'_p}\sqrt{\frac{r'_{Sch\ddot{u}S}-r'_p}{r'_{Sch\ddot{u}S}+r'_p}}. \quad (90)$$

Let us now compute the polar angle $\varphi_{Sch\ddot{u}S}$ at which the lower trajectory photon penetrates inside the Schücking sphere. Since φ decreases when r varies from $r_{Sch\ddot{u}S}$ to r_p and also when r varies from r_p to $r_{Sch\ddot{u}E}$, then

$$\varphi_{Sch\ddot{u}S}=\varphi_{Sch\ddot{u}E}+\int\limits_{r_p}^{r(t_{Sch\ddot{u}S})}\left|\frac{d\varphi}{dr}\right|dr+\int\limits_{r_p}^{r(t_{Sch\ddot{u}E})}\left|\frac{d\varphi}{dr}\right|dr. \quad (91)$$

To the linear order in the ratio s/r_p, one gets for $\varphi_{SchüS}$ [12]

$$\begin{aligned}\varphi_{SchüS} \simeq \varphi_{SchüE} + \pi - \arcsin\left(\frac{r_P}{r_{SchüE}}\right) - \arcsin\left(\frac{r_P}{r_{SchüS}}\right)\\ +\frac{1}{2}\frac{s}{r_{SchüE}}\sqrt{\frac{r^2_{SchüE}}{r^2_p}-1}+\frac{1}{2}\frac{s}{r_{SchüS}}\sqrt{\frac{r^2_{SchüS}}{r^2_p}-1}\\ +\frac{1}{2}\frac{s}{r_p}\sqrt{\frac{r_{SchüE}-r_p}{r_{SchüE}+r_p}}+\frac{1}{2}\frac{s}{r_p}\sqrt{\frac{r_{SchüS}-r_p}{r_{SchüS}+r_p}}.\end{aligned} \tag{92}$$

Step 3: Determination of t'_S and t_S.

One can now compute φ'_S

$$\varphi'_S = \varphi'_{SchüS} - \gamma'_{FS} + \arcsin\left(\frac{\chi_{Schü}}{\chi_{L,S}}\sin\gamma'_{FS}\right), \tag{93}$$

where $\varphi'_{SchüS}$ is given by (90) and where γ'_{FS} the smaller physical angle between the un-oriented direction of the photon and the direction towards the lens as the photon penetrates inside the Schücking sphere from the external side:

$$\gamma'_{FS} = \arctan\left(\left|\chi_{Schü}\frac{\dot{\varphi}'_{SchüS}}{\dot{\chi}'_{SchüS}}\right|\right) = \arctan\left(-\chi_{Schü}\frac{\dot{\varphi}'_{SchüS}}{\dot{\chi}'_{SchüS}}\right), \tag{94}$$

where $\dot{\varphi}'_{SchüS}$ and $\dot{\chi}'_{SchüS}$ are given respectively by

$$\dot{\varphi}'_{SchüS} = \frac{r'_p}{r'^2_{SchüS}\sqrt{B(r'_p)}} = \frac{r'_p}{a^2(t'_{SchüS})\chi^2_{Schü}\sqrt{B(r'_p)}}, \tag{95}$$

and

$$\dot{\chi}'_{SchüS} = -\frac{\sqrt{1-\frac{r'^2_p}{a^2(t'_{SchüS})\chi^2_{Schü}}\frac{B_{Schü}(t'_{SchüS})}{B(r'_p)}}}{a(t'_{SchüS})B_{Schü}(t'_{SchüS})} - \frac{C_{Schü}(t'_{SchüS})}{a(t'_{SchüS})}\frac{1}{B_{Schü}(t'_{SchüS})}, \tag{96}$$

with

$$C_{Schü}(t'_{SchüS}) = \sqrt{\frac{A}{a(t'_{SchüS})}\chi^2_{Schü} + \frac{\Lambda}{3}a^2(t'_{SchüS})\chi^2_{Schü}} \tag{97}$$

and

$$B_{Schü}(t'_{SchüS}) = 1 - \frac{A}{a(t'_{SchüS})}\chi^2_{Schü} - \frac{\Lambda}{3}a^2(t'_{SchüS})\chi^2_{Schü}. \tag{98}$$

To obtain (96) we have used the Jacobian of the coordinate transformation $(T, r) \rightarrow (t, \chi)$, (13), together with the expressions of $\dot{T}'_{Sch\ddot{u}S}$ and $\dot{r}'_{Sch\ddot{u}S}$

$$\begin{aligned} \dot{T}'_{Sch\ddot{u}S} &= \frac{1}{B(t'_{Sch\ddot{u}S})}, \\ \dot{r}'_{Sch\ddot{u}S} &= -\sqrt{1 - \frac{r'^2_p}{r'^2_{Sch\ddot{u}S}} \frac{B(t'_{Sch\ddot{u}S})}{B(r'_p)}}. \end{aligned} \tag{99}$$

$\chi_{L,S}$ is the geodesic distance between the source and the lens, which can be accurately approximated by

$$\chi_{L,S} \simeq \chi_S - \chi_L. \tag{100}$$

In the same manner, once $\varphi_{Sch\ddot{u}S}$ determined, one can compute the polar angle φ_S corresponding to the source by a relation analogous to (93)

$$\varphi_S = \varphi_{Sch\ddot{u}S} + \gamma_{FS} - \arcsin\left(\frac{\chi_{Sch\ddot{u}}}{\chi_{L,S}} \sin\gamma_{FS}\right), \tag{101}$$

where $\varphi_{Sch\ddot{u}S}$ is given by (92) and where γ_{FS} is the smaller physical angle between the un-oriented direction of the lower trajectory photon and the direction towards the lens as the photon penetrates inside the Schücking sphere from the external side:

$$\gamma_{FS} = \arctan\left(\left|\chi_{Sch\ddot{u}} \frac{\dot{\varphi}_{Sch\ddot{u}S}}{\dot{\chi}_{Sch\ddot{u}S}}\right|\right) = \arctan\left(\chi_{Sch\ddot{u}} \frac{\dot{\varphi}_{Sch\ddot{u}S}}{\dot{\chi}_{Sch\ddot{u}S}}\right). \tag{102}$$

Using

$$\dot{\chi}_{Sch\ddot{u}S} = -\frac{\sqrt{1 - \frac{r^2_p}{r^2_{Sch\ddot{u}S}} \frac{B_{Sch\ddot{u}}(t_{Sch\ddot{u}S})}{B(r_p)}}}{a(t_{Sch\ddot{u}S})B_{Sch\ddot{u}}(t_{Sch\ddot{u}S})} - \frac{C_{Sch\ddot{u}}(t_{Sch\ddot{u}S})}{a(t_{Sch\ddot{u}S})B_{Sch\ddot{u}}(t_{Sch\ddot{u}S})}, \tag{103}$$

obtained by using the Jacobian of the coordinate transformation $(T, r) \rightarrow (t, \chi)$ and

$$\dot{\varphi}_{Sch\ddot{u}S} = -\frac{r_p}{a^2(t_{Sch\ddot{u}S})\chi^2_{Sch\ddot{u}}\sqrt{B(r_p)}}, \tag{104}$$

then

$$\gamma_{FS} = \arctan\left\{\left(\frac{r_{Sch\ddot{u}S}\sqrt{B(r_p)}}{r_p B_{Sch\ddot{u}}(t_{Sch\ddot{u}S})}\left[C_{Sch\ddot{u}}(t_{Sch\ddot{u}S}) + \sqrt{1 - \frac{r^2_p}{r^2_{Sch\ddot{u}S}} \frac{B_{Sch\ddot{u}}(t_{Sch\ddot{u}S})}{B(r_p)}}\right]\right)^{-1}\right\}. \tag{105}$$

For a given M, one obtains in turn $t'_{Sch\ddot{u}E}$, $t'_{Sch\ddot{u}S}$, $\varphi'_{Sch\ddot{u}S}$, φ'_S, $t_{Sch\ddot{u}E}$, $t_{Sch\ddot{u}S}$, $\varphi_{Sch\ddot{u}S}$ and φ_S.

In general $\varphi'_S \neq \varphi_S$. To achieve $\varphi'_S = \varphi_S$, which corresponds to the fact that the upper and the lower trajectories photons are emitted by the same source, we have to adjust M, i.e., we have to vary M until the equality $\varphi'_S = \varphi_S$ is satisfied. We end up with values of M, $t'_{Sch\ddot{u}E}$, $t'_{Sch\ddot{u}S}$, $\varphi'_{Sch\ddot{u}S}$, $t_{Sch\ddot{u}E}$, $t_{Sch\ddot{u}S}$, $\varphi_{Sch\ddot{u}S}$ and $\varphi'_S = \varphi_S$.

We are now in a position to determine $t_S - t'_S$.

Using once again some elementary Euclidean geometry, similar to that used to obtain $\chi'_{Sch\ddot{u}E,E}$ (30), one obtains for the geodesic distance $\chi'_{Sch\ddot{u}S,S}$ between the source S and the photon of the upper trajectory as it crosses the Schücking sphere in the half space containing the source:

$$\chi'_{S,Sch\ddot{u}S} = \sqrt{\chi^2_{L,S} + \chi^2_{Sch\ddot{u}} - 2\chi_{L,S}\chi_{Sch\ddot{u}} \cos\left(\varphi'_{Sch\ddot{u}S} - \varphi_S\right)}. \tag{106}$$

Proceeding in the same manner, we get for the geodesic distance $\chi_{Sch\ddot{u}S,S}$ between the source S and the photon of the lower trajectory as it crosses the Schücking sphere in the half space containing the source

$$\chi_{Sch\ddot{u}S,S} = \sqrt{\chi^2_{L,S} + \chi^2_{Sch\ddot{u}} - 2\chi_{L,S}\chi_{Sch\ddot{u}} \cos\left(\varphi_{Sch\ddot{u}S} - \varphi_S\right)}. \tag{107}$$

Making use of the approximations

$$\cos x \simeq 1 - x^2/2 \quad \text{and} \quad \sqrt{1+x} \simeq 1 + x/2, \tag{108}$$

valid for $|x| \ll 1$, one gets

$$\chi'_{Sch\ddot{u}S,S} \simeq \left(\chi_{L,S} - \chi_{Sch\ddot{u}}\right) - \frac{\left(\varphi'_{Sch\ddot{u}S} - \varphi_S\right)^2}{2\left(\chi^{-1}_{L,S} - \chi^{-1}_{Sch\ddot{u}}\right)}, \tag{109}$$

$$\chi_{Sch\ddot{u}S,S} \simeq \left(\chi_{L,S} - \chi_{Sch\ddot{u}}\right) - \frac{\left(\varphi_{Sch\ddot{u}S} - \varphi_S\right)^2}{2\left(\chi^{-1}_{L,S} - \chi^{-1}_{Sch\ddot{u}}\right)}. \tag{110}$$

Then

$$\chi'_{Sch\ddot{u}S,S} - \chi_{Sch\ddot{u}S,S} \simeq \frac{\left(\varphi_{Sch\ddot{u}S} - \varphi_S\right)^2 - \left(\varphi'_{Sch\ddot{u}S} - \varphi_S\right)^2}{2\left(\chi^{-1}_{L,S} - \chi^{-1}_{Sch\ddot{u}}\right)}. \tag{111}$$

On the other hand, Eq. (18) with the final condition $\chi(t'_{Sch\ddot{u}S}) = 0$ gives

$$0 - \chi'_{Sch\ddot{u}S,S} = -\int_{t'_S}^{t'_{Sch\ddot{u}S}} \frac{dt}{a(t)}. \tag{112}$$

In a similar manner, from (18) with the final condition $\chi(t_{SchüS}) = 0$, one gets

$$0 - \chi_{SchüS,S} = -\int_{t_S}^{t_{SchüS}} \frac{dt}{a(t)}. \tag{113}$$

From (112) and (113), one deduces an expression for $\chi_{SchüS,S} - \chi'_{SchüS,S}$

$$\begin{aligned}\chi_{SchüS,S} - \chi'_{SchüS,S} &= \int_{t_S}^{t_{SchüS}} \frac{dt}{a(t)} - \int_{t'_S}^{t'_{SchüS}} \frac{dt}{a(t)} \\ &= \int_{t_S}^{t'_S} \frac{dt}{a(t)} + \int_{t'_S}^{t_{SchüS}} \frac{dt}{a(t)} - \int_{t'_S}^{t'_{SchüS}} \frac{dt}{a(t)} \\ &= \int_{t_S}^{t'_S} \frac{dt}{a(t)} + \int_{t'_{SchüS}}^{t_{SchüS}} \frac{dt}{a(t)}. \end{aligned} \tag{114}$$

But since $a(t)$ varies significantly only on time intervals of cosmological nature,

$$\int_{t_S}^{t'_S} \frac{dt}{a(t)} \simeq \frac{t'_S - t_S}{a(t'_S)} \quad \text{and} \quad \int_{t'_{SchüS}}^{t_{SchüS}} \frac{dt}{a(t)} \simeq \frac{t_{SchüS} - t'_{SchüS}}{a(t'_{SchüS})}. \tag{115}$$

Then

$$\chi_{SchüS,S} - \chi'_{SchüS,S} \simeq \frac{t'_S - t_S}{a(t'_S)} + \frac{t_{SchüS} - t'_{SchüS}}{a(t'_{SchüS})}. \tag{116}$$

Equating the right hand sides of (111) and (116), one arrives at

$$\frac{t_S - t'_S}{a(t'_S)} - \frac{t_{SchüS} - t'_{SchüS}}{a(t'_{SchüS})} \simeq \frac{(\varphi_{SchüS} - \varphi_S)^2 - \left(\varphi'_{SchüS} - \varphi_S\right)^2}{2\left(\chi_{L,S}^{-1} - \chi_{Schü}^{-1}\right)}. \tag{117}$$

Then one deduces an expression for $t_S - t'_S$

$$t_S - t'_S \simeq a(t'_S)\left[\frac{t_{SchüS} - t'_{SchüS}}{a(t'_{SchüS})} + \frac{(\varphi_{SchüS} - \varphi_S)^2 - \left(\varphi'_{SchüS} - \varphi_S\right)^2}{2\left(\chi_{L,S}^{-1} - \chi_{Schü}^{-1}\right)}\right]. \tag{118}$$

Hereafter are displayed the results for $\varphi_S (= \varphi'_S)$ the deflexion angle, M the mass of the cluster of galaxies (the lens) and $\Delta t = t_S - t'_S$, the time delay, in the case of the

Table 1 Upper limit value of Λ: $\Lambda = 0.77 \times 3\ \mathrm{am}^{-2} + 20\%$

$\alpha_E \pm 10\%$	± 0	± 0	± 0	$+$	$+$	$+$	$-$	$-$	$-$
$\alpha'_E \pm 10\%$	± 0	$+$	$-$	± 0	$+$	$-$	± 0	$+$	$-$
$-\varphi_S['']$	09.03	08.13	09.94	10.84	09.94	**11.74**	07.23	**06.32**	08.13
$M\,[10^{13} M_\odot]$	01.80	01.98	01.62	01.98	**02.18**	01.78	01.62	01.78	**01.46**
Δt[years]	09.76	09.18	10.25	12.35	11.81	**12.77**	07.38	**06.74**	07.91

Bold values correspond to minimal and maximal values

Table 2 Central value of Λ: $\Lambda = 0.77 \times 3\ \mathrm{am}^{-2}$

$\alpha_E \pm 10\%$	± 0	± 0	± 0	$+$	$+$	$+$	$-$	$-$	$-$
$\alpha'_E \pm 10\%$	± 0	$+$	$-$	± 0	$+$	$-$	± 0	$+$	$-$
$-\varphi_S['']$	09.97	08.98	10.97	11.97	10.97	**12.97**	07.98	**06.98**	08.98
$M\,[10^{13} M_\odot]$	01.82	02.00	01.64	02.00	**02.21**	01.80	01.64	01.80	**01.48**
Δt[years]	09.72	09.14	10.19	12.28	11.76	**12.68**	07.35	**06.73**	07.87

Bold values correspond to minimal and maximal values

Table 3 Lower limit value of Λ: $\Lambda = 0.77 \times 3\ \mathrm{am}^{-2} - 20\%$

$\alpha_E \pm 10\%$	± 0	± 0	± 0	$+$	$+$	$+$	$-$	$-$	$-$
$\alpha'_E \pm 10\%$	± 0	$+$	$-$	± 0	$+$	$-$	± 0	$+$	$-$
$-\varphi_S['']$	10.57	09.51	11.63	12.68	11.63	**13.74**	08.46	**07.40**	09.51
$M\,[10^{13} M_\odot]$	01.80	01.98	01.62	01.98	**02.18**	01.79	01.62	01.79	**01.46**
Δt[years]	09.53	08.97	09.98	12.03	11.53	**12.41**	7.21	**06.60**	07.72

Bold values correspond to minimal and maximal values

Table 4 $\Lambda = 0$

$\alpha_E \pm 10\%$	± 0	± 0	± 0	$+$	$+$	$+$	$-$	$-$	$-$
$\alpha'_E \pm 10\%$	± 0	$+$	$-$	± 0	$+$	$-$	± 0	$+$	$-$
$-\varphi_S['']$	11.86	10.67	13.05	14.23	13.05	**15.42**	09.49	**08.30**	10.67
$M\,[10^{13} M_\odot]$	01.68	01.84	01.51	01.84	**02.03**	01.66	01.51	01.66	**01.36**
Δt[years]	08.70	08.20	09.10	10.97	10.53	**11.30**	06.59	**06.04**	07.05

Bold values correspond to minimal and maximal values

lensed quasar SDSS J1004+4112 [17–19], where '±0' stands for the central value, '+' and '−' stand respectively for the upper and the lower experimental limits.

$$\begin{aligned} \alpha' &= 5'' \pm 10\%, \quad & \alpha &= 10'' \pm 10\%, \\ z_L &= 0.68, & z_S &= 1.734. \end{aligned} \tag{119}$$

The cluster mass M comes from a fitting: the angles φ_S and φ'_S are calculated as a function of M which is varied until the equality $\varphi'_S = \varphi_S$ is satisfied. Tables 1,2,3.

For the cosmological Λ, we take the experimentally favored $\Lambda = 0.77 \cdot 3\ \mathrm{am}^{-2} \pm 20\%$. We have also considered the case without cosmological constant $\Lambda = 0$ in Table 4.

4 Conclusion

In this paper, we have computed the time delay caused by a spherical mass, a cluster of galaxies, in presence of a cosmological constant Λ using the Einstein-Straus solution, which is the appropriate framework for taking into account the precession of the observer and the effect of the other masses of the universe in the form of a homogeneous isotropic dust, the observer being taken comoving with the dust. We have applied our results to the lensed quasar SDSS J1004+4112. We have computed the time delay between the images C and D of the quasar, which are the most aligned images with the lens, and obtained results compatible with the lower bound given by Fohlmeister [20]. Our predictions of the time delay range from **6** to **13** years.

It is worthwhile to compare our results with those of previous computations performed by Schücker and Zaimen [21] and by Kawano and Oguri [22]. Schücker and Zaimen computed the time delay in the framework of the Kottler solution and obtained predictions ranging from **13** to **28** years, using however a different mass of the lens. On the other hand, Kawano and Oguri obtained a time delay of **10** years.

In addition to the hypothesis of sphericity we have made in our calculations an additional assumption: we have supposed implicitly that the photons don't penetrate the mass distribution region, since we have used only the exterior Kottler solution inside the vacuole. It is then worthwhile to repeat the calculations taking into account that the photons can penetrate the interior of the mass distribution, where an interior Kottler solution must be used [23].

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