الجمهورية الجزائرية الديمقراطية الشعبية
PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
وزارة التعليم العالي و البحث العلمي
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH
جامعة عباس لغرور خنشلة
ABBES LAGHROUR- KHENCHELA UNIVERSITY

Faculty of Sciences and Technology

Department of Mathematics and Computer Science

Serial number: ……………

THESIS Submitted for
obtain **Master** degree in mathematics

Field of study: **Mathematics**
Specialty: **Applied Mathematics**

Entitled by:

Fixed-Point Theorems in Complex-Valued Metric Spaces, With Applications

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2021-2022

Acknowledgement

We express our deep gratitude and thanks to our teachers who have always been there for us. Their unconditional support and encouragement has been a great help. You provided us with the tools that we needed to choose the right direction and successfully complete our dissertation.

We are deeply grateful to our esteemed supervisor **Dr. Abdelkrim Abdelhalim** for his invaluable supervision, support and tutelage.

We would like also to express gratitude to **Mr. Y. Mefteh** for having accepted the presidency of our jury and **Mr. A. Achichi** who agreed to participate in our jury, for their guidance.

This page probably could not have been written without the moral support of our families members and our thanks also go to our friends especially for their company.

Finally We reserve a special mention to all those who gave us the support and help we expected.

Dedication

We dedicate this fruit of our long years of study first of all to :

Our dear parents, who are the light of our life, who suffered and sacrificed so much to make us happy, for their advice, affection and encouragement.

Thank you for all your efforts for us, may God keep you, protect you and bless your life.

All the members of our families, our sisters and brothers.

Our teachers and colleagues in the class of the second year Master 2021/2022.

Special greeting to ***M. Laid, H. Sabah, T. Malika and B. Manal****.*

ملخص

في هذا العمل سنثبت وجود ووحدانية النقطة الثابتة المشتركة، لمجموعة من التطبيقات تحت شرط تقلص كسري في فضاءات مترية مركبة، ايضا نستعمل خاصيتي (E, A) والتطبيقات المتوافقة بضعف لايجاد النقطة الثابتة في فضاءات مترية مركبة. بعد ذلك، نستعمل بعض النتائج السابقة في اثبات وجود النقطة الثابتة المشتركة لمجموعات منتهية، والحصول على حلول لجملة معادلات تكاملية من نوع اوريزون.

الكلمات المفتاحية: النقطةالثابتة، النقطة الثابتة المشتركة، فضاء متري مركب، فضاء ب-متري مركب، التوافق بضعف، العلاقات الضمنية، خاصية (E, A)، معادلات أوريزون التكاملية.

Abstract

In this work, we have proved the existence and uniqueness of common fixed point theorems under rational contraction condition in complex valued b-metric space. Also, using (E.A) property and weakly compatible mappings in order to find the fixed point in complex valued metric space.
In the setting of that, we use the results to prove the existence of common fixed point finite families of maps and common solution of the Urysohn's integral equations in complex-valued metric space.

Key words: fixed point, common fixed point, complex-valued metric space, complex-valued b-metric space, weakly compatible, implicit relations, (E.A) property, Urysohn integral equations.

Résumé

Dans ce travail, nous avons prouvé l'existence et l'unicité de théorèmes de point fixe commune sous condition de contraction rationnelle dans un espace b-métrique à valeurs complexes. En outre, en utilisant la propriété (E.A) et des applications faiblement compatibles afin de trouver le point fixe dans l'espace métrique à valeurs complexes.
Aprés, nous utilisons les résultats pour prouver un théorème de point fixe commun pour des familles finies d'applications et une solution commune des équations intégrales d'Urysohn dans un espace métrique complexe.

Mots clés: Point fixe, Point fixe commune, Espace métrique complexe, Espace b-métrique complexe, Faiblement compatible, Relations implicite, (E,A) propriété, Equations intégrales d'Urysohn.

Contents

Introduction

The Fixed point theory is a well known research field in mathematical sciences. Also, is an important tool in the area of the non-linear analysis [1].

It is well known that the Banach contraction principle which establishes the existence and uniqueness of a solution of an operator problem $Tx = x$ using adequate and simple conditions is the most powerful result in the field of metric fixed point theory. This principle provides distinctive solution to various mathematical models such as integral equations, differential equations and functional equations.. The majority of these findings are generalizations of various contractive conditions in metric spaces. In addition, Bakhtin [4] developed the notion of b-metric space.

After this classical result, many fixed point results have been developed, Azam et al. [3] introduced the notion of complex valued metric space and proved common fixed point theorems for two self-mappings satisfying a rational contraction type inequality[6].

In 1998, Jungck and Rhoades [5] introduced the notion of weakly compatible. Afterward, Bhatt et al. [5] initiated the concept of weakly compatible maps to study common fixed point theorem for weakly compatible maps in complex valued metric spaces.

Moreover, Verma and Pathak [14] introduced the notion of property $(E.A)$ and established common fixed point theorems using this property in complex valued metric space.

We looked at three chapters in this work:

In **chapter 1**, we will recall some known definitions and results of the metric spaces and complex-valued metric spaces which are useful for our work. Also, we discuss fixed point theorems in complex-valued metric spaces, thus, we recalled the (E,A) property and some integral equations linear and nonlinear type.

In **chapter 2**, we have proved the existence and uniqueness of common fixed point theorems for a pair of mappings satisfying the under rational contraction condition in complex-valued b-metric space. Also, using $(E.A)$ property weakly compatible mappings in order to find fixed point in complex-valued metric space, as well as the same for self-mappings satisfying contractive defined by implicit relations.

Finally, in the **last chapter**, we use the result, to prove the existence of common fixed point theorem for finite families of maps and find the common solution of the Urysohn's integral equations in complex-valued metric space.

CHAPTER

1

PRELIMINARIES

In this chapter, we will recall some known definitions and results which will be used in this work.

1.1 Metric Spaces, Banach Fixed Point and b-Metric Spaces

Definition 1.1.1 *: Let X be a set, we call distance on X any mapping $d : X \to X$ in $\mathbb{R}^+$ such as:*

(1) $d(x, y) = 0 \Leftrightarrow x = y,$

(2) $d(x, y) = d(y, x),$

(3) $d(x, y) \leq d(x, z) + d(z, y).$

The pair (X, d) is called a metric space.

Definition 1.1.2 *: Given (X, d) a metric space, a function $T : X \to X$ is said to be a contraction mapping if there is a constant θ with $0 \leq \theta \leq 1$ such that for all $x,\ y \in X$*

$$d(T(x), T(y)) \leq \theta \cdot d(x, y) \tag{1.1}$$

Theorem 1.1.1 *: **(Banach Fixed Point Theorem)***

*Let (X, d) be a complete metric space with a contraction mapping $F : X \to X$ (that is $d(F(x), F(y)) \leq \theta d(x, y), \forall\ x, y \in X$ with $\theta \in [0, 1]$). Then F admits a **unique fixed point** $F(x^*) = x^*$.*

Furthermore, x^* can be found by starting with an arbitrary element $x_0 \in X$ and defining the sequence $\{x_n\},\ n \geq 0$ by $x_n = F(x_{n-1})$. Then, $x_n \to x^*$.

Definition 1.1.3 *: Let X be a set and let d be a function from X into $[0, \infty)$. Then (X, d) is said to be a b-metric space if the following hold:*

(1) $d(x, y) = 0$ *if and only if* $x = y$;

(2) $d(x, y) = d(y, x)$ *(symmetry);*

There exists $s \geq 1$ satisfying $d(x, z) \leq s(d(x, y) + d(y, z))$ for any $x, y, z \in X$ (s-relaxed triangle inequality).

1.2 Complex Valued b-Metric Spaces

Definition 1.2.1 *: Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$. Thus $z_1 \precsim z_2$ if one of the following holds:*

(1) $Re(z_1) = Re(z_2)$ *and* $Im(z_1) = Im(z_2)$;

(2) $Re(z_1) < Re(z_2)$ *and* $Im(z_1) = Im(z_2)$;

(3) $Re(z_1) = Re(z_2)$ *and* $Im(z_1) < Im(z_2)$;

(4) $Re(z_1) < Re(z_2)$ *and* $Im(z_1) < Im(z_2)$.

We will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(2), (3)$ and (4) is satisfied; also we will write $z_1 \prec z_2$ if only (4) is satisfied.

It follows that

(1) $0 \precsim z_1 \precnsim z_2$ *implies* $|z_1| < |z_2|$;

(2) $z_1 \precsim z_2$ *and* $z_2 \prec z_3$ *imply* $z_1 \prec z_3$;

(3) $0 \precsim z_1 \precsim z_2$ *implies* $|z_1| \leq |z_2|$;

(4) if $a, b \in \mathbb{R}, 0 \leq a \leq b$ *and* $z_1 \precsim z_2$*, then* $az_1 \precsim bz_2$ *for all* $z_1, z_2 \in \mathbb{C}$.

The following definition is recently introduced by Rao et al. [10].

Definition 1.2.2 *: Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \cdot X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:*

(1) $0 \precsim d(x, y)$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \precsim s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a complex valued b-metric space.

Example 1.2.1 *: If $X = [0, 1]$, define the mapping $d : X \cdot X \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$, for all $x, y \in X$. Then (X, d) is complex valued b-metric space with $s = 2$.*

Definition 1.2.3 *: Let (X, d) be a complex valued b-metric space.*

(1) A point $x \in X$ is called interior point of a set $\mathbb{A} \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq \mathbb{A}$.

(2) A point $x \in X$ is called a limit point of a set $\mathbb{A}$ whenever for every $0 \prec r \in \mathbb{C}$, $B(x, r) \cup (\mathbb{A} - \{x\}) \neq \emptyset$.

(3) A subset $\mathbb{A} \subseteq X$ is called open whenever each element of $\mathbb{A}$ is an interior point of $\mathbb{A}$.

(4) A subset $\mathbb{A} \subseteq X$ is called closed whenever each element of $\mathbb{A}$ belongs to $\mathbb{A}$.

(5) A sub-basis for a Hausdorff topology τ on X is a family $F = \{B(x, r) : x \in X\}$ and $0 \prec r$.

Definition 1.2.4 *: Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in $\mathbb{X}$ and $x \in \mathbb{X}$.*

(1) If for every $c \in \mathbb{C}$, with $0 \prec r$, there is $N \in \mathbb{N}$ such that for all $n > \mathbb{N}$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x, and x is the limit point of $\{x_n\}$. We denote this by $\lim x_n = x$ or $\{x_n\} \to x$ as $n \to \infty$.

(2) If for every $c \in \mathbb{C}$, with $0 \prec r$, there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.

(3) If every Cauchy sequence in X is convergent, then (X, d) is said to be a complete complex valued b-metric space.

Lemma 1.2.1 *: Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.2.2 *: Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

1.3 Weakly Compatible Mappings

Definition 1.3.1 *: A pair of self-maps A and S of a complex-valued metric space (X, d) are weakly compatible if $ASx = SAx$ for all $x \in X$ at which $Ax = Sx$.*

Example 1.3.1 *: Defined complex metric $d : X \cdot X \to C$ by*
$d(z_1, z_2) = e^{ia}|z_1 - z_2|$, where a is any real constant. Then (X, d) is a complex-valued metric space. Suppose self-maps A and S be defined as:
$Az = 2e^{i\pi/4}$ *if* $Re(z) \neq 0$,
$Az = 3e^{i\pi/3}$ *if* $Re(z) = 0$,
and
$Sz = 2e^{i\pi/4}$ if $Re(z) \neq 0$,
$Sz = 4e^{i\pi/6}$ if $Re(z) = 0$.
Then maps A and S are weakly compatible at all $z \in \mathbb{C}$ with $Re(z) \neq 0$.

1.4 (E,A) Property in Complex-valued Metric Space

Definition 1.4.1 *: A pair of self maps A and S on a complex-valued metric space (X, d) satisfies the property $(E.A)$ if there exist a sequence $\{x_n\}$ and in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$*

Example 1.4.1 *: Let $X = C$ and d be any complex-valued metric. Define self maps A and S by $Az = z^2$ and $Sz = z$, for all $z \in X$. Consider a sequence in X as $\{x_n\} = \left\{\frac{1}{n}\right\}$ where $n = 1, 2, 3, ...$ then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 0$. Hence, the pair (A, S) satisfies property $(E.A)$ for the sequences $\{x_n\}$ in X.*

Definition 1.4.2 *: Two pairs of self maps (A, S) and (B, T) on a complex-valued metric space (X, d) satisfies commons property $(E.A)$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = p$ for some $p \in X$.*

1.5 Pairwise Commuting in Finite Families

Definition 1.5.1 *: Two finite families of self maps* $\{A_i\}_{i-1}^{m}$ *and* $\{B_j\}_{j-1}^{n}$ *on a set* X *are pairwise commuting if*

(1) $A_iA_j = A_jA_i,\ i, j \in \{1, 2, 3, ..., m\}$,

(2) $B_iB_j = B_jB_i,\ i, j \in \{1, 2, 3, ..., n\}$,

(3) $A_iB_j = B_jA_i,\ i \in \{1, 2, 3, ..., m\}, j \in \{1, 2, 3, ..., n\}$.

Lemma 1.5.1 *: Let* (X, d) *be a complex valued metric space. Then a sequence* $\{z_r\}$ *in* X *converges to* z *if and only if* $|d(z_r, z)| \to 0$ *as* $r \to \infty$.

Lemma 1.5.2 *Let* (X, d) *be a complex valued metric space. Then a sequence* $\{z_r\}$ *in* X *is a Cauchy sequence if and only if* $|d(z_r, z_{r+s})| \to 0$ *as* $r \to \infty$*,where* $s \in \mathbb{N}$.

Definition 1.5.2 *: Let* $\{z_r\}$ *be a sequence in a complex valued metric space* (X, d) *and* $z \in X$*. Then,*

(1) $\{z_r\}$ *is called C-Cauchy sequence if for any* $w \in \mathbb{C}$ *with* $0 \prec w$ *there is* $r_0 \in \mathbb{N}$ *such that* $d(z_r, z_s) \prec w$ *for all* $r, s > r_0$.

(2) (X, d) *is called C-complete complex valued metric space if every C-Cauchy sequence is convergent in* (X, d).

Definition 1.5.3 *: Let* K *and* L *be two self-mappings on a nonempty set* X*. Then*

(1) A point $z \in X$ *is called a fixed point of* L *if* $Lz = z$.

(2) A point $z \in X$ *is called a coincidence point of* K *and* L *if* $Kz = Lz$.

(3) A point $z \in X$ *is a common fixed point of* K *and* L *if* $Kz = Lz = z$.

Next, we give the useful concept and lemma for proving the existence results of common fixed point in this field.

Definition 1.5.4 *: Let* X *be a complex valued metric space. Then a pair of self-mappings* $K, L : X \to X$ *is said to be weakly compatible if they commute at their coincidence points, that is* $z \in X$ *with* $Kz = Lz$ *implies that* $KLz = LKz$.

Lemma 1.5.3 *: Let X be a nonempty set and $T : X \to X$ be a function. Then there exists a subset $E \subseteq X$ such that $T(E) = T(X)$ and $T|_E : E \to X$ is one-to-one.*

1.6 Hausdorff topology

In topology and related branches of mathematics [7], a Hausdorff space, separated space is a topological space where for any two distinct points there exist neighborhoods of each which are disjoint from each other. Of the many separation axioms that can be imposed on a topological space, the "Hausdorff condition" is the most frequently used and discussed. It implies the uniqueness of limits of sequences, nets, and filters.

Hausdorff spaces are named after Felix Hausdorff, one of the founders of topology. Hausdorff's original definition of a topological space (in 1914) included the Hausdorff condition as an axiom.

Definition 1.6.1 *: A topological space X is called Hausdorff if for each pair x_1 and x_2 distinct in X, there exists neighborhoods U_1 and U_2 of x_1 and x_2, respectively, such that U_1 and U_2 are disjoint.*

Theorem 1.6.1 *: Every finite point set in a Hausdorff space is closed. This condition is called the T1 axiom.*

Theorem 1.6.2 *: Let X be a topological space satisfying the T_1 axiom; let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Definition 1.6.2 *: A sequence (x_n) of an arbitrary topological space X converges to the point $x \in X$ if for any neighborhood U of x, there exists a positive integer N such that $x_n \in U$ if $n \geq N$.*

Theorem 1.6.3 *: If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.*

Remark 1.6.1 *: In mathematics, a Hausdorff space X is called a fixed-point space if every continuous function $f : X \to X$ has a fixed point.*
For example, any closed interval $[a, b]$ in $\mathbb{R}$ is a fixed point space, and it can be proved

from the intermediate value property of real continuous function. The open interval (a, b)*, however, is not a fixed point space. To see it, consider the function, for example*

$$f(x) = a + \frac{1}{b-a} \cdot (x-a)^2 f(x) = a + \frac{1}{b-a} \cdot (x-a)^2.$$

Any linearly ordered space that is connected and has a top and a bottom element is a fixed point space.
Note that, in the definition, we could easily have disposed of the condition that the space is Hausdorff.

1.7 Integral Equations

An equation in which the unknown function of one or more variables appears under the integral sign is said Integral equation. This general definition takes into account many naturally occurring forms of modeling different problems of mechanics and mathematical physics or by reworking an important class of problems previously formulated by differential operators, especially the boundary problems and those of Cauchy.

1.7.1 Linear integral equations

The ordinary form of a linear integral equation is given by

$$\alpha(t)z(t) = f(t) + \lambda \int K(t,s)z(s)ds \tag{1.2}$$

where $\alpha(t), f(t), K(t,s)$ are given functions, the function $z(x)$ which appears inside and outside of the integral sign is the unknown to be determined, λ is a real or complex parameter different than zero. The function $K(x,t)$ is called kernel of the integral equation.

Fredholm's integral equations

An equation of the form $(\lim d(STz_n, TSz_n) = 0$ whose integration terminals are fixed is called Fredholm's linear integral equation.

- If $\alpha(t) = 0$, the equation is written

$$f(t) + \lambda \int_a^b K(t,s)z(s)ds = 0 \tag{1.3}$$

  It is said of the first kind.

- If $\alpha(t) = 1$, the equation is written

$$z(t) = f(t) + \lambda \int_a^b K(t,s)z(s)ds \tag{1.4}$$

  It is said of the second kind.

- If $\alpha(t)$ is continuous and vanishes at some points, but not at all points of $[a,b]$, it is said of the first kind.

- If $f(t) = 0$, the equation is written

$$z(t) = \lambda \int_a^b K(t,s)z(s)ds \tag{1.5}$$

  It is said homogeneous.

Volterra's integral equations

The integral equations of Volterra of the first, second kind or homogeneous are defined in the same way previous except that the upper bound of integration is variable, i.e., $b = t$

1.7.2 Non linear integral equations

The ordinary form of a non linear integral equation is given by

$$\alpha(t)z(t) = f(t) + \lambda \int K(t,s,z(s))ds \tag{1.6}$$

where $\alpha(t), f(t), K(t,s)$ are given functions, the function $z(x)$ which appears inside and outside of the integral sign is the unknown to be determined, λ is a real or complex parameter different than zero. The function $K(t,s,z(s))$ is called kernel of the integral equation.

Fredholm's integral equations

An equation of the form (1.4) whose integration terminals are fixed is called Fredholm's non linear integral equation.

- If $\alpha(t) = 0$, the equation is written

$$f(t) + \lambda \int_a^b K(t,s,z(s))ds = 0 \tag{1.7}$$

  It is said of the first kind.

- If $\alpha(t) = 1$ constant, the equation is written

$$z(t) = f(t) + \lambda \int_a^b K(t, s, z(s))ds \tag{1.8}$$

  It is said of the second kind.

- If $\alpha(t)$ is continuous and vanishes at some points, but not at all points of $[a, b]$, it is said of the first kind.

- If $f(t) = 0$, the equation is written

$$z(t) = \lambda \int_a^b K(t, s, z(s))ds \tag{1.9}$$

  It is said homogeneous.

The equation

$$\alpha(t)z(t) = f(t) + \lambda \int_a^b K(t, s, z(s))ds \tag{1.10}$$

It is said of the third kind.

Volterra's integral equations

The integral equations of Volterra of the first, second kind or homogeneous are defined in the same way previous except that the upper bound of integration is variable, i.e., $b = t$

Urysohn's integral equations

An equation of the form (1.4) whose integration terminals are fixed is called Urysohn non linear integral equation.

(1) If $\alpha(t) = 0$ and $\lambda = 1$, the equation is written

$$f(t) + \int_a^b K(t, s, z(s))ds \tag{1.11}$$

It is said of the first kind.

(2) If $\alpha(t) = 1$ and $\lambda = 1$, the equation is written

$$z(t) = f(t) + \int_a^b K(t, s, z(s))ds \tag{1.12}$$

It is said of the second kind.

(3) If $\lambda = 1$ and $\alpha(t)$ is continuous and vanishes at some points, but not at all points of $[a, b]$, it is said of the first kind.

(4) If $f(t) = 0$, the equation is written

$$z(t) = \int_a^b K(t, s, z(s))ds \tag{1.13}$$

It is said homogeneous.

CHAPTER

2

FIXED POINT THEOREMS IN COMPLEX-VALUED METRIC SPACES

In this chapter, we will discus the common fixed point for mappings in a complex-valued metric space.

2.1 Common Fixed Point Theorems under Rational Contractions

Theorem 2.1.1 *: Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$ be mappings satisfying*

$$d(Sx, Ty) \precsim \alpha d(x, y) + \frac{\beta d(x, Sx) d(y, Ty)}{d(x, Ty) + d(y, Sx) + d(x, y)} \tag{2.1}$$

For all $x, y \in X$, such that $x \neq y$, $d(x, Ty) + d(y, Sx) + d(x, y) \neq 0$, where α, β are nonnegative reals with $\alpha + s\beta < 1$ or $d(Sx, Ty) = 0$ if $d(x, Ty) + d(y, Sx) + d(x, y) = 0$. Then S and T have a unique common fixed point.

Proof: For any arbitrary point $x_0 \in X$, define sequence $\{x_n\}$ in X such that

$$\begin{aligned} x_{2n+1} &= Sx_{2n} \\ x_{2n+2} &= Tx_{2n+1}, \end{aligned} \tag{2.2}$$

for n=0,1,2,3,....

Now, we show that the sequence $\{x_n\}$ is Cauchy. Let $x = x_{2n}$ and $y = x_{2n+1}$ in (2.1); we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\precsim \alpha d(x_{2n}, x_{2n+1}) + \frac{\beta d(x_{2n}, Sx_{2n}) d(x_{2n+1}, Tx_{2n+1})}{d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n}) + d(x_{2n}, x_{2n+1})} \\ &= \alpha d(x_{2n}, x_{2n+1}) + \frac{\beta d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}) + d(x_{2n}, x_{2n+1})}. \end{aligned} \tag{2.3}$$

So that

$$|d(x_{2n+1}, x_{2n+2})| \leq \alpha |d(x_{2n}, x_{2n+1})| + \frac{\beta |d(x_{2n}, x_{2n+1})||d(x_{2n+1}, x_{2n+2})|}{|d(x_{2n}, x_{2n+2})| + |d(x_{2n}, x_{2n+1})|}. \tag{2.4}$$

As (owing to triangular inequality)

$$|d(x_{2n+1}, x_{2n+2})| \leq s|d(x_{2n+1}, x_{2n})| + s|d(x_{2n}, x_{2n+1})|. \tag{2.5}$$

Therefore

$$\begin{aligned} |d(x_{2n+1}, x_{2n+2})| &\leq \alpha |d(x_{2n}, x_{2n+1})| + s\beta |d(x_{2n}, x_{2n+1})| \\ &= (\alpha + s\beta)|d(x_{2n}, x_{2n+1})| \\ |d(x_{2n+1}, x_{2n+2})| &\leq (\alpha + s\beta)|d(x_{2n}, x_{2n+1})|. \end{aligned} \tag{2.6}$$

Similarly, we obtain

$$|d(x_{2n+2}, x_{2n+3})| \leq (\alpha + s\beta)|d(x_{2n+1}, x_{2n+2})|. \tag{2.7}$$

With $\delta = \alpha + s\beta$, and for all $n \geq 0$, and consequently, we have

$$\begin{aligned} |d(x_{2n+1}, x_{2n+2})| &\leq \delta|d(x_{2n}, x_{2n+1})| \\ &\leq \delta^2|d(x_{2n-1}, x_{2n})| \leq ... \\ &\leq \delta^{2n+1}|d(x_0, x_1)|. \end{aligned} \tag{2.8}$$

That is,

$$\begin{aligned} |d(x_{n+1}, x_{n+2})| &\leq \delta|d(x_n, x_{n+1})| \\ &\leq \delta^2|d(x_{n-1}, x_n)| \leq ... \\ &\leq \delta^{n+1}|d(x_0, x_1)|. \end{aligned} \tag{2.9}$$

Thus, for any $m > n, m, n \in \mathbb{N}$, we have

$$\begin{aligned} |d(x_n, x_m)| &\leq s|d(x_n, x_{n+1})| + s|d(x_{n+1}, x_m)| \\ &\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^2|d(x_{n+2}, x_m)| \\ &\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^3|d(x_{n+2}, x_{n+3})| + s^3|d(x_{n+3}, x_m)| \\ &\leq ... \\ &\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^3|d(x_{n+2}, x_{n+3})| + ... \\ &+ s^{m-n-2}|d(x_{m-3}, x_{m-2})| + s^{m-n-1}|d(x_{m-2}, x_{m-1})| + s^{m-n}|d(x_{m-1}, x_m)|. \end{aligned} \tag{2.10}$$

By using (2.9), we get

$$\begin{aligned} |d(x_n, x_m)| &\leq s\delta^n|d(x_0, x_1)| + s^2\delta^{n+1}|d(x_0, x_1)| + s^3\delta^{n+2}|d(x_0, x_1)| + ... \\ &+ s^{m-n-2}\delta^{m-3}|d(x_0, x_1)| + s^{m-n-1}\delta^{m-2}|d(x_0, x_1)| + s^{m-n}\delta^{m-1}|d(x_0, x_1)| \\ &= \sum_{i=1}^{m-n} s^i\delta^{i+n-1}|d(x_0, x_1)|. \end{aligned} \tag{2.11}$$

Therefore,

$$\begin{aligned}|d(x_n, x_m)| &\leq \sum_{i=1}^{m-n} s^{i+n-1}\delta^{i+n-1}|d(x_0, x_1)| \\ &= \sum_{t=n}^{m-1} s^t\delta^t|d(x_0, x_1)| \\ &\leq \sum_{t=n}^{\infty}(s\delta)^t|d(x_0, x_1)| \\ &= \frac{(s\delta)^t}{1-s\delta}|d(x_0, x_1)|.\end{aligned} \tag{2.12}$$

and hence

$$|d(x_n, x_m)| \leq \frac{(s\delta)^n}{1-s\delta}|d(x_0, x_1)| \to 0 \ as \ m, n \to \infty. \tag{2.13}$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $x_n \to u$ us $n \to \infty$. Assume not, then there exists $z \in X$ such that

$$|d(u, Su)| = |z| > 0. \tag{2.14}$$

So by using the triangular inequality and (2.1), we get

$$\begin{aligned}z = d(u, Su) &\precsim sd(u, x_{2n+2}) + sd(x_{2n+2}, Su) \\ &= sd(u, x_{2n+2}) + sd(Tx_{2n+1}, Su) \\ &\precsim sd(u, x_{2n+2}) + s\alpha d(u, x_{2n+1}) + \frac{s\beta d(u, Su)d(x_{2n+1}, Tx_{2n+1})}{d(u, Tx_{2n+1}) + d(x_{2n+1}, Su) + d(u, x_{2n+1})} \\ &= sd(u, x_{2n+2}) + s\alpha d(u, x_{2n+1}) + \frac{s\beta d(u, Su)d(x_{2n+1}, x_{2n+2})}{d(u, x_{2n+2}) + d(x_{2n+1}, Su) + d(u, x_{2n+1})}.\end{aligned} \tag{2.15}$$

Which implies that

$$\begin{aligned}z =& |d(u, Su)| \\ &\leq s|d(u, x_{2n+2})| + s\alpha|d(u, x_{2n+1})| + \frac{s\beta|d(u, Su)||d(x_{2n+1}, x_{2n+2})|}{|d(u, x_{2n+2})| + |d(x_{2n+1}, Su)| + |d(u, x_{2n+1})|}.\end{aligned} \tag{2.16}$$

Taking the limit of (2.16) as $n \to \infty$, we obtain that $|z| = |d(u, Su)| \leq 0$, a contradiction with (2.14). So $|z| = 0$. Hence $Su = u$.

Similarly, we obtain $Tu = u$.

Now we show that S and T have unique common fixed point of S and T. To prove this,

assume that u^* is another common fixed point of S and T.
Then

$$d(u, u^*) = d(Su, Tu^*) \precsim \alpha d(u, u^*) + \frac{\beta d(u, Su)d(u^*, Tu^*)}{d(u, Tu^*) + d(u^*, Su) + d(u, u^*)}. \tag{2.17}$$

So that

$$\begin{aligned} |d(u, u^*)| &\leq \alpha|d(u, u^*)| + \frac{\beta|d(u, Su)||d(u^*, Tu^*)|}{|d(u, Tu^*)| + |d(u^*, Su)| + |d(u, u^*)|} \\ &\precsim \alpha|d(u, u^*)|, \end{aligned} \tag{2.18}$$

so that $u = u^*$ which proves the uniqueness of common fixed point.

Now, we consider the second case: $d(x, Ty) + d(y, Sx) + d(x, y) = 0$. Put $x = x_{2n}$ and $y = x_{2n+1}$ in this expression; we get

$$d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n}) + d(x_{2n}, x_{2n+1}) = 0$$

(for any n) which implies $d(Sx_{2n}, Tx_{2n+1}) = 0$ so that $x_{2n} = Sx_{2n} = x_{2n+1} = Tx_{2n+1} = x_{2n+2}$.
Thus, we have $x_{2n+1} = Sx_{2n} = x_{2n}$, so there exists K_1 and L_1 such that $K_1 = SL_1 = L_1$, where $K_1 = x_{2n+1}$ and $L_1 = x_{2n}$.
Using foregoing arguments, one can also show that there exist K_2 and L_2 such that $K_2 = TL_2 = L_2$, where $K_2 = x_{2n+2}$ and $L_2 = x_{2n+1}$. As $d(L_1, TL_2) + d(L_2, SL_1) + d(L_1, L_2) = 0$ (due to definition) implies $d(SL_1, TL_2) = 0$, therefore $K_1 = SL_1 = TL_2 = K_2$.
Thus we obtain that $K_1 = SL_1 = SK_1$. Similarly, one can also have $K_2 = TK_2$. As $K_1 = K_2$ implies $SK_1 = TK_1 = K_1$, therefore $K_1 = K_2$ is common fixed point of S and T.
For uniqueness of common fixed point, assume that K_1^* in X is another common fixed point of S and T. Then we have $SK_1^* = TK_1^* = K_1^*$.
As $d(K_1, TK_1^*) + d(K_1^*, SK_1) + d(K_1, K_1^*) = 0$, therefore $d(K_1, K_1^*) = d(SK_1, TK_1^*) = 0$.
This implies that $K_1 = K_1^*$. This completes the proof of the theorem. ■

Corollary 2.1.1 : *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$. Let $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) \precsim \alpha d(x, y) + \frac{\beta d(x, Tx)d(y, Ty)}{d(x, Ty) + d(y, Tx) + d(x, y)} \tag{2.19}$$

for all $x, y \in X$ *such that* $x \neq y, d(x, Ty) + d(y, Tx) + d(x, y) \neq 0$, *where* α, β *are non-negative reals with* $\alpha + s\beta < 1$ *or* $d(Tx, Ty) = 0$ *if* $d(x, Ty) + d(y, Tx) + d(x, y) = 0$. *Then* T *has a unique fixed point in* X.

Proof: We can prove this result by applying Theorem (2.1.1) with $S = T$. ■

Corollary 2.1.2 : *Let* (X, d) *be a complete complex valued b-metric space with the coefficient* $s \geq 1$ *and let* $T : X \to X$ *be a mapping satisfying (for some fixed n)*

$$d(T^n x, T^n y) \precsim \alpha d(x, y) + \frac{\beta d(x, T^n x) d(y, T^n y)}{d(x, T^n y) + d(y, T^n x) + d(x, y)} \tag{2.20}$$

for all $x, y \in X$ *such that* $x \neq y, d(x, Ty) + d(y, Tx) + d(x, y) \neq 0$, *where* α, β *are nonnegative reals with* $\alpha + s\beta < 1$ *or* $d(T^n x, T^n y) = 0$ *if* $d(x, T^n y) + d(y, T^n x) + d(x, y) = 0$. *Then* T *has a unique fixed point in* X.

Theorem 2.1.2 : *Let* (X, d) *be a complete complex valued b-metric space with the coefficient* $s \geq 1$ *and let* $S, T : X \to X$ *be mappings satisfying*

$$d(Sx, Ty) \precsim \alpha d(x, y) + \frac{\beta [d^2(x, Tx) d^2(y, Sx)]}{d(x, Ty) + d(y, Sx)} + \gamma [d(x, Sx) + d(y, Ty)] \tag{2.21}$$

for all $x, y \in X$ *such that* $x \neq y$, *where* α, β *and* γ *are nonnegative reals with* $\alpha + 2s\beta + 2\gamma < 1$ *or* $d(Sx, Ty) = 0$ *if* $d(x, Ty) + d(y, Sx) = 0$. *Then* S *and* T *have a unique common fixed point in* X.

Proof: For any arbitrary point $x_0 \in X$, define sequence $\{x_n\}$ in X such that

$$\begin{aligned} x_{2n+1} &= Sx_{2n}, \\ x_{2n+2} &= Tx_{2n+1}, \end{aligned} \tag{2.22}$$

for $n = 0, 1, 2, ...$

Now, we show that the sequence $\{x_n\}$ is a Cauchy sequence.

Let $x = x_{2n}$ and $y = x_{2n+1}$ in (2.21); we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\precsim \alpha d(x_{2n}, x_{2n+1}) + \frac{\beta [d^2(x_{2n}, Tx_{2n+1}) + d^2(x_{2n+1}, Sx_{2n})]}{d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})} + \gamma [d(x_{2n}, Sx_{2n}) \\ &+ d(x_{2n+1}, Tx_{2n+1})] \\ &= \alpha d(x_{2n}, x_{2n+1}) + \frac{\beta [d^2(x_{2n}, x_{2n+2}) + d^2(x_{2n+1}, x_{2n+1})]}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})} + \gamma [d(x_{2n}, x_{2n+1}) \\ &+ d(x_{2n+1}, x_{2n+2})], \end{aligned} \tag{2.23}$$

so that

$$\begin{aligned}|d(x_{2n+1},x_{2n+2})| &\leq \alpha|d(x_{2n},x_{2n+1})| + \frac{\beta|d^2(x_{2n},x_{2n+2})|}{|d(x_{2n},x_{2n+2})|} \\ &\quad + \gamma[|d(x_{2n},x_{2n+1})| + |d(x_{2n+1},x_{2n+2})|] \\ |d(x_{2n+1},x_{2n+2})| &\leq \alpha|d(x_{2n},x_{2n+1})| + \beta|d(x_{2n},x_{2n+2})| + \gamma[|d(x_{2n},x_{2n+1})| \\ &\quad + |d(x_{2n+1},x_{2n+2})|].\end{aligned} \tag{2.24}$$

As

$$|d(x_{2n},x_{2n+2})| \leq s[|d(x_{2n},x_{2n+1})| + |d(x_{2n+1},x_{2n+2})|], \tag{2.25}$$

therefore

$$\begin{aligned}|d(x_{2n+1},x_{2n+2})| &\leq \alpha|d(x_{2n},x_{2n+1})| + s\beta[|d(x_{2n},x_{2n+1})| + |d(x_{2n+1},x_{2n+2})|] \\ &\quad + \gamma[|d(x_{2n},x_{2n+1})| + |d(x_{2n+1},x_{2n+2})|] \\ |d(x_{2n+1},x_{2n+2})| &\leq \left(\frac{\alpha+s\beta+\gamma}{1-s\beta-\gamma}\right)|d(x_{2n},x_{2n+1})|.\end{aligned} \tag{2.26}$$

Similarly, we obtain,

$$|d(x_{2n+2},x_{2n+3})| \leq \left(\frac{\alpha+s\beta+\gamma}{1-s\beta-\gamma}\right)|d(x_{2n+1},x_{2n+2})|. \tag{2.27}$$

Since $\alpha + 2s\beta + 2\gamma < 1$ and $s \geq 1$ we get $\alpha + 2\beta + \gamma < 1$.
Therefore, with $\delta = (\alpha+\beta+\gamma)/(1-\beta-\gamma) < 1$, and for all $n \geq 0$, and consequently, we have

$$\begin{aligned}|d(x_{2n+1},x_{2n+2})| &\leq \delta|d(x_{2n},x_{2n+1})| \\ &\leq \delta^2|d(x_{2n-1},x_{2n})| \leq ... \\ &\leq \delta^{2n+1}|d(x_0,x_1)|.\end{aligned} \tag{2.28}$$

That is,

$$|d(x_{n+1},x_{n+2})| \leq \delta|d(x_n,x_{n+1}) \leq \delta^2|d(x_{n-1},x_n)| \leq ... \leq \delta^{n+1}|d(x_0,x_1)|. \tag{2.29}$$

Thus, for any $m > n, m, n \in \mathbb{N}$, we get

$$\begin{aligned}
|d(x_n, x_m)| &\leq s|d(x_n, x_{n+1})| + s|d(x_{n+1}, x_m)| \\
&\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^2|d(x_{n+2}, x_m)| \\
&\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^3|d(x_{n+2}, x_{n+3})| + s^3|d(x_{n+3}, x_m)| \\
&\leq ... \\
&\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^3|d(x_{n+2}, x_{n+3})| + ... \\
&+ s^{m-n-2}|d(x_{m-3}, x_{m-2})| + s^{m-n-1}|d(x_{m-2}, x_{m-1})| + s^{m-n}|d(x_{m-1}, x_m)|.
\end{aligned} \tag{2.30}$$

By using (2.29) we get

$$\begin{aligned}
|d(x_n, x_m)| &\leq s\delta^n|d(x_0, x_1)| + s^2\delta^{n+1}|d(x_0, x_1)| \\
&+ s^3\delta^{n+2}|d(x_0, x_1)| + ... + s^{m-n-2}\delta^{m-3}|d(x_0, x_1)| \\
&+ s^{m-n-1}\delta^{m-2}|d(x_0, x_1)| + s^{m-n}\delta^{m-1}|d(x_0, x_1)| \\
&= \sum_{i=1}^{m-n} s^i\delta^{i+n-1}|d(x_0, x_1)|.
\end{aligned} \tag{2.31}$$

Therefore

$$\begin{aligned}
|d(x_n, x_m)| &\leq \sum_{i=1}^{m-n} s^{i+n-1}\delta^{i+n-1}|d(x_0, x_1)| \\
&= \sum_{t=n}^{m-1} s^t\delta^t|d(x_0, x_1)| \\
&\leq \sum_{t=n}^{\infty}(s\delta)^t|d(x_0, x_1)| \\
&= \frac{(s\delta)^n}{1 - s\delta}|d(x_0, x_1)|
\end{aligned} \tag{2.32}$$

and hence

$$|d(x_n, x_m)| \leq \frac{(s\delta)^n}{1 - s\delta}|d(x_0, x_1)| \to 0 \ as \ m, n \to \infty. \tag{2.33}$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Assume not, then there exists $z \in X$ such that

$$|d(u, Su)| = |z| > 0. \tag{2.34}$$

So by using the triangular inequality and (2.21), we get

$$\begin{aligned}
z = d(u, Su) &\precsim sd(u, x_{2n+2}) + sd(x_{2n+2}, Su) \\
&= sd(u, x_{2n+2}) + sd(Su, Tx_{2n+1}) \\
&\precsim sd(u, x_{2n+2}) + s\alpha d(u, x_{2n+1}) + \frac{s\beta[d^2(u, Tx_{2n+1}) + d^2(x_{2n+1}, Su)]}{d(u, Tx_{2n+1}) + d(x_{2n+1}, Su)} \\
&+ s\gamma[d(u, Su) + d(x_{2n+1}, Tx_{2n+1})] \\
&= sd(u, x_{2n+2}) + s\alpha d(u, x_{2n+1}) + \frac{s\beta[d^2(u, x_{2n+2}) + d^2(x_{2n+1}, Su)]}{d(u, x_{2n+2}) + d(x_{2n+1}, Su)} \\
&+ s\gamma[z + d(x_{2n+1}, x_{2n+2})],
\end{aligned} \tag{2.35}$$

which implies that

$$\begin{aligned}
|z| &= |d(u, Su)| \\
&\leq s|d(u, x_{2n+2})| + s\alpha|d(u, x_{2n+1})| + \frac{s\beta[|d^2(u, x_{2n+2})| + |d^2(x_{2n+1}, Su)|]}{|d(u, x_{2n+2})| + |d(x_{2n+1}, Su)|} \\
&+ s\gamma[|z| + |d(x_{2n+1}, x_{2n+2})|].
\end{aligned} \tag{2.36}$$

Taking the limit of (2.36) as $n \to \infty$, we obtain that $|z| = |d(u, Su)| \leq 0$, a contradiction with (2.34). So $|z| = 0$; hence $Su = u$.
Similarly, we obtain that $Tu = u$. Now we show that S and T have a unique common fixed point of S and T. To show this, assume that u^* is another common fixed point of S and T. Then

$$\begin{aligned}
d(u, u^*) &= d(Su, Tu^*) \\
&\precsim \alpha d(u, u^*) + \frac{\beta[d^2(u, Tu^*) + d^2(u^*, Su)]}{d(u, Tu^*) + d(u^*, Su)} \\
&+ \gamma[d(u, Su) + d(u^*, Tu^*],
\end{aligned} \tag{2.37}$$

$\prec d(u, u^*)$, a contradiction. So $u = u^*$, which proves the uniqueness of common fixed point in X. For the second case $d(Sx, Ty) = 0$ if $d(x, Ty) + d(y, Sx) = 0$, the proof of unique common fixed point can be completed in the line of Theorem (2.1.1). This completes the proof of the theorem. ■

Corollary 2.1.3 *: Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) \precsim \alpha d(x, y) + \frac{\beta[d^2(x, Ty) + d^2(y, Tx)]}{d(x, Ty) + d(y, Tx)} + \gamma[d(x, Tx) + d(y, Ty)] \tag{2.38}$$

for all $x, y \in X$ such that $x \neq y$, where α, β and γ are nonnegative reals with $\alpha+2s\beta+2\gamma < 1$ or $d(Tx, Ty) = 0$ if $d(x, Ty) + d(y, Tx) = 0$. Then T has a unique fixed point.

Proof: We can prove this result by applying Theorem (2.1.2) with $S = T$. ■

Corollary 2.1.4 *: Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping satisfying (for some fixed n)*

$$d(T^n x, T^n y) \precsim \alpha d(x, y) + \frac{\beta[d^2(x, T^n y) + d^2(y, T^n x)]}{d(x, T^n y) + d(y, T^n x)} + \gamma[d(x, T^n x) + d(y, T^n y)] \quad (2.39)$$

for all $x, y \in X$, such that $x \neq y$, where α, β, and γ are nonnegative reals with $\alpha + 2s\beta + 2\gamma < 1$ or $d(T^n x, T^n y) = 0$ if $d(x, T^n y) + d(y, T^n x) = 0$. Then T has a unique fixed point in X.

Proof: From Corollary (2.1.3), we obtain that $u \in X$ such that $T^n u = u$. The uniqueness follows from

$$\begin{aligned} d(Tu, u) = d(TT^n u, T^n u) &= d(T^n Tu, T^n u) \\ &\precsim \alpha d(Tu, u) + \frac{\beta[d^2(Tu, T^n u) + d^2(u, T^n Tu)]}{d(Tu, T^n u) + d(u, T^n Tu)} \\ &+ \gamma[d(Tu, T^n Tu) + d(u, T^n u)] \\ &= (\alpha + 2\beta) d(Tu, u). \end{aligned} \quad (2.40)$$

By Taking modulus of (2.40) and since $(\alpha + 2\beta) < 1$, we obtain
$|d(Tu, u)| \leq (\alpha + 2\beta)|d(Tu, u)| < |d(Tu, u)|$, a contradiction. So $Tu = u$.
Hence $Tu = T^n u = u$. Therefore, the fixed point of T is unique. This completes the proof. ■

2.2 Common Fixed Point Theorems Using Implicit Relation

Implicit relations play important role in establishing of common fixed point results. Let M_6 be the set of all continuous functions satisfying the following conditions:

(A) $\phi(u,0,u,0,0,u) \precsim 0 \Rightarrow u \precsim 0$.

(B) $\phi(u,0,0,u,u,0) \precsim 0 \Rightarrow u \precsim 0$.

(C) $\phi(u,u,0,0,u,u) \precsim 0 \Rightarrow u \precsim 0$, for all $0 \precsim u$.

Example 2.2.1 *: Define* $\phi : (C)^6 \to C$ *as* $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \phi_1(\min\{t_2,t_3,t_4,t_5,t_6\})$ *where* $\phi_1 : C \to C$ *is increasing and continuous function such that* $\phi_1(s) > s$ *for all* $s \in C$. *Clearly,* ϕ *satisfies all conditions* $(A),(B)$ *and* (C). *Therefore,* $\phi \in M_6$.

Now, we begin with following observation:

Lemma 2.2.1 *: Let* A, B, S *and* T *be self mappings of a complex-valued metric space* (X,d) *satisfies the following:*

$$\text{the pair } (A,S) \text{ or } (B,T) \text{ satisfies the property } (E.A). \tag{2.41}$$

for any $x, y \in X,\ \phi \in M_6$,

$$\phi\left(d(Ax,By), d(Sx,Ty), d(Sx,Ax), d(Ty,By), d(Sx,By), d(Ty,Ax)\right) \precsim 0 \tag{2.42}$$

$$A(X) \subset T(X) \text{ or } B(X) \subset S(X). \tag{2.43}$$

Then the pairs (A,S) *and* (B,T) *share the common* $(E.A)$ *property.*

Proof: Suppose that the pair (A,S) satisfy property $(E.A)$, then there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$. Since $A(X) \subset T(X)$, hence for each $\{x_n\}$ there exist $\{y_n\}$ in X such that $Ax_n = Ty_n$. Therefore $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = z$.

Now we claim that $\lim_{n\to\infty} By_n = z$. Suppose that $\lim_{n\to\infty} By_n \neq z$, then applying the inequality (2.42), we obtain

$$\phi\left(d(Ax_n,By_n), d(Sx_n,Ty_n), d(Sx_n,Ax_n), d(Ty_n,By_n), d(Sx_n,By_n), d(Ty_n,Ax_n)\right) \precsim 0$$

which on making $n \to \infty$ reduced to

$$\phi\left(d(z, \lim_{n\to\infty} By_n), d(z, z, t), d(z, z), d(z, \lim_{n\to\infty} By_n), d(z, \lim_{n\to\infty} By_n), d(z, z)\right) \precsim 0$$

$$\phi\left(d(z, \lim_{n\to\infty} By_n), 0, 0, d(z, \lim_{n\to\infty} By_n), d(z, \lim_{n\to\infty} By_n), 0\right) \precsim 0$$

which is contradiction to using (B), we get
$d(z, \lim_{n\to\infty} By_n) \precsim 0$ which gives, $|d(z, \lim_{n\to\infty} By_n)| \leq 0$, a contradiction and therefore, $\lim_{n\to\infty} By_n = z$.
Hence, the pairs (A, S) and (B, T) share the common $(E.A.)$ property. ■

Theorem 2.2.1 *: Let A, B, S and T be self mappings of a complex-valued metric space (X, d) satisfying the condition (2.42), and*

$$\textit{the pairs } (A, S) \textit{ and } (B, T) \textit{ share the common } (E.A.) \textit{ propert} \tag{2.44}$$

$$S(X) \textit{ and } T(X) \textit{ are closed subsets of } X. \tag{2.45}$$

Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, $A, S, B,$ and T have a unique fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof: In view of (4), there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$. for some $z \in X$.
Since $S(X)$ is a closed subset of X, therefore, there exists a point $u \in X$ such that $z = Su$.
We claim that $Au = z$. If $Au \neq z$, then by (2.42), take $x = u,$ *and* $y = y_n$,

$$\phi\left(d(Au, By_n), d(Su, Ty_n), d(Su, Au), d(Ty_n, By_n), d(Su, By_n), d(Ty_n, Au)\right) \precsim 0$$

taking the limit as $n \to \infty$ we get

$$\phi\left(d(Au, z), d(z, z), d(z, Au), d(z, z), d(z, z), d(z, Au)\right) \precsim 0$$

$$\phi\left(d(Au, z), 0, d(z, Au), 0, 0, d(z, Au)\right) \precsim 0$$

Using (A), we get $d(Au, z) \precsim 0$.
This gives, $|d(Au, z)| \leq 0$, a contradiction. Therefore, $Au = z = Su$ which shows that u is a coincidence point of the pair (A, S).

Since $T(X)$ is also a closed subset of X, therefore, $\lim_{n\to\infty} Ty_n = z$ in $T(X)$ and hence there exists a point $v \in X$ such that $Tv = z = Au = Su$. Now we show that $Bv = z$. If $Bv \neq z$, then by (2.42), take $x = u,\ and\ y = v$, we have

$$\phi\left(d(Au, Bv), d(Su, Tv), d(Su, Au), d(Tv, Bv), d(Su, Bv), d(Tv, Au)\right) \precsim 0$$

$$\phi\left(d(z, Bv), 0, 0, d(z, Bv), d(z, Bv), 0\right) \precsim 0$$

Using (B), we get $d(z, Bv) \precsim 0$. which gives $|d(z, Bv)| \leq 0$, a contradiction. Therefore, $Bv = z = Tv$ which shows that v is a coincidence point of the pair (B, T).
Since, (A, S) and (B, T) are weakly compatible and $Au = Su,\ Bv = Tv$, therefore, $Az = ASu = SAu = Sz,\ Bz = BTv = TBv = Tz$.
If $Az \neq z$ then by (2.42), we have

$$\phi\left(d(Az, Bv), d(Sz, Tv), d(Sz, Az), d(Tv, Bv), d(Sz, Bv), d(Tv, Az)\right) \precsim 0$$

$$\phi\left(d(Az, z), d(Az, z), d(Az, Az), d(Bv, Bv), d(Az, z), d(z, Az)\right) \precsim 0$$

$$\phi\left(d(Az, z), d(Az, z), 0, 0, d(Az, z), d(z, Az)\right) \precsim 0$$

Using (C), we get $d(Az, z) \precsim 0$. which gives $|d(Az, z)| \leq 0$, a contradiction.
Hence, $Az = z = Sz$.
Similarly, one can prove that $Bz = Tz = z$. Hence, $Az = Sz = Tz = Bz$, and z is a common fixed point of $A, S, B\ and\ T$.
Uniqueness: Let z and w be two common fixed points of $A, S, B\ and\ T$. If $z \neq w$, then by using inequality (2.42), we have

$$\phi\left(d(Az, Bw), d(Sz, Tw), d(Sz, Az), d(Tw, Bw), d(Sz, Bw), d(Tw, Az)\right) \precsim 0$$

$$\phi\left(d(z, w), d(z, w), d(z, z), d(w, w), d(z, w), d(w, z)\right) \precsim 0$$

$$\phi\left(d(z, w), d(z, w), 0, 0, d(z, w), d(z, w)\right) \precsim 0$$

Using (C), we get $d(z, w) \precsim 0$. which gives $|d(z, w)| \leq 0$, a contradiction.
Hence, $z = w$. ■

By choosing $A, S, B\ and\ T$ suitably, one can derive corollaries involving two or more mappings. As a sample, we deduce the following natural result for a pair of self mappings by setting $A = B$ and $S = T$ in above theorem:

Corollary 2.2.1 : *Let A and S be self mappings of a complex-valued metric space (X, d) satisfies the following:*

$$\text{the pair } (A, S) \text{ satisfies the property } (E.A). \tag{2.46}$$

for any $x, y \in X,\ \phi \in M_6$,

$$\phi\left(d(Ax, Ay), d(Sx, Sy), d(Sx, Ax), d(Sy, Ay), d(Sx, Ay), d(Sy, Ax)\right) \precsim 0 \tag{2.47}$$

$$S(X) \text{ is a closed subset of } X. \tag{2.48}$$

Then A and S have a point of coincidence each. Moreover, if the pair (A, S) is weakly compatible, then A and S have a unique common fixed point.

2.3 Common Fixed Point Theorems Using Weakly Compatible

Throughout this section, $\mathbb{R}$ denotes a set of real numbers, $\mathbb{C}^+$ denotes a set $\{c \in \mathbb{C} : 0 \precsim c\}$ and Γ denotes the class of all functions γ: $\mathbb{C}_+ \to [0,1)$ which satisfies the condition: for any sequences $\{x_n\}$ in $\mathbb{C}^+$, such that,

$$\gamma(x_n) \to 1 \;\Rightarrow\; |x_n| \to 0.$$

The following are examples of the function in Γ:

(1) $\gamma_1(x) = k$, where $k \in [\text{O},\ 1)$;

(2) $\gamma_2(x) = \dfrac{1}{1+k|x|}$, where $k \in (\text{O},\ \infty)$.

Theorem 2.3.1 *: Let K, L, M, N be four self-mappings on a C- complete complex-valued metric space (X, d) and $\lambda_1, \lambda_2, \lambda_3 : \mathbb{C}^+ \to [0,1)$ be given mappings. Suppose that the following conditions hold:*

(1) $\lambda_1(z) + \lambda_2(z) + \lambda_3(z) < 1$ for all $z \in \mathbb{C}^+$ and the mapping $\gamma : \mathbb{C}^+ \to [0,1)$, which is defined by

$$\gamma(z) = \frac{\lambda_1(z)}{1 - [\lambda_2(z) + \lambda_3(z)]} \quad for\ all\ z \in \mathbb{C}^+,$$

belongs to Γ;

(2) for each $z, w \in X$, we have:

$$\begin{aligned} d(Kz, Lw) \precsim\ & \lambda_1(d(Mz, Nw))d(Mz, Nw) \\ & + \lambda_2(d(Mz, Nw))\frac{d(Kz, Mz)d(Lw, Nw)}{1 + d(Mz, Nw)} \\ & + \lambda_3(d(Mz, Nw))\frac{d(Kz, Nw)d(Lw, Nw)}{1 + d(Mz, Nw)}. \end{aligned}$$

If $K(X) \subseteq N(X)$ and $L(X) \subseteq M(X)$ and the pairs (K, M) and (L, N) are weakly compatible, then K, L, M and N have a unique common fixed point in X.

Proof: Let $z_0 \in X$. Since $K(X) \subseteq N(X)$ and $L(X) \subseteq M(X)$, we can construct two sequences $\{z_n\}$ and $\{w_n\}$ by the rule

$$K_{z_{2n-2}} = N_{z_{2n-1}} = w_{2n-1} \quad \text{and} \quad L_{z_{2n-1}} = M_{z_{2n}} = w_{2n}, \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.49}$$

First we will show that $\{w_n\}$ is a C-Cauchy sequence in X. By using condition (2.42) of Theorem (2.3.1) with $z = z_{2n}$ and $w = z_{2n+1}$, we have

$$\begin{aligned} d(K_{Z_{2n}}, L_{Z_{2n+1}}) &\precsim \lambda_1(d\ (M_{Z_{2n}}, N_{Z_{2n+1}}))d(M_{Z_{2n}}, N_{Z_{2n+1}}) \\ &+ \lambda_2(d(M_{Z_{2n}}, N_{Z_{2n+1}}))\frac{d(K_{Z_{2n}}, M_{Z_{2n}})d(L_{Z_{2n+1}}, N_{Z_{2n+1}})}{1 + d(M_{Z_{2n}}, N_{Z_{2n+1}})} \\ &+ \lambda_3(d(M_{Z_{2n}}, N_{Z_{2n+1}}))\frac{d(K_{Z_{2n}}, N_{Z_{2n+1}})d(L_{Z_{2n+1}}, N_{Z_{2n+1}})}{1 + d(M_{Z_{2n}}, N_{Z_{2n+1}})} \end{aligned}$$

for all $n \in \mathbb{N}$. By using (2.49) , we get

$$\begin{aligned} |d(w_{2n+1}, w_{2n+2})| &\leq \lambda_1(d(w_{2n}, w_{2n+1}))|d(w_{2n}, w_{2n+1})| \\ &+ \lambda_2(d(w_{2n}, w_{2n+1}))\left|\frac{d(w_{2n+1}, w_{2n})d(w_{2n+2}, w_{2n+1})}{1 + d(w_{2n}, w_{2n+1})}\right| \\ &+ \lambda_3(d(w_{2n}, w_{2n+1}))\left|\frac{d(w_{2n+1}, w_{2n+1})d(w_{2n+2}, w_{2n+1})}{1 + d(w_{2n}, w_{2n+1})}\right| \\ &\leq \lambda_1(d(w_{2n}, w_{2n+1}))|d(w_{2n}, w_{2n+1})| \\ &+ \lambda_2(d(w_{2n}, w_{2n+1}))|d(w_{2n+2}, w_{2n+1})| \\ \Rightarrow |d(w_{2n+1}, w_{2n+2})| &\leq \frac{\lambda_1(d(w_{2n}, w_{2n+1}))}{1 - \lambda_2(d(w_{2n}, w_{2n+1}))}|d(w_{2n}, w_{2n+1})| \\ &\leq \frac{\lambda_1(d(w_{2n}, w_{2n+1}))}{1 - [\lambda_2(d(w_{2n}, w_{2n+1})) + \lambda_3(d(w_{2n}, w_{2n+1}))]}|d(w_{2n},\ w_{2n+1})| \end{aligned}$$

for all $n \in \mathbb{N}$. Applying condition (2.41) of Theorem (2.3.1) , we get

$$|d(w_{2n+1}\ ,\ w_{2n+2})| \leq \gamma(d(w_{2n},\ w_{2n+1}))|d(w_{2n},\ w_{2n+1})|$$

for all $n \in \mathbb{N}$. Similarly, we obtain that

$$|d(w_{2n},\ w_{2n+1})| \leq \gamma(d(w_{2n-1},\ w_{2n}))|d(w_{2n-1},\ w_{2n})|$$

for all $n \in \mathbb{N}$. Consequently,

$$|d(w_n, w_{n+1})| \leq \gamma(d(w_{n-1}, w_n))|d(w_{n-1}, w_n)| \leq |d(w_{n-1}, w_n)|, \text{for all} \quad n \in \mathbb{N}\backslash\{1\}. \quad (2.50)$$

Thus the sequence $\{|d(w_n, w_{n+1})|\}_{n\in\mathbb{N}\backslash\{1\}}$ is monotone non-increasing and bounded below. Therefore $|d(w_n, w_{n+1})| \to l$ for some $l \geq 0$. Now, we claim that $l = 0$. To support the claim, suppose that $l > 0$. Then taking limit as $n \to \infty$ in (2.50) , we have

$$1 \leq \lim_{n\to\infty} \gamma(d(w_{n-1},\ w_n)) \leq 1 \ \Rightarrow \lim_{n\to\infty} \gamma(d(w_{n-1},\ w_n)) = 1.$$

But $\gamma \in \Gamma$, so we can write $|d(w_{n-1}, w_n)| \to 0$, which is contradiction to the fact that $l > 0$. Thus $l = 0$ and hence

$$\lim_{n\to\infty} |d(w_{n-1}, w_n)| = 0. \tag{2.51}$$

Next, to show that $\{w_n\}$ is a C-Cauchy sequence, it is enough to show that $\{w_{2n}\}$ is a C-Cauchy sequence. Let on contrary, $\{w_{2n}\}$ is not a C-Cauchy sequence. Then there is $c \in \mathbb{C}$ with $c \succ 0$ for which there exists $2m_k > 2n_k \geq k$ for all $k \in \mathbb{N}$ such that

$$d(w_{2n_k},\ w_{2m_k}) \gtrsim c. \tag{2.52}$$

Now, corresponding to n_k, we can choose m_k in such a way that it is the smallest integer with $2m_k > 2n_k \geq k$ satisfying (2.52). Then

$$d(w_{2n_k}, w_{2m_k-2}) \prec c \tag{2.53}$$

In the light of (2.52) , (2.53) and triangular inequality, we have

$$c \precsim d(w_{2n_k}, w_{2m_k}) \precsim d(w_{2n_k}, w_{2m_k-2}) + d(w_{2m_k-2}, w_{2m_k-1}) + d(w_{2m_k-1}, w_{2m_k})$$

$$\prec c + d(w_{2m_k-2}, w_{2m_k-1}) + d(w_{2m_k-1}, w_{2m_k}),$$

which implies that,

$$|c| \leq |d(w_{2n_k} w_{2m_k})| < |c| + |d(w_{2m_k-2}, w_{2m_k-1})| + |d(w_{2m_k-1}, w_{2n_k})|.$$

Taking limit as $k \to \infty$ and using (2.51), we have

$$|c| \leq \lim_{k\to\infty} |d(w_{2n_k}, w_{2m_k})| < |c| \Rightarrow \lim_{k\to\infty} |d(w_{2n_k}, w_{2m_k})| = |c|. \tag{2.54}$$

Now, using triangular inequality, we have

$$|d(w_{2n_k}, w_2 m_k)| \leq |d(w_{2n_k}, w_{2m_k+1})| + |d(w_{2m_k+1}, w_{2m_k})|$$

$$\leq |d(w_{2n_k}, w_{2m_k})| + |d(w_{2m_k}, w_{2m_k+1})| + |d(w_{2m_k+1}, w_{2m_k})|$$

letting $k \to \infty$ and using (2.51), (2.54), we get

$$\lim_{k\to\infty} |d(w_{2n_k}, w_{2m_k+1})| = |c| \tag{2.55}$$

Next, we have

$$\begin{aligned} d(w_{2n_k},\ w_{2m_k+1}) &\precsim d(w_{2n_k}, w_{2n_k+1}) + d(w_{2n_k+1}, w_{2m_k+2}) + d(w_{2m_k+2}, w_{2m_k+1}) \\ &= d(w_{2n_k}, w_{2m_k+1}) + d(Kz_{2n_k}, Lz_{2n_k+1}) + d(w_{2n_k+2},\ w_{2n_k+1}). \end{aligned}$$

On using condition (2.42) of Theorem (2.3.1) with $z = z_{2n_k}$ and $w = z_{2m_k+1}$, one can write

$$\begin{aligned} d(w_{2n_k},\ w_{2m_k+1}) &\precsim d(w_{2n_k},\ w_{2n_k+1}) + \lambda_1(d(Mz_{2n_k},\ Nz_{2m_k+1}))d(Mz_{2n_k},\ Nz_{2m_k+1}) \\ &+ \lambda_2(d(Mz_{2n_k}, Nz_{2m_k+1}))\frac{d(Kz_{2n_k}, Mz_{2n_k})d(Lz_{2m_k+1}, Nz_{2m_k+1})}{1 + d(Mz_{2n_k}, Nz_{2m_k+1})} \\ &+ \lambda_3(d(Mz_{2n_k},\ Nz_{2m_k+1}))\frac{d(Kz_{2n_k}, Nz_{2m_k+1})d(Lz_{2m_k+1}, Nz_{2m_k+1})}{1 + d(Mz_{2n_k}, Nz_{2m_k+1})} \\ &+ d(w_{2m_k+2}, w_{2m_k+1}). \end{aligned}$$

By using (2.49), we get

$$\begin{aligned} |d(w_{2n_k},\ w_{2m_k+1})| &\leq |d(w_{2n_k},\ w_{2n_k+1})| + \lambda_1(d(w_{2n_k},\ w_{2m_k+1}))|d(w_{2n_k},\ w_{2m_k+1})| \\ &+ \lambda_2(d(w_{2n_k},\ w_{2m_k+1}))\left|\frac{d(w_{2n_k+1}, w_{2n_k})d(w_{2m_k+2}, w_{2m_k+1})}{1 + d(w_{2n_k},\ w_{2m_k+1})}\right| \\ &+ \lambda_3(d(w_{2n_k},\ w_{2m_k+1}))\left|\frac{d(w_{2n_k+1}, w_{2m_k+1})d(w_{2m_k+2}, w2m_k + 1)}{1 + d(w_{2n_k},\ w_{2m_k+1})}\right| \\ &+ |(d(w_{2m_k+2},\ w_{2m_k+1}))|. \end{aligned}$$

In the light of condition (2.41) of Theorem (2.3.1) , we get

$$\begin{aligned} |d(w_{2n_k}, w_{2m_k+1})| &\leq |d(w_{2n_k},\ w_{2n_k+1})| + \gamma(d(w_{2n_k},\ w_{2m_k+1}))|d(w_{2n_k},\ w_{2m_k+1})| \\ &+ \left|\frac{d(w_{2n_k+1}, w_{2n_k})d(w_{2m_k+2}, w_{2m_k+1})}{1 + d(w_{2n_k}, w_{2m_k+1})}\right| \\ &+ \left|\frac{d(w_{2n_k+1}, w_{2m_k+1})d(w_{2m_k+2}, w_{2m_k+1})}{1 + d(w_{2n_k},\ w_{2m_k+1})}\right| + |(d(w_{2m_k+2},\ w_{2m_k+1})| \\ &\leq |d(w_{2n_k}, w_{2n_k+1})| + |d(w_{2m_k+1}\ ,\ w_{2n_k})| \\ &+ \left|\frac{d(w_{2n_k+1}, w_{2n_k})d(w_{2m_k+2}, w_{2m_k+1})}{1 + d(w_{2m_k+1}, w_{2n_k})}\right| \\ &+ \left|\frac{d(w_{2n_k+1}, w_{2m_k+1})d(w_{2m_k+2}, w_{2m_k+1})}{1 + d(w_{2n_k},\ w_{2m_k+1})}\right| + |(d(w_{2m_k+2},\ w_{2m_k+1})| \end{aligned}$$

Taking limit as $k \to \infty$ and using (2.51), (2.55), we get

$$|c| \leq \lim_{k\to\infty} \gamma(d(w_{2n_k}, w_{2m_k+1}))|c| \leq |c| \ \Rightarrow\ \lim_{k\to\infty} \gamma(d(w_{2n_k}, w_{2m_k+1})) = 1.$$

Since $\gamma \in \Gamma$, we obtain that $|d(w_{2n_k},\ w_{2m_k+1})| \to 0$ as $k \to \infty$, which is a contradiction. Thus $\{w_{2n}\}$ is a C-Cauchy sequence and hence $\{w_n\}$ is a C-Cauchy sequence. But X is complete, so there exists a point $t \in X$ such that $w_n \to t$ as $n \to \infty$. Therefore in the light of (2.49), one can get

$$\lim_{n\to\infty} Kz_{2n} = \lim_{n\to\infty} Lz_{2n+1} = \lim_{n\to\infty} Mz_{2n} = \lim_{n\to\infty} Nz_{2n+1} = t \tag{2.56}$$

Next, since $K(X) \subseteq N(X)$, there exists $u \in X$ such that $Nu = t$. Thus (2.56) becomes

$$\lim_{n\to\infty} Kz_{2n} = \lim_{n\to\infty} Lz_{2n+1} = \lim_{n\to\infty} Mz_{2n} = \lim_{n\to\infty} Nz_{2n+1} = t = Nu \tag{2.57}$$

We will show that $Lu = Nu$. For this, consider

$$d(t, Lu) \precsim d(t, Kz_{2n}) + d(Kz_{2n}, Lu),$$

by condition (2.42) of Theorem (2.3.1) with $z = z_{2n}$ and $w = u$, we have

$$\begin{aligned} d(t,\ Lu) \precsim\ & d(t, Kz_{2n}) + \lambda_1(d(Mz_{2n}, Nu))d(Mz_{2n}, Nu) \\ & + \lambda_2(d(Mz_{2n}, Nu))\frac{d(Kz_{2n}, Mz_{2n})d(Lu, Nu)}{1 + d(Mz_{2n}, Nu)} \\ & + \lambda_3(d(Mz_{2n}, Nu))\frac{d(Kz_{2n}, Nu)d(Lu, Nu)}{1 + d(Mz_{2n}, Nu)} \end{aligned}$$

In the light of condition (2.41) of Theorem 2.3.1, we can write

$$d(Lu, t)) \precsim d(Mz_{2n}, Nu) + \frac{d(Kz_{2n}, Mz_{2n})d(Lu, Nu)}{1 + d(Mz_{2n}, Nu)} + \frac{d(Kz_{2n}, Nu)d(Lu, Nu)}{1 + d(Mz_{2n}, Nu)} + d(Kz_{2n}, t)$$

Taking limit as $n \to \infty$ and using (2.57), we get $d(Lu, t) \precsim 0$, which is possible if $d(Lu, t) = 0$. Thus $Lu = t$ and hence from (2.56), we obtained that

$$Lu = Nu = t. \tag{2.58}$$

On the other hand, since $L(X) \subseteq M(X)$, there exists $v \in X$ such that $Mv = t$. Thus (2.56) becomes

$$\lim_{n\to\infty} Kz_{2n} = \lim_{n\to\infty} Lz_{2n+1} = \lim_{n\to\infty} Mz_{2n} = \lim_{n\to\infty} Nz_{2n+1} = t = Mv \tag{2.59}$$

Now, we will show that $Kv = Mv$. For this, consider

$$d(Kv, t) \precsim d(Kv,\ Lz_{2n+1}) + d(Lz_{2n+1}\ ,\ t),$$

setting $z = v$, $w = z_{2n+1}$ in condition (2.42) of Theorem (2.3.1) and proceeding the same way as above, one can get $d(Kv,\ t) \precsim 0$. It is possible if $d(Kv,\ t) = 0$. Thus $Kv = t$ and hence from (2.59), we have

$$Kv = Mv = t. \tag{2.60}$$

Therefore from (2.58) and (2.60) , we get

$$Kv = Mv = Lu = Nu = t. \tag{2.61}$$

Now, using the weak compatibility of the pairs $(K,\ M), (L,\ N)$ and (2.61) it follows that

$$Mv = Kv \ \Rightarrow\ KMv = MKv \ \Rightarrow\ Kt = Mt \tag{2.62}$$

and

$$Nu = Lu \ \Rightarrow\ LNu = NLu \ \Rightarrow\ Lt = Nt \tag{2.63}$$

That is t is the coincident point of each pair $(K,\ M)$ and $(L,\ N)$ in X.

Next, we show that t is a common fixed point of K, L , M and N. For this, let $Kt = t$. If not, then on using condition (2.42) of Theorem (2.3.1) with $z = t$ and $w = u$, we have

$$\begin{aligned} d(Kt,\ Lu) &\precsim \lambda_1(d(Mt,\ Nu))d(Mt,\ Nu) \\ &+ \lambda_2(d(Mt,\ Nu))\frac{d(Kt, Mt)d(Lu, Nu)}{1 + d(Mt, Nu)} \\ &+ \lambda_3(d(Mt,\ Nu))\frac{d(Kt, Nu)d(Lu, Nu)}{1 + d(Mt, Nu)} \end{aligned}$$

with the help of (2.61) and (2.62), it follows that

$$\begin{aligned} d(Kt,\ t) &\precsim \lambda_1(d(Kt,\ t))d(Kt, t) \\ &+ \lambda_2(d(Kt,\ t))\frac{d(Kt, Kt)d(t, t)}{1 + d(Kt, t)} \\ &+ \lambda_3(d(Kt, t))\frac{d(Kt, t)d(t, t)}{1 + d(Kt, t)} \\ &= \lambda_1(d(Kt, t))d(Kt, t). \end{aligned}$$

Which is not possible as $\lambda_1(d(t,\ Kt)) < 1$, thus $Kt = t$ and hence from (2.62) , we get

$$Kt = Mt = t \tag{2.64}$$

Similarly, setting $z = v, w = t$ in condition (2.42) of Theorem (2.3.1) and using (2.61), (2.63), one can get

$$Lt = Nt = t \tag{2.65}$$

Therefore from (2.64) and (2.65) , it follows that

$$Kt = Lt = Mt = Nt = t \tag{2.66}$$

That is t is a common fixed point of K, L , M and N.

Finally, we check the uniqueness of a common fixed point of K, L , M and N. For this, assume that $t^* \neq t$ be another fixed point of K, L , M and N. Then on setting $z = t$ and $w = t^*$ in condition (2.42) of Theorem (2.3.1), we have

$$\begin{aligned} d(Kt, Lt^*) \precsim\ & \lambda_1(d(Mt, Nt^*))d(Mt, Nt^*) \\ & + \lambda_2(d(Mt, Nt^*))\frac{d(Kt, Mt)d(Lt^*, Nt^*)}{1 + d(Mt, Nt^*)} \\ & + \lambda_3(d(Mt, Nt^*))\frac{d(Kt, Nt^*)d(Lt^*, Nt^*)}{1 + d(Mt, Nt^*)} \end{aligned}$$

which implies that

$$\begin{aligned} d(t, t^*) \precsim\ & \lambda_1(d(t, t^*))d(t, t^*) \\ & + \lambda_2(d(t, t^*))\frac{d(t, t)d(t^*, t^*)}{1 + d(t, t^*)} \\ & + \lambda_3(d(t, t^*))\frac{d(t, t^*)d(t^*, t^*)}{1 + d(t, t^*)} \\ & = \lambda_1(d(t, t^*))d(t, t^*), \end{aligned}$$

which is a contradiction. Thus, $t^* = t$ and so t is a unique common fixed point of K, L , M and N. This completes the proof. ■

From Theorem (2.3.1) we can derive the following corollaries and the proof of which easily follows from Theorem (2.3.1), so we omit it.

Corollary 2.3.1 *: Let (X,d) be a C-complete complex valued metric space and $K, L, M, N : X \to X$ be four mappings satisfying*

$$d(Kz, Lw) \precsim \lambda_1 d(M_Z, Nw) + \lambda_2\frac{d(Kz, Mz)d(Lw, Nw)}{1 + d(Mz, Nw)} + \lambda_3\frac{d(Kz, Nw)d(Lw, Nw)}{1 + d(Mz, Nw)}$$

for all $z, w \in X$, where $\lambda_1 + \lambda_2 + \lambda_3 \in \mathbb{R}_+$ with $\lambda_1 + \lambda_2 + \lambda_3 < 1$. If $K(X) \subseteq N(X)$ and $L(X) \subseteq M(X)$ and the pairs (K, M) and (L, N) are weakly compatible, then K, L, M and N have a unique common fixed point in X.

Corollary 2.3.2 : *Let K, L be two self-mappings on a C-complete complex valued metric space (X,d) and $\lambda_1, \lambda_2, \lambda_3 : \mathbb{C}^+ \to [0,1)$ be given mappings. Suppose that the following conditions hold:*

(1) $\lambda_1(z) + \lambda_2(z) + \lambda_3(z) < 1$ for all $z \in \mathbb{C}^+$ and the mapping $\gamma : \mathbb{C}^+ \to [0,1)$, which is defined by

$$\gamma(z) := \frac{\lambda_1(z)}{1 - [\lambda_2(z) + \lambda_3(z)]} \text{ for all } z \in \mathbb{C}^+,$$

belongs to Γ;

(2) for each $z, w \in X$, we have

$$\begin{aligned} d(Kz, Lw) \precsim\ & \lambda_1(d(z,w))d(z,w) \\ & + \lambda_2(d(z,w))\frac{d(Kz,z)d(Lw,w)}{1+d(z,w)} \\ & + \lambda_3(d(z,w))\frac{d(Kz,w)d(Lw,w)}{1+d(z,w)}. \end{aligned}$$

If the pair (K, M) and (L, N) are weakly compatible, then K and L have a unique common fixed point in X.

Corollary 2.3.3 : *Let K be a self- mapping on a C- complete complex valued metric space (X,d) and $\lambda_1, \lambda_2, \lambda_3 : \mathbb{C}^+ \to [0,1)$ be given mappings. Suppose that the following conditions hold:*

(1) $\lambda_1(z) + \lambda_2(z) + \lambda_3(z) < 1$ for all $z \in \mathbb{C}^+$and the mapping $\gamma : \mathbb{C}^+ \to [0,1)$, which is defined by

$$\gamma(z) := \frac{\lambda_1(z)}{1 - [\lambda_2(z) + \lambda_3(z)]} \text{ for all } z \in \mathbb{C}^+,$$

belongs to Γ;

(2) for each $z, w \in X$, we have

$$\begin{aligned} d(Kz, Kw) \precsim\ & \lambda_1(d(z,w))d(z,w) \\ & + \lambda_2(d(z,w))\frac{d(Kz,z)d(Kw,w)}{1+d(z,w)} \\ & + \lambda_3(d(z,w))\frac{d(Kz,w)d(Kw,w)}{1+d(z,w)}. \end{aligned}$$

Then K has a unique fixed point in X.

Remark 2.3.1 *:*

- *By substituting* $\lambda_3(z) = 0$ *in Corollary* (2.3.2)*, we get Theorem* (2.3.2)*.*
- *By substituting* $M = N = I$*, where* I *is an identity mapping on* X*, and* $\lambda_3(z) = 0$ *in Corollary* (2.3.1)*, we get Corollary* (2.3.3)
- *By substituting* $\lambda_3(z) = 0$ *in Corollary* (2.3.3)*, we get Corollary* 3.4 *of [12]*
- *By substituting* $\lambda_3(z) = 0$ *and* $\lambda_1(z) = \lambda_1, \lambda_2(z) = \lambda_2$*, where* λ_1, λ_2 *are non-negative real numbers such that* $\lambda_1 + \lambda_2 < 1$*, in Corollary* (2.3.3)*, we get Corollary* 3.5 *of [12].*

Theorem 2.3.2 *: Let* P *be a self-mapping on a* C*-complete complex valued metric space (X,d) and* $\lambda_1, \lambda_2, \lambda_3 : \mathbb{C}^+ \to [0, 1)$ *be given mappings. Suppose that the following conditions hold:*

(1) $\lambda_1(z) + \lambda_2(z) + \lambda_3(z) < 1$ *for all* $z \in \mathbb{C}^+$ *and the mapping* $\gamma : \mathbb{C}^+ \to [0, 1)$*, which is defined by*

$$\gamma(z) = \frac{\lambda_1(z)}{1 - [\lambda_2(z) + \lambda_3(z)]} \text{ for all } z \in \mathbb{C}+,$$

belongs to Γ*;*

(2) for each $z, w \in X$*, we have:*

$$\begin{aligned} d(P^n z, P^n w) \precsim\ & \lambda_1(d(z, w))d(z, w) \\ & + \lambda_2(d(z, w))\frac{d(P^n z, z)d(P^n w, w)}{1 + d(z, w)} \\ & + \lambda_3(d(z, w))\frac{d(P^n z, w)d(P^n w, w)}{1 + d(z, w)}, \end{aligned}$$

for some $n \in \mathbb{N}$*.*

Then P *has a unique fixed point in* X*.*

Proof: Using the same technique in the proof of Theorem 3.6 of [12] with Corollary (2.3.3) by setting $K = P^n$, we get this result. ■

Remark 2.3.2 *:*

- *By substituting* $\lambda_3(z) = 0$ *in Theorem* (2.3.2)*, we get Theorem* 3.6 *of [12].*
- *By substituting* $\lambda_3(z) = 0$ *and* $\lambda_1(z) = \lambda_1, \lambda_2(z) = \lambda_2$*, where* λ_1, λ_2 *are non-negative real numbers such that* $\lambda_1 + \lambda_2 < 1$*, in Theorem* (2.3.2)*, we get Corollary* 3.7 *of [12] .*

Theorem 2.3.3 : *Let K, L be two self-mappings on a complex valued metric space (X,d) and $\lambda_1, \lambda_2, \lambda_3 : \mathbb{C}^+ \to [0,1)$ be given mappings. Suppose that the following conditions hold:*

(1) $\lambda_1(z) + \lambda_2(z) + \lambda_3(z) < 1$ for all $z \in \mathbb{C}_+$ and the mapping $\gamma : \mathbb{C}^+ \to [0,1)$, which is defined by

$$\gamma(z) := \frac{\lambda_1(z)}{1 - [\lambda_2(z) + \lambda_3(z)]} \text{ for all } z \in \mathbb{C}^+,$$

belongs to Γ;

(2) for each $z, w \in X$, we have

$$\begin{aligned} d(Kz, Kw) &\precsim \lambda_1(d(Lz, Lw))d(Lz, Lw) \\ &+ \lambda_2(d(Lz, Lw))\frac{d(Kz, Lz)d(Kw, Lw)}{1 + d(Lz, Lw)} \\ &+ \lambda_3(d(Lz, Lw))\frac{d(Kz, Lw)d(Kw, Lw)}{1 + d(Lz, Lw)}. \end{aligned}$$

If $K(X) \subseteq L(X)$ such that $L(X)$ is C-complete and the pair (K, L) is weakly compatible, then K and L have a unique common fixed point in X.

Proof: The proof easily follows from Theorem 4.3 of [12], so we omit it. ■

Remark 2.3.3 : *By substituting $\lambda_3(z) = 0$ in Theorem (2.3.3), we get Theorem 4.3 of [12].*

CHAPTER

3

APPLICATIONS

In this chapter, we apply some theorems of Chapter two to find the common fixed point for finite families, and to find solutions to the Urysohn's integral equations system.

3.1 Application in finite families

The following example illustrates Theorem (2.2.1)

As an application of Theorem (2.2.1), we prove a common fixed point theorem for four finite families of maps on metric space. While proving our result, we utilize Definition 1.5.1 which is a natural expansion of commutativity condition to two finite families.

Theorem 3.1.1 *: Let* $\{A_1, A_2, ..., A_m\}, \{B_1, B_2, ..., B_n\}, \{S_1, S_2, ..., S_p\}$ *and* $\{T_1, T_2, ..., T_q\}$ *be four finite families of self maps of a complex-valued metric space* (X, d) *such that* $A = A_1.A_2...A_m, B = B_1.B_2...B_n, S = S_1.S_2....S_p$ *and* $T = T_1.T_2....T_q$ *satisfy the condition (2.42) of Lemma* 2.2.1 *and*

(1) $A(X) \subset T(X)$ *(or* $B(X) \subset S(X)$*)*

(2) the pair (A, S) *(or* (B, T)*) satisfy* $(E.A)$ *property.*

Then the pairs (A, S) *and* (B, T) *have a point of coincidence each. Moreover, finite families of self maps* A_i, B_r, S_k *and* T_t *have a unique common fixed point provided that the pairs of families* $(\{A_i\}, \{S_k\})$ *and* $(\{B_r\}, \{T_t\})$ *commute pairwise for all* $i = 1, 2, ..., m, k = 1, 2, ..., p, r = 1, 2, ..., n$ *and* $t = 1, 2, ..., q$*.*

Proof: Since self maps A, B, S, T satisfy all the conditions of Theorem (2.2.1), the pair (A, S) and (B, T) have a point of coincidence. Also the pairs of families $(\{A_i\}, \{S_k\})$ and $(\{B_r\}, \{T_t\})$ commute pairwise, we first show that $AS = SA$ as

$$\begin{aligned} AS &= (A_1.A_2...A_m)(S_1.S_2....S_p) = (A_1.A_2...A_{m-1})(A_m S_1.S_2....S_p) \\ &= (A_1.A_2...A_{m-1})(S_1.S_2....S_p A_m) = (A_1.A_2...A_{m-2})(A_{m-1} S_1.S_2....S_p A_m) \\ &= (A_1.A_2...A_{m-2})(S_1.S_2....S_p A_{m-1} A_m) = ... = A_1(S_1.S_2....S_p A_2...A_m) \\ &= (S_1.S_2....S_p)(A_1.A_2...A_m) = SA. \end{aligned}$$

Similarly one can prove that $BT = TB$. Hence, obviously the pair (A, S) and (B, T) are weakly compatible. Now using Theorem (2.2.1), we conclude that A, B, S *and* T, have a unique common fixed point in X, say z.

Now, one needs to prove that z remains the fixed point of all the component maps.

For this consider

$$\begin{aligned}
A(A_iz) &= ((A_1.A_2...A_m)A_i)z = (A_1.A_2...A_{m-1})(A_mA_i)z \\
&= (A_1.A_2...A_{m-1})(A_iA_m)z = (A_1.A_2...A_{m-2})(A_{m-1}A_iA_m)z \\
&= (A_1.A_2...A_{m-2})(A_iA_{m-1}A_m)z = ... = A_1(A_iA_2...A_m)z \\
&= (A_1A_i)(A_2...A_m)z \\
&= (A_iA_1)(A_2....A_m) = A_i(A_1.A_2...A_m)z = A_iAz = A_iz.
\end{aligned}$$

Similarly, one can prove that

$$\begin{aligned}
A(S_kz) = S_k(Az) = S_kz, S(S_kz) = S_k(Sz) = S_kz \\
S(A_iz) = A_i(Sz) = A_iz, B(B_iz) = B_i(Bz) = B_iz \\
B(T_tz) = T_t(Bz) = T_tz, T(T_tz) = T_t(Tz) = T_tzn,
\end{aligned}$$

and $T(B_rz) = B_r(Tz) = B_rz$, which shows that (for all i, r, k and t) A_iz and S_kz are other fixed points of pair (A, S) whereas B_rz and T_tz are other fixed points of pair (B, T). As A, B, S and T have a unique fixed point, so, we get $z = A_iz = S_kz = B_rz = T_tz$, for all $i = 1, 2, ..., m$, $k = 1, 2, ..., p$, $r = 1, 2, ..., n$, $t = 1, 2, ..., q$.
Which shows that z is a unique fixed point of $\{A_i\}_{i=1}^m, \{B_r\}_{r=1}^n, \{S_k\}_{k=1}^p$ and $\{T_t\}_{t=1}^q$. ■

3.2 Application to Urysohn integral equations

Over the years, many mathematicians tried successfully to studied the existence and uniqueness of solution for the several nonlinear integral equations since it has been widely used in the various fields of science and engineering. In this section, we use our theoretical result (Theorem 2.3.1) in order to prove the existence and uniqueness of common solution for the following Urysohn integral equations:

$$z(t) = \phi_i(t) + \int_a^b k_i(t, s, z(s))ds, \tag{3.1}$$

Where $i = 1, 2, 3, 4, a, b \in \mathbb{R}$ with $a \leq b, t \in [a, b], z, \phi_i \in C([a, b], \mathbb{R}^n)$ and
$k_i : [a, b] \cdot [a, b] \cdot \mathbb{R}^n \to \mathbb{R}^n$ is a given mapping for each $i = 1, 2, 3, 4$.
Throughout this section, for each $i = 1, 2, 3, 4$ and k_i in (3.1), we use the following symbol:

$$\Delta_i(z(t)) = \int_a^b k_i(t, s, z(s))ds,$$

Theorem 3.2.1 *: Consider the Urysohn integral equations (3.1). Suppose that the following assumptions hold for each $t \in [a,b]$:*

- (C_1) *:* $\phi_1(t) + \phi_4(t) + \Delta_1 z(t) - \Delta_4 \left(\Delta_1 z(t) + \phi_1(t) + \phi_4(t)\right) = 0$
  and $\phi_2(t) + \phi_3(t) + \Delta_2 z(t) - \Delta_3 \left(\Delta_2 z(t) + \phi_2(t) + \phi_3(t)\right) = 0$*;*

- (C_2) *:* $\phi_1(t) + 3\phi_3(t) + \Delta_1 \left(\Delta_1 z(t) + \phi_1(t)\right) + 2\Delta_3 z(t) + \Delta_3 \left(2z(t) - \Delta_3 z(t) - \phi_3(t)\right) = 4z(t)$
  and $\phi_2(t) + 3\phi_4(t) + \Delta_2 \left(\Delta_2 z(t) + \phi_2(t)\right) + 2\Delta_4 z(t) + \Delta_4 \left(2z(t) - \Delta_4 z(t) - \phi_4(t)\right) = 4z(t)$.

If there exist mappings $\lambda_1, \lambda_2, \lambda_3 : \mathbb{C}^+ \to [0,1)$ satisfying the following conditions:

- $\lambda_1(z) + \lambda_2(z) + \lambda_3(z) < 1$ *for all $z \in \mathbb{C}^+$ and the mapping $\gamma : \mathbb{C}^+ \to [0,1)$, which is defined by*
  $$\gamma(z) := \frac{\lambda_1(z)}{1 - [\lambda_2(z) + \lambda_3(z)]} \ \textit{for all } z \in \mathbb{C}_+$$
  belongs to Γ;

- *for each $z, w \in X$ and $t \in [a,b]$, we have*
  $$A_{zw}(t)\sqrt{1+a^2}e^{i\arctan a} \precsim \lambda_1\left(\max_{t\in[a,b]} B_{zw}(t)\right) B_{zw}(t) + \lambda_2\left(\max_{t\in[a,b]} B_{zw}(t)\right) C_{zw}(t)$$
  $$+\lambda_3\left(\max_{t\in[a,b]} B_{zw}(t)\right) D_{zw}(t),$$
  where
  $$A_{zw}(t) = \left\|\Delta_1 z(t) + \phi_1(t) - \Delta_2 w(t) - \phi_2(t)\right\|_\infty,$$
  $$B_{zw}(t) = \left\|2z(t) - \Delta_3 z(t) - \phi_3(t) - 2w(t) + \Delta_4 w(t) + \phi_4(t)\right\|_\infty \sqrt{1+a^2}\exp^{i\arctan(a)},$$
  $$C_{zw}(t) = \left(\frac{\left\|\Delta_1 z(t) + \phi_1(t) - 2z(t) + \Delta_3 z(t) + \phi_3(t)\right\|_\infty}{1 + \max_{t\in[a,b]} B_{zw}(t)}\right)$$
  $$\cdot \left\|\Delta_2 w(t) + \phi_2(t) - 2w(t) + \Delta_4 w(t) + \phi_4(t)\right\|_\infty \sqrt{1+a^2}e^{i\arctan(a)},$$
  $$D_{zw}(t) = \left(\frac{\left\|\Delta_1 z(t) + \phi_1(t) - 2w(t) + \Delta_4 w(t) + \phi_4(t)\right\|_\infty}{1 + \max_{t\in[a,b]} B_{zw}(t)}\right)$$
  $$\cdot \left\|\Delta_2 w(t) + \phi_2(t) - 2w(t) + \Delta_4 w(t) + \phi_4(t)\right\|_\infty \sqrt{1+a^2}e^{i\arctan(a)}.$$

  Then the system (3.1) *has a unique common solution.*

Proof: Let $X = C([a,b], \mathbb{R}^n)$ and $d : X \cdot X \to \mathbb{C}$ be defined by

$$d(z,w) = \max_{t\in[a,b]} \|z(t) - w(t)\|_\infty \sqrt{1+a^2} \exp^{i \arctan a}.$$

Then (X, d) is a C-complete complex valued metric space.

Define four mappings $K, L, M, N : X \to X$ by

- $Kz(t) = \Delta_1 z(t) + \phi_1(t) = \int_a^b k_1(t, s, z(s))ds + \phi_1(t)$,
- $Lz(t) = \Delta_2 z(t) + \phi_2(t) = \int_a^b k_2(t, s, z(s))ds + \phi_2(t)$,
- $Mz(t) = 2z(t) - \Delta_3 z(t) - \phi_3(t) = 2z(t) - \int_a^b k_3(t, s, z(s))ds - \phi_3(t)$,
- $Nz(t) = 2z(t) - \Delta_4 z(t) - \phi_4(t) = 2z(t) - \int_a^b k_4(t, s, z(s))ds - \phi_4(t)$.

Let $z, w \in X$. Then we obtain that

$$\begin{aligned}
d(Kz, Lw) &= \max_{t\in[a,b]} \|\Delta_1 z(t) + \phi_1(t) - \Delta_2 w(t) - \phi_2(t)\|_\infty \sqrt{1+a^2} e^{i \arctan a} \\
d(Mz, Nw) &= \max_{t\in[a,b]} \|2z(t) - \Delta_3 z(t) - \phi_3(t) - 2w(t) + \Delta_4 w(t) + \phi_4(t)\|_\infty \sqrt{1+a^2} e^{i \arctan a} \\
d(Kz, Mz) &= \max_{t\in[a,b]} \|\Delta_1 z(t) + \phi_1(t) - 2z(t) + \Delta_3 z(t) + \phi_3(t)\|_\infty \sqrt{1+a^2} e^{i \arctan a} \\
d(Lw, Nw) &= \max_{t\in[a,b]} \|\Delta_2 w(t) + \phi_2(t) - 2w(t) + \Delta_4 w(t) + \phi_4(t)\|_\infty \sqrt{1+a^2} e^{i \arctan a} \\
d(Kz, Nw) &= \max_{t\in[a,b]} \|\Delta_1 z(t) + \phi_1(t) - 2w(t) + \Delta_4 w(t) + \phi_4(t)\|_\infty \sqrt{1+a^2} e^{i \arctan a}.
\end{aligned} \tag{3.2}$$

From condition (2) of Theorem 3.2.1, for each $t \in [a,b]$ we have

$$\begin{aligned}
A_{zw}(t)\sqrt{1+a^2} e^{i \arctan a} &\precsim \lambda_1 \left(\max_{t\in[a,b]} B_{zw}(t)\right) B_{zw}(t) + \lambda_2 \left(\max_{t\in[a,b]} B_{zw}(t)\right) C_{zw}(t) \\
&\quad + \lambda_3 \left(\max_{t\in[a,b]} B_{zw}(t)\right) D_{zw}(t) \\
&\precsim \lambda_1 \left(\max_{t\in[a,b]} B_{zw}(t)\right) \max_{t\in[a,b]} B_{zw}(t) + \lambda_2 \left(\max_{t\in[a,b]} B_{zw}(t)\right) \max_{t\in[a,b]} C_{zw}(t) \\
&\quad + \lambda_3 \left(\max_{t\in[a,b]} B_{zw}(t)\right) \max_{t\in[a,b]} D_{zw}(t).
\end{aligned}$$

This yields that

$$\begin{aligned}
\max_{t\in[a,b]} A_{zw}(t)\sqrt{1+a^2} e^{i \arctan a} &\precsim \lambda_1 \left(\max_{t\in[a,b]} B_{zw}(t)\right) \max_{t\in[a,b]} B_{zw}(t) + \lambda_2 \left(\max_{t\in[a,b]} B_{zw}(t)\right) \max_{t\in[a,b]} C_{zw}(t) \\
&\quad + \lambda_3 \left(\max_{t\in[a,b]} B_{zw}(t)\right) \max_{t\in[a,b]} D_{zw}(t).
\end{aligned}$$

By using (3.2), we get

$$d(Kz,Lw) \precsim \lambda_1(d(Mz,Nw))d(Mz,Nw) + \lambda_2(d(Mz,Nw))\frac{d(Kz,Mz)d(Lw,Nw)}{1+d(Mz,Nw)}$$
$$+ \qquad \lambda_3(d(Mz,Nw))\frac{d(Kz,Nw)d(Lw,Nw)}{1+d(Mz,Nw)}.$$

Now, we will show that $K(X) \subseteq N(X)$. For this, consider

$$\begin{aligned} N\left(Kz(t)+\phi_4(t)\right) &= 2\left[Kz(t)+\phi_4(t)\right] - \Delta_4\left(Kz(t)+\phi_4(t)\right) - \phi_4(t) \\ &= Kz(t) + Kz(t) + \phi_4(t) - \Delta_4\left(Kz(t)+\phi_4(t)\right) \\ &= Kz(t) + \Delta_1 z(t) + \phi_1(t) + \phi_4(t) - \Delta_4\left(\Delta_1 z(t) + \phi_1(t) + \phi_4(t)\right) \\ &= Kz(t) + \phi_1(t) + \phi_4(t) + \Delta_1 z(t) - \Delta_4\left(\Delta_1 z(t) + \phi_1(t) + \phi_4(t)\right). \end{aligned}$$

Using (C_1), we get $N\left(Kz(t)+\phi_4(t)\right) = Kz(t)$. This shows that $K(X) \subseteq N(X)$.
Similarly, we can show that $L(X) \subseteq M(X)$.
Next, we will show that the pairs (K,M) and (L,N) are weakly compatible. Note that for each $t \in [a,b]$, we get

$$\begin{aligned} \|MKz(t) - KMz(t)\| &= \|M\left(\Delta_1 z(t) + \phi_1(t)\right) - K\left(2z(t) - \Delta_3 z(t) - \phi_3(t)\right)\| \\ &= \|2\left(\Delta_1 z(t) + \phi_1(t)\right) - \Delta_3\left(\Delta_1 z(t) + \phi_1(t)\right) - \phi_3(t) - \Delta_1\left(2z(t) - \Delta_3 z(t) - \phi_3(t)\right) - \phi_1(t)\|. \end{aligned} \tag{3.3}$$

If $Kz = MZ$ for some $z \in X$, then we have

$$\Delta_1 z(t) + \phi_1(t) = 2z(t) - \Delta_3 z(t) + \phi_3(t)$$

For all $t \in [a,b]$. Thus (3.3) becomes

$$\begin{aligned} &\|MKz(t) - KMz(t)\| \\ &= \|2\left(2z(t) - \Delta_3 z(t) - \phi_3(t)\right) - \Delta_3\left(2z(t) - \Delta_3 z(t) - \phi_3(t)\right) - \phi_3(t) - \Delta_1\left(\Delta_1 z(t) + \phi_1(t)\right) - \phi_1(t)\| \\ &= \|4z(t) - 2\Delta_3 z(t) - 3\phi_3(t) - \Delta_3\left(2z(t) - \Delta_3 z(t) - \phi_3(t)\right) - \Delta_1\left(\Delta_1 z(t) + \phi_1(t)\right) - \phi_1(t)\|, \end{aligned}$$

for all $t \in [a,b]$. From (C_2), we get $\|MKz(t) - KMz(t)\| = 0$, that is, $MKz(t) = KMz(t)$ for all $t \in [a,b]$. Therefore, $MKz = KMz$ whenever $Kz = Mz$. Hence the pair (K,M) is weakly compatible. Similarly, one can show that (L,N) is weakly compatible.
Now all the condition of Theorem (2.3.1) are satisfied. Therefore, there exists a unique common fixed point of K, L, M, N in X and consequently, there exists a unique common solution of system 3.1. ■

Conclusion

In this thesis, we attempted to present a comprehensive study of fixed point theory in complex-valued metric spaces and we used it to derive enough examples to demonstrate our findings.

Our research began with three chapters:

- In Chapter 1, we defined some known definitions and results of the metric spaces and complex-valued metric spaces. Also, we discussed fixed point theorems in complex-valued metric spaces, thus, we recalled the (E, A) property and some integral equations linear and nonlinear type.

- In Chapter 2, we proved the existence and uniqueness of common fixed point theorems under rational contraction condition in complex valued b-metric space. Also, using $(E.A)$ property and weakly compatible mappings in order to find the fixed point in complex valued metric space.

- In Chapter 3, we used the results to prove the existence of common fixed point theorem for finite families of maps and common solution of the Urysohn's integral equations in complex valued metric space.

In this light, this theory, as well as the search for more important and diverse fields of application in mathematics and other sciences.

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