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# Optimized Lower Bounds for Five-Body Hamiltonians

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**Abstract** A methodology recently proposed for deriving optimized lower bounds for the energies of the ground states of $N$-body systems is applied to the five-body case. Such lower bounds result from an optimization process over a certain number of parameters. The so-called universal dynamical constraints, i.e., relations between the values of the parameters corresponding to the optimized lower bound and which have the properties to be of dynamical nature and to be independent of the particular form of the potential, are computed in their totality and great calculational details are given. The optimized lower bound obtained in this way proves to be saturated, i.e., identical to the exact result, in the harmonic oscillator case. Particular mass configurations are considered with the corresponding simplifications in the universal dynamical constraints. In relation to the property of saturability in the case of harmonic interactions, particular attention is devoted to five-body systems with harmonic interactions. Various mass configurations are considered and for each case the corresponding ground state energy is derived.

## 1 Introduction

As is well-known, the $N$-body problems are very involved. Very few of them are exactly solvable. Even the simplest cases of the one-body problem with a central potential or the two-body problem in the case of a translationally and rotationally invariant interaction are exactly solvable only for very particular forms of the interaction potential. Furthermore, the complexity of the problem grows quickly as $N$ increases. The numerical resolution, very simple in the case of one-body and two-body problems under the above mentioned conditions, complicates quickly as the number of particles grows requiring thereby considerable calculational facilities. An alternative to numerical computations are exact results. Among them, the lower bounds for the ground state energies of $N$-body systems. Recently, a lower bound, known as "the optimized lower bound" since resulting from an optimization over free parameters, and derived initially for the three-body [1] and the four-body [2] systems, has been generalized to $N$-body systems [3] for arbitrary $N$ with the two requirements of non-relativistic kinematics and translationally invariant two-body forces. Our main concern here is to consider in detail the next system in the degree of complexity after the three- and the four-body cases, i.e., the five-body case. In Sect. 2, we will briefly recall the methodology when particularized to the five-body case. In Sect. 3, the universal dynamical constraints, relations among the values of the parameters corresponding to the optimized lower bound and which have the two properties to be of dynamical nature an to be independent of the form of the potential, are derived. Section 4 is devoted to particular mass configurations and the corresponding simplifications in the universal dynamical constraints. In Sect. 5, some numerical results are presented, showing

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in particular a numerical evidence of saturability in the case of harmonic forces. Related to this, Sect. 5 is devoted to the five-body harmonic oscillator, also called a system of five oscillators, where we derive explicit analytical expressions for the ground state energy of the system for different mass configurations. Finally great calculational details about the derivation of the universal dynamical constraints in the five-body case are gathered in an appendix.

## 2 Optimized Lower Bound

Let be a five-body system obeying non-relativistic kinematics with translationally invariant two-body interactions, that is a five-body system governed by a Hamiltonian $H$ of the form

$$H = \sum_{i=1}^{5} \frac{1}{2m_i} \boldsymbol{p}_i^2 + \sum_{i<j=1}^{5} V^{(ij)}(\boldsymbol{r}_{ij}), \tag{1}$$

where $m_i$, $\boldsymbol{r}_i$, $\boldsymbol{p}_i$ stand respectively for the mass, the position and the linear momentum of the $i$th particle. $\boldsymbol{r}_{ij} := \boldsymbol{r}_i - \boldsymbol{r}_j$, $i \neq j = 1, \ldots, 5$. It will be noted that the potential $V^{(ij)}$ from which derives the two-body force may depend on the pair. Our starting point is the following decomposition

$$\sum_{i=1}^{5} \frac{1}{2m_i} \boldsymbol{p}_i^2 = \left( \sum_{j=1}^{5} b_j \boldsymbol{p}_j \right) \left( \sum_{i=1}^{5} \boldsymbol{p}_i \right) + \sum_{i<j=1}^{5} a_{ij} \boldsymbol{p}_{ij}^2 \tag{2}$$

of the kinetic part of the Hamiltonian involving the parameters $b_j$, $j = 1, \ldots, 5$, and the necessary positive parameters $a_{ij}$, $i < j = 1, \ldots, 5$. $\boldsymbol{p}_{ij}$ is a linear combination of the various linear momenta $\boldsymbol{p}_k$

$$\boldsymbol{p}_{ij} = \sum_{k=1}^{5} \frac{\omega_{ij,k}}{2} \boldsymbol{p}_k, \tag{3}$$

with the coefficients $\omega_{ij,k}$ of the linear combination chosen such that $\boldsymbol{r}_{ij}$ and $\boldsymbol{p}_{ij}$ are conjugate variables of one another, that is satisfying canonical commutation relations

$$[r_{ij,k}, p_{ij,\ell}] = i\hbar \delta_{k,\ell} \qquad k, \ell = 1, 2, 3, \tag{4}$$

where $r_{ij,k}$ and $p_{ij,\ell}$ stand respectively for the $k$th component of $\boldsymbol{r}_{ij}$ and the $\ell$th component of $\boldsymbol{p}_{ij}$. The factor $1/2$ in the right-hand side of (3) is a matter of convenience. Replacing the momenta $\boldsymbol{p}_{ij}$ by their expressions, Eqs. (3) and (2) can be rewritten as

$$\sum_{i=1}^{5} \frac{1}{2m_i} \boldsymbol{p}_i^2 = \left( \sum_{j=1}^{5} b_j \boldsymbol{p}_j \right) \left( \sum_{i=1}^{5} \boldsymbol{p}_i \right) + \sum_{i<j=1}^{5} \frac{a_{ij}}{4} \left( \sum_{k=1}^{5} \omega_{ij,k} \boldsymbol{p}_k \right)^2. \tag{5}$$

Let us notice that the parameters $b_j$, $a_{ij}$ and $\omega_{ij,k}$ are constrained by relations obtained by identifying the two sides of Eq. (5). To be more precise, this identification provides us with $15 = 5 + 10$ constraints. If one remarks that the $b_j$ are in number of 5 and the $a_{ij}$ are in number of 10, then these constraints may be used to eliminate the $b_j$ and the $a_{ij}$ in favor of the $\omega_{ij,k}$. From now on the $b_j$ and the $a_{ij}$ are considered as implicit functions of the $\omega_{ij,k}$. We may without loss of generality take the $\omega_{ij,i}$ to be equal to one by a redefinition of $a_{ij}$ and of the $\omega_{ij,k}$ for $k \neq i = 1, \ldots, 5$. Then imposing the canonical commutation relations, Eq. (4), one ends with $\omega_{ij,j} = -1$. Thus the number of parameters $\omega_{ij,k}$ is 30. The decomposition of the Hamiltonian, (1), corresponding to (5) is

$$H = \left( \sum_{j=1}^{5} b_j \boldsymbol{p}_j \right) \left( \sum_{i=1}^{5} \boldsymbol{p}_i \right) + \sum_{i<j=1}^{5} \left[ a_{ij} \boldsymbol{p}_{ij}^2 + V^{(ij)}(\boldsymbol{r}_{ij}) \right]. \tag{6}$$

Let $|\Psi\rangle$ be the normalized ground state of the system and $E$ the corresponding energy. We have

$$\begin{aligned} E &= \langle\Psi|H|\Psi\rangle \\ &= \langle\Psi\left|\left(\sum_{j=1}^{5} b_j \boldsymbol{p}_j\right)\left(\sum_{i=1}^{5} \boldsymbol{p}_i\right)|\Psi\rangle + \sum_{i<j=1}^{5} \langle\Psi|\left[a_{ij}\boldsymbol{p}_{ij}^2 + V^{(ij)}(\boldsymbol{r}_{ij})\right]\right|\Psi\rangle. \end{aligned} \tag{7}$$

Since the ground state $|\Psi\rangle$ is invariant under translation, then

$$\left(\sum_{i=1}^{5} \boldsymbol{p}_i\right)|\Psi\rangle = \boldsymbol{0}, \tag{8}$$

and thus the contribution of the first term in the right-hand side of (7) vanishes. It results that

$$E = \sum_{i<j=1}^{5} \left\langle\Psi\left|\left[a_{ij}\boldsymbol{p}_{ij}^2 + V^{(ij)}(\boldsymbol{r}_{ij})\right]\right|\Psi\right\rangle. \tag{9}$$

But by virtue of the variational principle

$$\langle\Psi|\left[a_{ij}\boldsymbol{p}_{ij}^2 + V^{(ij)}(\boldsymbol{r}_{ij})\right]|\Psi\rangle \geq E_{ij}^{(2)}[a_{ij}(\{\omega_{k\ell,m}\})], \tag{10}$$

where $E_{ij}^{(2)}[a_{ij}(\{\omega_{k\ell,m}\})]$ stands for the ground state energy of the two-particle Hamiltonian

$$H_{ij}^{(2)}[a_{ij}(\{\omega_{k\ell,m}\})] = a_{ij}\boldsymbol{p}_{ij}^2 + V^{(ij)}(\boldsymbol{r}_{ij}). \tag{11}$$

It follows that

$$E \geq \sum_{i<j=1}^{5} E_{ij}^{(2)}[a_{ij}(\{\omega_{k\ell,m}\})]. \tag{12}$$

Thus one obtains a family of lower bounds for $E$, a lower bound $E\left(\{\omega_{k\ell,m}\}\right)$

$$E\left(\{\omega_{k\ell,m}\}\right) := \sum_{i<j=1}^{5} E_{ij}^{(2)}[a_{ij}(\{\omega_{k\ell,m}\})], \tag{13}$$

for each set of values of the parameters $\omega_{k\ell,m}$. Let us define $E_{\text{olb}}$ as

$$E_{\text{olb}} := \max_{\{\omega_{k\ell,m}\}} E\left(\{\omega_{k\ell,m}\}\right). \tag{14}$$

$E_{\text{olb}}$, which is obviously the best of the just mentioned lower bounds, (13), is called the optimized lower bound.

## 3 Universal Dynamical Constraints

When $\sum_{i<j=1}^{5} E_{ij}^{(2)}[a_{ij}(\{\omega_{k\ell,m}\})]$ reaches its maximum, all the derivatives with respect to the $\omega_{k\ell,m}$ must vanish, that is

$$\sum_{i<j=1}^{5} \frac{\partial E_{ij}^{(2)}}{\partial a_{ij}} \frac{\partial a_{ij}}{\partial \omega_{k\ell,m}} = 0 \quad m \neq k, m \neq \ell, k < \ell = 1, \ldots, 5. \tag{15}$$

We may consider (15) as a linear system of 30 equations involving 10 unknowns, the $\frac{\partial E_{ij}^{(2)}}{\partial a_{ij}}$. Since the $\frac{\partial E_{ij}^{(2)}}{\partial a_{ij}}$ are not all zero, the rectangular $(10 \times 30)$ matrix $\widetilde{\mathbf{B}}$ with matrix elements $\frac{\partial a_{ij}}{\partial \omega_{k\ell,m}}$, where $ij$ correspond to the line index and $k\ell, m$ to the column index, must be at most of rank 9. This means that every square $(10 \times 10)$

matrix extracted from the matrix $\widetilde{\mathbf{B}}$, by selecting 10 of its columns, must be of determinant zero. This will result in $30 - 20 + 1 = 21$ conditions between the values of the parameters $\omega_{k\ell,m}$ at the maximum. Before going further, let us stress that these conditions, which we will call "universal dynamical constraints", are extremely important. Indeed, with these 21 conditions at hand, in order to obtain the optimized lower bound, (14), one can choose 9 parameters among the thirty $\omega_{k\ell,m}$, express the remaining 21 in terms of them, and then maximize over only 9 parameters, instead of maximizing over 30 parameters. Thus, the maximization procedure is made in this way much more easier, since we deal with non linear optimization. But, it is worthwhile to emphasize that the above mentioned conditions are not, properly speaking, constraints. In other words, we can work with the 30 parameters without taking these conditions into account, that is we can ignore the universal dynamical constraints, perform the maximization procedure, and verify, a posteriori, that the universal dynamical constraints are numerically verified at the maximum. Let us now derive the relations connecting the $a_{ij}$ to the $\omega_{k\ell,m}$. The identification of both sides of (5) gives a linear system of 15 equations with the ten $a_{ij}$ and the five $b_k$ as unknowns and the thirty $\omega_{k\ell,m}$ as parameters, then the elimination of the $b_k$ in profit of the $a_{ij}$ results in a linear system of 10 equations with the $a_{ij}$ as unknowns and the $\omega_{k\ell,m}$ as parameters, which can be written in matrix form as

$$\widetilde{\mathbf{D}}\mathbf{A} = \boldsymbol{\alpha}, \tag{16}$$

where $\widetilde{\mathbf{D}}$ is a $(10 \times 10)$ square matrix given, after the following change of notation,

$$\begin{array}{llllll}
c_3 = \omega_{12,3}, & c_4 = \omega_{12,4}, & c_5 = \omega_{12,5}, & d_2 = \omega_{13,2}, & d_4 = \omega_{13,4}, & d_5 = \omega_{13,5}, \\
e_2 = \omega_{14,2}, & e_3 = \omega_{14,3}, & e_5 = \omega_{14,5}, & f_2 = \omega_{15,2}, & f_3 = \omega_{15,3}, & f_4 = \omega_{15,4}, \\
g_1 = \omega_{23,1}, & g_4 = \omega_{23,4}, & g_5 = \omega_{23,5}, & h_1 = \omega_{24,1}, & h_3 = \omega_{24,3}, & h_5 = \omega_{24,5}, \\
j_1 = \omega_{25,1}, & j_3 = \omega_{25,3}, & j_4 = \omega_{25,4}, & k_1 = \omega_{34,1}, & k_2 = \omega_{34,2}, & k_5 = \omega_{34,5}, \\
l_1 = \omega_{35,1}, & l_2 = \omega_{35,2}, & l_4 = \omega_{35,4}, & n_1 = \omega_{45,1}, & n_2 = \omega_{45,2}, & n_3 = \omega_{45,3},
\end{array} \tag{17}$$

by

$$\widetilde{\mathbf{D}} = \begin{pmatrix}
1 & d_{2-}^2 & e_{2-}^2 & f_{2-}^2 & g_{1-}^2 & h_{1-}^2 & j_{1-}^2 & k_{12}^2 & l_{12}^2 & n_{12}^2 \\
c_{3-}^2 & 1 & e_{3-}^2 & f_{3-}^2 & g_{1+}^2 & h_{13}^2 & j_{13}^2 & k_{1-}^2 & l_{1-}^2 & n_{13}^2 \\
c_{4-}^2 & d_{4-}^2 & 1 & f_{4-}^2 & g_{14}^2 & h_{1+}^2 & j_{14}^2 & k_{1+}^2 & l_{14}^2 & n_{1-}^2 \\
c_{5-}^2 & d_{5-}^2 & e_{5-}^2 & 1 & g_{15}^2 & h_{15}^2 & j_{1+}^2 & k_{15}^2 & l_{1+}^2 & n_{1+}^2 \\
c_{3+}^2 & d_{2+}^2 & e_{23}^2 & f_{23}^2 & 1 & h_{3-}^2 & j_{3-}^2 & k_{2-}^2 & l_{2-}^2 & n_{23}^2 \\
c_{4+}^2 & d_{24}^2 & e_{2+}^2 & f_{24}^2 & g_{4-}^2 & 1 & j_{4-}^2 & k_{2+}^2 & l_{24}^2 & n_{2-}^2 \\
c_{5+}^2 & d_{25}^2 & e_{25}^2 & f_{2+}^2 & g_{5-}^2 & h_{5-}^2 & 1 & k_{25}^2 & l_{2+}^2 & n_{2+}^2 \\
c_{34}^2 & d_{4+}^2 & e_{3+}^2 & f_{34}^2 & g_{4+}^2 & h_{3+}^2 & j_{34}^2 & 1 & l_{4-}^2 & n_{3-}^2 \\
c_{35}^2 & d_{5+}^2 & e_{35}^2 & f_{3+}^2 & g_{5+}^2 & h_{35}^2 & j_{3+}^2 & k_{5-}^2 & 1 & n_{3+}^2 \\
c_{45}^2 & d_{45}^2 & e_{5+}^2 & f_{4+}^2 & g_{45}^2 & h_{5+}^2 & j_{4+}^2 & k_{5+}^2 & l_{4+}^2 & 1
\end{pmatrix}, \tag{18}$$

with $c_{j+} := (c_j + 1)/2$, $c_{j-} := (c_j - 1)/2$ and $c_{jk} := (c_j - c_k)/2$ and so on.

It is worthwhile to notice that every parameter is present in one column and only one column of the matrix $\widetilde{\mathbf{D}}$ and that parameters come by three in each column. $\mathbf{A}$ and $\boldsymbol{\alpha}$ in Eq. (16) are two column matrices with 10 lines given by

$$\mathbf{A} := \begin{pmatrix} a_{12} \\ a_{13} \\ a_{14} \\ a_{15} \\ a_{23} \\ a_{24} \\ a_{25} \\ a_{34} \\ a_{35} \\ a_{45} \end{pmatrix}, \qquad \boldsymbol{\alpha} := \begin{pmatrix} \alpha_{12} \\ \alpha_{13} \\ \alpha_{14} \\ \alpha_{15} \\ \alpha_{23} \\ \alpha_{24} \\ \alpha_{25} \\ \alpha_{34} \\ \alpha_{35} \\ \alpha_{45} \end{pmatrix}, \tag{19}$$

with

$$\alpha_{ij} := \frac{1}{2m_i} + \frac{1}{2m_j}. \tag{20}$$

The matrix equation (16) can in principle be inverted, thus giving the ten $a_{ij}$ as functions of the thirty $\omega_{k\ell,m}$:

$$\mathbf{A} = \widetilde{\mathbf{D}}^{-1}\boldsymbol{\alpha}. \tag{21}$$

However, in practice due to the presence of parameters (the $\omega_{k\ell,m}$ and the masses $m_i$), the analytical inversion of the matrix $\widetilde{\mathbf{D}}$ is far from being an easy task in the most general case, i.e, for the most general mass configuration. Fortunately, the derivation of the universal dynamical constraints do not require the knowledge of the explicit expressions of the $a_{ij}$ in terms of the $\omega_{k\ell,m}$. Indeed, one can show [3] that the rank condition on the matrix $\widetilde{\mathbf{B}}$ is equivalent to the same rank condition on a much simpler matrix $\widetilde{\mathbf{M}}$. The columns of the matrix $\widetilde{\mathbf{M}}$ are obtained from the matrix $\widetilde{\mathbf{D}}$ in the following way: take the unique (sole) column of the matrix $\widetilde{\mathbf{D}}$ depending on a given parameter, derive it with respect to this parameter and then repeat the procedure for each of the parameters $\omega_{k\ell,m}$, ending with a rectangular (10 × 30) matrix.

$$\widetilde{\mathbf{M}} = \left(\begin{array}{ccccccccccccccc}
0 & 0 & 0 & d_{2-} & 0 & 0 & e_{2-} & 0 & 0 & f_{2-} & 0 & 0 & g_{1-} & 0 & 0 \\
c_{3-} & 0 & 0 & 0 & 0 & 0 & 0 & e_{3-} & 0 & 0 & f_{3-} & 0 & g_{1+} & 0 & 0 \\
0 & c_{4-} & 0 & 0 & d_{4-} & 0 & 0 & 0 & 0 & 0 & 0 & f_{4-} & g_{14} & g_{41} & 0 \\
0 & 0 & c_{5-} & 0 & 0 & d_{5-} & 0 & 0 & e_{5-} & 0 & 0 & 0 & g_{15} & 0 & g_{51} \\
c_{3+} & 0 & 0 & d_{2+} & 0 & 0 & e_{23} & e_{32} & 0 & f_{23} & f_{32} & 0 & 0 & 0 & 0 \\
0 & c_{4+} & 0 & d_{24} & d_{42} & 0 & e_{2+} & 0 & 0 & f_{24} & 0 & f_{42} & 0 & g_{4-} & 0 \\
0 & 0 & c_{5+} & d_{25} & 0 & d_{52} & e_{25} & 0 & e_{52} & f_{2+} & 0 & 0 & 0 & 0 & g_{5-} \\
c_{34} & c_{43} & 0 & 0 & d_{4+} & 0 & 0 & e_{3+} & 0 & 0 & f_{34} & f_{43} & 0 & g_{4+} & 0 \\
c_{35} & 0 & c_{53} & 0 & 0 & d_{5+} & 0 & e_{35} & e_{53} & 0 & f_{3+} & 0 & 0 & 0 & g_{5+} \\
0 & c_{45} & c_{54} & 0 & d_{45} & d_{54} & 0 & 0 & e_{5+} & 0 & 0 & f_{4+} & 0 & g_{45} & g_{54}
\end{array}\right.$$

$$\left.\begin{array}{ccccccccccccccc}
h_{1-} & 0 & 0 & j_{1-} & 0 & 0 & k_{12} & k_{21} & 0 & l_{12} & l_{21} & 0 & n_{12} & n_{21} & 0 \\
h_{13} & h_{31} & 0 & j_{13} & j_{31} & 0 & k_{1-} & 0 & 0 & l_{1-} & 0 & 0 & n_{13} & 0 & n_{31} \\
h_{1+} & 0 & 0 & j_{14} & 0 & j_{41} & k_{1+} & 0 & 0 & l_{14} & 0 & l_{41} & n_{1-} & 0 & 0 \\
h_{15} & 0 & h_{51} & j_{1+} & 0 & 0 & k_{15} & 0 & k_{15} & l_{1+} & 0 & 0 & n_{1+} & 0 & 0 \\
0 & h_{3-} & 0 & 0 & j_{3-} & 0 & 0 & k_{2-} & 0 & 0 & l_{2-} & 0 & 0 & n_{23} & n_{32} \\
0 & 0 & 0 & 0 & 0 & j_{4-} & 0 & k_{2+} & 0 & 0 & l_{24} & l_{42} & 0 & n_{2-} & 0 \\
0 & 0 & h_{5-} & 0 & 0 & 0 & 0 & k_{25} & k_{52} & 0 & l_{2+} & 0 & 0 & n_{2+} & 0 \\
0 & h_{3+} & 0 & 0 & j_{34} & j_{43} & 0 & 0 & 0 & 0 & 0 & l_{4-} & 0 & 0 & n_{3-} \\
0 & h_{35} & h_{53} & 0 & j_{3+} & 0 & 0 & 0 & k_{5-} & 0 & 0 & 0 & 0 & 0 & n_{3+} \\
0 & 0 & h_{5+} & 0 & 0 & j_{4+} & 0 & 0 & k_{5+} & 0 & 0 & l_{4+} & 0 & 0 & 0
\end{array}\right) \tag{22}$$

The determination of the matrix $\widetilde{\mathbf{M}}$ is the first step of the three-step recipe of reference [3]. Then, in the second step, we have to make a choice of 9 independent parameters, which we take here to be $c_3$, $c_4$, $c_5$, $d_2$, $d_4$, $d_5$, $e_2$, $e_5$, $j_1$. Remains the explicit computation of the universal dynamical constraints, which is by far the most difficult step. The full details of the computations are given in the appendix. We content ourselves here to give the expressions of the universal dynamical constraints, which are in number of 21. We have

$$e_3 = \frac{c_3d_4e_2 - c_3d_4 - c_3e_2 + c_3 - d_4e_2 + d_4 + c_4d_2 - d_2 + e_2 - c_4}{(1-c_4)\,(1-d_2)}, \tag{23}$$

$$f_2 = \frac{c_5 + j_1}{1 + j_1c_5}, \tag{24}$$

$$f_3 = \frac{j_1c_5 - c_5d_2j_1 - c_3d_5 + j_1c_3d_5 + c_3 - d_5j_1 - c_3j_1 - d_2 + d_5 + j_1}{(1-d_2)\,(1+j_1c_5)}, \tag{25}$$

$$f_4 = \frac{j_1c_4e_5 - c_5e_2j_1 + c_4 - e_5c_4 - c_4j_1 - e_5j_1 + j_1c_5 - e_2 + e_5 + j_1}{(1-e_2)\,(1+j_1c_5)}, \tag{26}$$

$$g_1 = \frac{d_2 - c_3}{1 - c_3d_2}, \tag{27}$$

$$g_4 = \frac{c_4 d_2 - c_3 d_4 - c_4 + d_4}{1 - c_3 d_2}, \tag{28}$$

$$g_5 = \frac{d_2 c_5 - c_3 d_5 - c_5 + d_5}{1 - c_3 d_2}, \tag{29}$$

$$h_1 = \frac{e_2 - c_4}{1 - c_4 e_2}, \tag{30}$$

$$h_3 = \frac{c_4 d_2 - c_4 + c_3 d_2 + e_2 + d_4 - c_3 e_2 d_2 - c_3 d_4 + c_3 d_4 e_2 - d_4 e_2 - d_2}{(1 - d_2)\,(1 - c_4 e_2)}, \tag{31}$$

$$h_5 = \frac{c_5 e_2 - c_4 e_5 - c_5 + e_5}{1 - c_4 e_2}, \tag{32}$$

$$j_3 = \frac{c_3 d_5 j_1 - c_3 d_2 j_1 - c_3 d_5 + c_3 d_2 + j_1 + c_5 j_1 - c_5 d_2 j_1 - d_5 j_1 + d_5 - d_2}{1 + c_5 - d_2 - d_2 c_5}, \tag{33}$$

$$j_4 = \frac{j_1 + j_1 c_5 - e_5 j_1 + c_4 e_5 j_1 - c_4 e_2 j_1 - c_5 e_2 j_1 - c_4 e_5 + e_5 - e_2 + c_4 e_2}{1 + c_5 - e_2 - c_5 e_2}, \tag{34}$$

$$k_1 = \frac{c_3 - c_4 - d_2 + e_2 - c_3 e_2 + c_4 d_2}{1 - c_4 - d_2 + d_4 - c_3 d_4 + c_4 d_2 - d_4 e_2 + c_3 d_4 e_2}, \tag{35}$$

$$k_2 = \frac{e_2 - d_2 + c_3 d_2 - c_4 e_2 - c_3 d_2 e_2 + c_4 d_2 e_2}{1 - c_4 - d_2 + d_4 - c_3 d_4 + c_4 d_2 - d_4 e_2 + c_3 d_4 e_2}, \tag{36}$$

$$k_5 = \frac{e_5 - d_5 + c_3 d_5 - c_4 e_5 - d_2 e_5 + d_5 e_2 - c_3 d_5 e_2 + c_4 d_2 e_5}{1 - c_4 - d_2 + d_4 - c_3 d_4 + c_4 d_2 - d_4 e_2 + c_3 d_4 e_2}, \tag{37}$$

$$l_1 = \frac{c_3 - d_2 + j_1 - c_3 j_1 + c_5 j_1 - c_5 d_2 j_1}{1 - d_2 + d_5 - c_3 d_5 + c_5 j_1 - d_5 j_1 + c_3 d_5 j_1 - c_5 d_2 j_1}, \tag{38}$$

$$l_2 = \frac{c_5 - d_2 + j_1 + c_3 d_2 - c_5 d_2 - c_3 d_2 j_1}{1 - d_2 + d_5 - c_3 d_5 + c_5 j_1 - d_5 j_1 + c_3 d_5 j_1 - c_5 d_2 j_1}, \tag{39}$$

$$l_4 = \frac{d_4\,(c_3-1)(e_2-1)(j_1-1)+(1-d_2)\,[(c_4-1)(e_5-1)(j_1-1)-c_5\,(e_2-1)(j_1-1)-(1+c_5)(e_2-1)]}{(1-e_2)\,[j_1\,(c_5+1)(1-d_2)+d_5\,(1-c_3)(1-j_1)+(1-d_2)(1-j_1)]}, \tag{40}$$

$$n_1 = \frac{c_4 - e_2 + j_1 - c_4 j_1 + c_5 j_1 - c_5 e_2 j_1}{1 - e_2 + e_5 - c_4 e_5 + c_5 j_1 - e_5 j_1 + c_4 e_5 j_1 - c_5 e_2 j_1}, \tag{41}$$

$$n_2 = \frac{c_5 - e_2 + j_1 + c_4 e_2 - c_5 e_2 - c_4 e_2 j_1}{1 - e_2 + e_5 - c_4 e_5 + c_5 j_1 - e_5 j_1 + c_4 e_5 j_1 - c_5 e_2 j_1}, \tag{42}$$

$$n_3 = \frac{(e_2-1)\,[(c_3-1)(d_4-d_5)(j_1-1)+j_1\,(c_5-1)(d_2-1)+(d_2-1)(1+j_1)]+(c_4-1)(d_2-1)(j_1-1)}{(1-d_2)(1-e_2)(1+c_5 j_1)+e_5\,(1-c_4)(1-d_2)(1-j_1)}. \tag{43}$$

## 4 Particular Mass Configurations

Let us now see how the universal dynamical constraints simplify for particular mass configurations. In order to preserve the symmetries of the problem as implied by the mass configurations, we will assume hereafter that the two-body interaction between particles $i$ and $j$ can depend only on their respective masses $m_i$ and $m_j$. We will examine in turn configurations with one distinct mass $(m_1, m_1, m_1, m_1, m_1)$, two distinct masses $(m_1, m_1, m_1, m_1, m_5)$, $(m_1, m_1, m_1, m_4, m_4)$, three distinct masses $(m_1, m_1, m_1, m_4, m_5)$, $(m_1, m_1, m_3, m_3,$

$m_5$) and four distinct masses ($m_1, m_1, m_3, m_4, m_5$). We will make use of the rule

$$\omega_{ij,k} = 0, \qquad k \neq i, j \tag{44}$$

when $m_i = m_j$.

### 4.1 Configuration $(m_1, m_1, m_1, m_1, m_1)$

Here all the $\omega_{ij,k}$ must be zero. By replacing the $\omega_{ij,k}$ by zero in the expressions of the universal dynamical constraints, one see that the universal dynamical constraints are identically satisfied.

### 4.2 Configuration $(m_1, m_1, m_1, m_1, m_5)$

Here $c_3 = c_4 = c_5 = 0$, $d_2 = d_4 = d_5 = 0$, $e_2 = e_3 = e_5 = 0$, $g_1 = g_4 = g_5 = 0$, $h_1 = h_3 = h_5 = 0$ and $k_1 = k_2 = k_5 = 0$. The universal dynamical constraints reduce to

$$f_2 = f_3 = f_4 = j_3 = j_4 = l_1 = l_2 = l_4 = n_1 = n_2 = n_3 = j_1,$$

with one independent parameter $j_1$.

### 4.3 Configuration $(m_1, m_1, m_1, m_4, m_4)$

Here $c_3 = c_4 = c_5 = 0$, $d_2 = d_4 = d_5 = 0$, $g_1 = g_4 = g_5 = 0$, and $n_1 = n_2 = n_3 = 0$. The universal dynamical constraints simplify to

$$\begin{aligned}
&f_2 = f_3 = j_3 = j_1 = h_1 = h_3 = k_1 = k_2 = l_1 = l_2 = e_3 = e_2,\\
&f_4 = j_4 = l_4 = h_5 = k_5 = e_5,
\end{aligned}$$

with the number of independent parameters reducing from 9 to 2: $e_2$ and $e_5$.

### 4.4 Configuration $(m_1, m_1, m_1, m_4, m_5)$

We have in this case $c_3 = c_4 = c_5 = 0$, $d_2 = d_4 = d_5 = 0$, $g_1 = g_4 = g_5 = 0$. The universal dynamical constraints are

$$\begin{aligned}
&e_3 = h_1 = h_3 = k_1 = k_2 = e_2,\\
&f_2 = f_3 = j_3 = l_1 = l_2 = j_1,\\
&h_5 = k_5 = e_5,\\
&f_4 = j_4 = l_4 = \frac{e_2 - e_5 - j_1 + e_5 j_1}{e_2 - 1},\\
&n_1 = n_2 = n_3 = \frac{e_2 - j_1}{e_2 - e_5 + e_5 j_1 - 1},
\end{aligned}$$

with three independent parameters: $e_2$, $e_5$ and $j_1$.

### 4.5 Configuration $(m_1, m_1, m_3, m_3, m_5)$

Here $c_3 = c_4 = c_5 = 0$, $k_1 = k_2 = k_5 = 0$, then $e_2 = d_2$ and $d_5 = e_5$. So the number of independent parameters reduces from 9 to 4: $d_2$, $d_4$, $d_5$ and $j_1$, and the universal dynamical constraints read

$$\begin{aligned}
&e_3 = d_4,\\
&f_2 = j_1,\\
&f_3 = f_4 = j_3 = j_4 = \frac{d_2 - d_5 - j_1 + d_5 j_1}{d_2 - 1},\\
&g_1 = h_1 = d_2,\\
&g_4 = h_3 = d_4,\\
&g_5 = h_5 = d_5,\\
&l_1 = l_2 = n_1 = n_2 = \frac{d_2 - j_1}{d_2 - d_5 + d_5 j_1 - 1},\\
&l_4 = n_3 = \frac{d_2 + d_4 - d_5 - j_1 - d_4 j_1 + d_5 j_1}{d_2 - d_5 + d_5 j_1 - 1}.
\end{aligned}$$

### 4.6 Configuration $(m_1, m_1, m_3, m_4, m_5)$

We have $c_3 = c_4 = c_5 = 0$. Thus the number of independent parameters reduces to six $d_2$, $d_4$, $d_5$, $e_2$, $e_5$ and $j_1$, and the universal dynamical constraints become

$$\begin{aligned}
j_1 &= f_2, \quad g_1 = d_2, \quad h_1 = e_2, \quad g_4 = d_4, \quad g_5 = d_5, \quad h_5 = e_5,\\
j_3 &= f_3 = \frac{d_2 - d_5 - j_1 + d_5 j_1}{d_2 - 1},\\
j_4 &= f_4 = \frac{e_2 - e_5 - j_1 + e_5 j_1}{e_2 - 1},\\
h_3 &= e_3 = \frac{e_2 + d_4 - d_4 e_2 - d_2}{1 - d_2},\\
k_2 &= k_1 = \frac{e_2 - d_2}{1 - d_2 + d_4 - d_4 e_2},\\
k_5 &= \frac{e_5 - d_5 - d_2 e_5 + d_5 e_2}{1 - d_2 + d_4 - d_4 e_2},\\
l_2 &= l_1 = \frac{j_1 - d_2}{1 - d_2 + d_5 - d_5 j_1},\\
l_4 &= \frac{-e_2 + e_5 - d_4 + j_1 + e_2 d_2 + e_2 d_4 - e_5 d_2 - e_5 j_1 - d_2 j_1 + d_4 j_1 - e_2 d_4 j_1 + e_5 d_2 j_1}{(d_2 - d_5 + d_5 j_1 - 1)(e_2 - 1)},\\
n_2 &= n_1 = \frac{-e_2 + j_1}{1 - e_2 + e_5 - e_5 j_1},\\
n_3 &= \frac{-e_2 - d_4 + d_5 + j_1 + e_2 d_2 + e_2 d_4 - e_2 d_5 - d_2 j_1 + d_4 j_1 - d_5 j_1 - e_2 d_4 j_1 + e_2 d_5 j_1}{(d_2 - 1)(e_2 - e_5 + e_5 j_1 - 1)}.
\end{aligned}$$

Two remarks are in order:

- For the particular mass configurations considered above, the whole or a part of the universal dynamical constraints may be obtained using symmetry arguments. In the cases with one and two distinct masses the totality of the universal dynamical constraints results from symmetry arguments. In the cases with three and four distinct masses, a number of universal dynamical constraints cannot result from symmetry arguments; there are respectively 2, 3 and 9 such relations for the mass configurations $(m_1, m_1, m_1, m_4, m_5)$, $(m_1, m_1, m_3, m_3, m_5)$ and $(m_1, m_1, m_3, m_4, m_5)$. We will call such relations as calculated universal dynamical constraints.

– There is another way to obtain the calculated universal dynamical constraints. One begins by using symmetry arguments to determine the independent $b_k$, $a_{ij}$ and $\omega_{ij,k}$. Then by identifying the kinetic energy term with its parameterized decomposition and eliminating the $b_k$ in favor of the $a_{ij}$, one obtains a linear system of equations for the independent $a_{ij}$ with the independent $\omega_{ij,k}$ as parameters, which can be written in a form similar to (16), with a reduced square matrix $\widetilde{\mathbf{D}}$ from which we construct a reduced matrix $\widetilde{\mathbf{M}}$ in much the same way as we construct the matrix $\widetilde{\mathbf{M}}$ from the matrix $\widetilde{\mathbf{D}}$. Imposing to the reduced matrix $\widetilde{\mathbf{M}}$ a rank condition, one arrives to the same calculated universal dynamical constraints obtained above.

## 5 The Five-Body System with Harmonic Interactions

Our investigation shows a numerical evidence of saturability of the optimized lower bound in the case of harmonic forces. In other words, the optimized lower bound and the exact ground state energy of five-body systems interacting via harmonic forces become equal. But a numerical evidence is not, strictly speaking, a rigorous proof of saturability and the situation is quite analogous to the three- and four-body cases, where we have a numerical evidence of saturability but where no rigorous analytical proof of saturability is available. Recently, we succeed to fill partially this lack for particular mass configurations but arbitrary $N$ [4], where $N$ stands for the number of bodies. In the five-body case, this amounts to prove saturability for the mass configurations $(m_1, m_1, m_1, m_1, m_1)$, $(m_1, m_1, m_1, m_1, m_5)$ and $(m_1, m_1, m_1, m_4, m_4)$. For the most general mass configuration, the optimized lower bound has to be calculated numerically and we have no general formula for the exact ground state energy in terms of the masses and the coupling constants. However, there are intermediate situations where the optimized lower bound is still to be calculated numerically, but where we can derive a general formula for the exact ground state energy of the five-body system in terms of the masses and the coupling constants. This occurs for the two mass configurations $(m_1, m_1, m_1, m_4, m_5)$ and $(m_1, m_1, m_3, m_3, m_5)$. Let us consider in turn these two mass configurations and derive an analytical expression for the corresponding ground state energy.

### 5.1 Mass Configurations $(m_1, m_1, m_1, m_4, m_5)$

The Hamiltonian reads

$$\begin{aligned} H = {} & \frac{\boldsymbol{p}_1^2}{2m_1} + \frac{\boldsymbol{p}_2^2}{2m_1} + \frac{\boldsymbol{p}_3^2}{2m_1} + \frac{\boldsymbol{p}_4^2}{2m_4} + \frac{\boldsymbol{p}_5^2}{2m_5} \\ & + k_{12}(\boldsymbol{r}_{12}^2 + \boldsymbol{r}_{13}^2 + \boldsymbol{r}_{23}^2) + k_{14}(\boldsymbol{r}_{14}^2 + \boldsymbol{r}_{24}^2 + \boldsymbol{r}_{34}^2) + k_{15}(\boldsymbol{r}_{15}^2 + \boldsymbol{r}_{25}^2 + \boldsymbol{r}_{35}^2) + k_{45}(\boldsymbol{r}_{45}^2). \end{aligned} \tag{45}$$

An appropriate choice of the Jacobi coordinates is the following:

$$\boldsymbol{\rho} = \boldsymbol{r}_2 - \boldsymbol{r}_1, \tag{46}$$

$$\boldsymbol{\lambda} = \boldsymbol{r}_3 - \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2), \tag{47}$$

$$\boldsymbol{\sigma} = \boldsymbol{r}_4 - \frac{1}{3}(\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3), \tag{48}$$

$$\boldsymbol{\zeta} = \boldsymbol{r}_5 - \frac{m_1(\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3) + m_4 \boldsymbol{r}_4}{3m_1 + m_4}. \tag{49}$$

Using an obvious notation, the corresponding conjugate momenta are respectively

$$\boldsymbol{p}_\rho = \frac{1}{2}(\boldsymbol{p}_2 - \boldsymbol{p}_1), \tag{50}$$

$$\boldsymbol{p}_\lambda = \frac{2}{3}\left(\boldsymbol{p}_3 - \frac{1}{2}(\boldsymbol{p}_1 + \boldsymbol{p}_2)\right), \tag{51}$$

$$\boldsymbol{p}_\sigma = \frac{3m_1 m_4}{3m_1 + m_4}\left(\frac{1}{m_4}\boldsymbol{p}_4 - \frac{1}{3m_1}(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3)\right), \tag{52}$$

$$\boldsymbol{p}_\zeta = \frac{3m_1 + m_4}{3m_1 + m_4 + m_5}\boldsymbol{p}_5 - \frac{m_5}{3m_1 + m_4 + m_5}(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 + \boldsymbol{p}_4). \tag{53}$$

The expression of the centre of mass coordinate **R**,

$$\mathbf{R} = \frac{1}{3m_1 + m_4 + m_5}\left(m_1(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3) + m_4\boldsymbol{p}_4 + m_5\boldsymbol{p}_5\right), \tag{54}$$

together with the relations (46–49) can be inverted to give the individual coordinates $\boldsymbol{r}_i$, $i = 1, ..., 5$, in terms of the Jacobi and the centre of mass coordinates, with the result

$$\begin{aligned}
\boldsymbol{r}_1 &= -\tfrac{1}{2}\boldsymbol{\rho} - \tfrac{1}{3}\boldsymbol{\lambda} - \frac{m_4}{3m_1 + m_4}\boldsymbol{\sigma} - \frac{m_5}{3m_1 + m_4 + m_5}\boldsymbol{\zeta} + \mathbf{R}, &(55)\\
\boldsymbol{r}_2 &= \tfrac{1}{2}\boldsymbol{\rho} - \tfrac{1}{3}\boldsymbol{\lambda} - \frac{m_4}{3m_1 + m_4}\boldsymbol{\sigma} - \frac{m_5}{3m_1 + m_4 + m_5}\boldsymbol{\zeta} + \mathbf{R}, &(56)\\
\boldsymbol{r}_3 &= -\tfrac{2}{3}\boldsymbol{\lambda} - \frac{m_4}{3m_1 + m_4}\boldsymbol{\sigma} - \frac{m_5}{3m_1 + m_4 + m_5}\boldsymbol{\zeta} + \mathbf{R}, &(57)\\
\boldsymbol{r}_4 &= - \frac{m_4}{3m_1 + m_4}\boldsymbol{\sigma} - \frac{m_5}{3m_1 + m_4 + m_5}\boldsymbol{\zeta} + \mathbf{R}, &(58)\\
\boldsymbol{r}_5 &= \frac{3m_1 + m_4}{3m_1 + m_4}\boldsymbol{\sigma} - \frac{m_5}{3m_1 + m_4 + m_5}\boldsymbol{\zeta} + \mathbf{R}. &(59)
\end{aligned}$$

In the same manner the relations (50–53) together with the total momentum **P** relation,

$$\mathbf{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 + \boldsymbol{p}_4 + \boldsymbol{p}_5, \tag{60}$$

can be inverted to express the individual momenta in terms of the Jacobi and total momenta. One obtains

$$\begin{aligned}
\boldsymbol{p}_1 &= -\boldsymbol{p}_\rho - \tfrac{1}{2}\boldsymbol{p}_\lambda - \tfrac{1}{3}\boldsymbol{p}_\sigma + \tfrac{m_1}{3m_1+m_4}\boldsymbol{p}_\zeta + \tfrac{m_1}{3m_1+m_4+m_5}\mathbf{R}, &(61)\\
\boldsymbol{p}_2 &= \boldsymbol{p}_\rho - \tfrac{1}{2}\boldsymbol{p}_\lambda - \tfrac{1}{3}\boldsymbol{p}_\sigma - \tfrac{m_1}{3m_1+m_4}\boldsymbol{p}_\zeta + \tfrac{m_1}{3m_1+m_4+m_5}\mathbf{R}, &(62)\\
\boldsymbol{p}_3 &= \boldsymbol{p}_\lambda - \tfrac{1}{3}\boldsymbol{p}_\sigma - \tfrac{m_1}{3m_1+m_4}\boldsymbol{p}_\zeta + \tfrac{m_1}{3m_1+m_4+m_5}\mathbf{R}, &(63)\\
\boldsymbol{p}_4 &= \boldsymbol{p}_\sigma - \tfrac{m_4}{3m_3+m_4}\boldsymbol{p}_\zeta + \tfrac{m_4}{3m_1+m_4+m_5}\mathbf{R}, &(64)\\
\boldsymbol{p}_5 &= \boldsymbol{p}_\zeta + \tfrac{m_5}{3m_1+m_4+m_5}\mathbf{R}. &(65)
\end{aligned}$$

Then the potential energy $V$ can be expressed in terms of the Jacobi coordinates. One ends with

$$\begin{aligned}
V = &\left(\frac{3}{2}k_{12} + \frac{1}{2}k_{14} + \frac{1}{2}k_{15}\right)\boldsymbol{\rho}^2 + \left(2k_{12} + \frac{2}{3}k_{14} + \frac{2}{3}k_{15}\right)\boldsymbol{\lambda}^2\\
&+ \left(3k_{14} + \frac{3m_1^2}{(3m_1 + m_4)^2}k_{15} + \frac{9m_1^2}{(3m_1 + m_4)^2}k_{45}\right)\boldsymbol{\sigma}^2\\
&+ (3k_{15} + k_{45})\,\boldsymbol{\zeta}^2 + \frac{6\,(k_{15}m_4 - k_{45}m_1)}{3m_1 + m_4}\,\boldsymbol{\sigma}.\boldsymbol{\zeta}.
\end{aligned} \tag{66}$$

Also the kinetic energy $T$ can be expressed in terms of the Jacobi momenta and the total momentum. One gets

$$T = \frac{\mathbf{P}^2}{2(3m_1 + m_4 + m_5)} + \frac{1}{m_1}\boldsymbol{p}_\rho^2 + \frac{3}{4m_1}\boldsymbol{p}_\lambda^2 + \frac{3m_1 + m_4}{6m_1m_4}\boldsymbol{p}_\sigma^2 + \frac{3m_1 + m_4 + m_5}{2(3m_1 + m_4)m_5}\boldsymbol{p}_\zeta^2. \tag{67}$$

Subtracting the centre of mass kinetic energy from the Hamiltonian $H = T + V$, one obtains the Hamiltonian of the relative motion $H_R$

$$\begin{aligned}
H_R = &\frac{1}{m_1}\boldsymbol{p}_\rho^2 + \left(\frac{3}{2}k_{12} + \frac{1}{2}k_{14} + \frac{1}{2}k_{15}\right)\boldsymbol{\rho}^2 + \frac{3}{4m_1}\boldsymbol{p}_\lambda^2 + \left(2k_{12} + \frac{2}{3}k_{14} + \frac{2}{3}k_{15}\right)\boldsymbol{\lambda}^2\\
&+ \frac{3m_1 + m_4}{6m_1m_4}\boldsymbol{p}_\sigma^2 + \frac{3m_1 + m_4 + m_5}{2(3m_1 + m_4)m_5}\boldsymbol{p}_\zeta^2\\
&+ \left(3k_{14} + \frac{3m_1^2}{(3m_1 + m_4)^2}k_{15} + \frac{9m_1^2}{(3m_1 + m_4)^2}k_{45}\right)\boldsymbol{\sigma}^2 + (3k_{15} + k_{45})\,\boldsymbol{\zeta}^2\\
&+ \frac{6\,(k_{15}m_4 - k_{45}m_1)}{3m_1 + m_4}\,\boldsymbol{\sigma}.\boldsymbol{\zeta}.
\end{aligned} \tag{68}$$

We recognize a system of 4 harmonic oscillators with a coupling between the $\sigma$- and the $\zeta$-oscillators. By performing a dilatation on the Jacobi coordinate $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} \to \boldsymbol{\sigma}' = \left(\frac{3m_1 m_4 (3m_1 + m_4 + m_5)}{m_5}\right)^{1/2} \frac{1}{3m_1 + m_4} \boldsymbol{\sigma}.$$

This means that the corresponding Jacobi momentum $\boldsymbol{p}_\sigma$ undergoes the inverse dilatation

$$\boldsymbol{p}_\sigma \to \boldsymbol{p}_{\sigma'} = (3m_1 + m_4) \left(\frac{m_5}{3m_1 m_4 (3m_1 + m_4 + m_5)}\right)^{1/2} \boldsymbol{p}_\sigma .$$

$H_R$ then may be cast in the form

$$\begin{aligned} H_R = {} & \frac{1}{m_1}\boldsymbol{p}_\rho^2 + \left(\frac{3}{2}k_{12} + \frac{1}{2}k_{14} + \frac{1}{2}k_{15}\right)\boldsymbol{\rho}^2 + \frac{3}{4m_1}\boldsymbol{p}_\lambda^2 + \left(2k_{12} + \frac{2}{3}k_{14} + \frac{2}{3}k_{15}\right)\boldsymbol{\lambda}^2 \\ & + \frac{3m_1+m_4+m_5}{2(3m_1+m_4)m_5}\left(\boldsymbol{p}_{\sigma'}^2 + \boldsymbol{p}_\zeta^2\right) + \frac{\left[(3m_1+m_4)^2 k_{14} + m_4^2 k_{15} + 3m_1^2 k_{45}\right] m_5}{m_1 m_4 (3m_1+m_4+m_5)} \boldsymbol{\sigma}'^2 \\ & + (3k_{15}+k_{45})\boldsymbol{\zeta}^2 + 6\,(k_{15}m_4 - k_{45}m_1)\left(\frac{m_5}{3m_1 m_4 (3m_1+m_4+m_5)}\right)^{1/2} \boldsymbol{\sigma}'.\boldsymbol{\zeta}. \end{aligned} \tag{69}$$

Let us now perform a rotation on the Jacobi coordinates $\boldsymbol{\sigma}'$ and $\boldsymbol{\zeta}$

$$\begin{aligned} \boldsymbol{\sigma}' &\to \widetilde{\boldsymbol{\sigma}} = \cos(\theta)\boldsymbol{\sigma}' + \sin(\theta)\boldsymbol{\zeta}, \\ \boldsymbol{\zeta} &\to \widetilde{\boldsymbol{\zeta}} = -\sin(\theta)\boldsymbol{\sigma}' + \cos(\theta)\boldsymbol{\zeta}. \end{aligned}$$

It is clear that $\boldsymbol{p}_{\sigma'}^2 + \boldsymbol{p}_\zeta^2$ remains invariant and $H_R$ can be expressed in terms of the new Jacobi coordinates $\widetilde{\boldsymbol{\sigma}}$ and $\widetilde{\boldsymbol{\zeta}}$ as

$$\begin{aligned} H_R = {} & \frac{1}{m_1}\boldsymbol{p}_\rho^2 + \left(\frac{3}{2}k_{12} + \frac{1}{2}k_{14} + \frac{1}{2}k_{15}\right)\boldsymbol{\rho}^2 + \frac{3}{4m_1}\boldsymbol{p}_\lambda^2 + \left(2k_{12} + \frac{2}{3}k_{14} + \frac{2}{3}k_{15}\right)\boldsymbol{\lambda}^2 \\ & + \frac{3m_1 + m_4 + m_5}{2(3m_1 + m_4)m_5}\left(\boldsymbol{p}_{\widetilde{\sigma}}^2 + \boldsymbol{p}_{\widetilde{\zeta}}^2\right) \\ & + \left(A\cos^2(\theta) + B\sin^2(\theta) + 2C\cos(\theta)\sin(\theta)\right)\widetilde{\boldsymbol{\sigma}}^2 \\ & + \left(A\cos^2(\theta) + B\sin^2(\theta) - 2C\cos(\theta)\sin(\theta)\right)\widetilde{\boldsymbol{\zeta}}^2 \\ & + \left(-2A\cos(\theta)\sin(\theta) + 2B\cos(\theta)\sin(\theta) + 2C\left(\cos^2(\theta) - \sin^2(\theta)\right)\right)\ \widetilde{\boldsymbol{\sigma}}.\widetilde{\boldsymbol{\zeta}}, \end{aligned} \tag{70}$$

where we have introduced the notation

$$\begin{aligned} A &:= \left((3m_1 + m_4)^2 k_{14} + m_4^2 k_{15} + 3m_1^2 k_{45}\right)\frac{m_5}{m_1 m_4 (3m_1 + m_4 + m_5)}, \\ B &:= 3k_{15} + k_{45}, \\ C &:= 3\left(\frac{m_5}{3m_1 m_4 (3m_1 + m_4 + m_5)}\right)^{1/2}(m_4 k_{15} - m_1 k_{45}). \end{aligned}$$

Let us choose $\theta$ so that the factor in front of $\widetilde{\boldsymbol{\sigma}}.\widetilde{\boldsymbol{\zeta}}$ vanishes. This corresponds to the value $\theta_0$ of $\theta$ such that

$$\tan(2\theta_0) = \frac{2C}{A - B}. \tag{71}$$

With this choice of $\theta$, $H_R$ is the sum of four independent uncoupled three-dimensional harmonic oscillators, and the ground state energy is then the sum of the ground state energies of the four oscillators:

$$\begin{aligned} E = 3\Bigg[ & \left(\frac{1}{m_1}\left(\frac{3}{2}k_{12} + \frac{1}{2}k_{14} + \frac{1}{2}k_{15}\right)\right)^{1/2} + \left(\frac{3}{4m_1}\left(2k_{12} + \frac{2}{3}k_{14} + \frac{2}{3}k_{15}\right)\right)^{1/2} \\ & + \left(\frac{3m_1 + m_4 + m_5}{2m_5(3m_1 + m_4)}\right)^{1/2}\Big(\left(A\cos^2(\theta_0) + B\sin^2(\theta_0) + 2C\cos(\theta_0)\sin(\theta_0)\right)^{1/2} \\ & + \left(A\sin^2(\theta_0) + B\cos^2(\theta_0) - 2C\cos(\theta_0)\sin(\theta_0)\right)^{1/2}\Big)\Bigg]. \end{aligned} \tag{72}$$

Replacing $\theta_0$ by its value (71), one gets after some manipulations

$$\begin{aligned}&\left(A\cos^2(\theta_0)+B\sin^2(\theta_0)+2C\cos(\theta_0)\sin(\theta_0)\right)^{1/2}\\&\quad+\left(A\sin^2(\theta_0)+B\cos^2(\theta_0)-2C\cos(\theta_0)\sin(\theta_0)\right)^{1/2}=\sqrt{A+B+2\sqrt{AB-C^2}}.\end{aligned}$$

But

$$\begin{aligned}A+B&=\frac{m_5\,(3m_1+m_4)^2}{m_1m_4\,(3m_1+m_4+m_5)}k_{14}+\frac{(3m_1+m_4)\,(3m_1+m_5)}{m_1\,(3m_1+m_4+m_5)}k_{15}+\frac{(3m_1+m_4)\,(m_4+m_5)}{m_4\,(3m_1+m_4+m_5)}k_{45},\\AB-C^2&=\frac{m_5\,(3m_1+m_4)^2}{m_1m_4\,(3m_1+m_4+m_5)}\,(3k_{14}k_{15}+k_{14}k_{45}+k_{15}k_{45})\,.\end{aligned}$$

Therefore,

$$\begin{aligned}E=3&\Bigg[2\sqrt{\frac{1}{2m_1}(3k_{12}+k_{14}+k_{15})}\\&+\sqrt{\frac{3m_1+m_4}{2m_1m_4}k_{14}+\frac{3m_1+m_5}{2m_1m_5}k_{15}+\frac{m_4+m_5}{2m_4m_5}k_{45}+\sqrt{\frac{3m_1+m_4+m_5}{m_1m_4m_5}\,(3k_{14}k_{15}+k_{14}k_{45}+k_{15}k_{45})}}\,\Bigg]\end{aligned}\tag{73}$$

### 5.2 Mass Configurations $(m_1, m_1, m_3, m_3, m_5)$

Here a suitable choice of Jacobi coordinates is

$$\boldsymbol{\rho}=\boldsymbol{r}_2-\boldsymbol{r}_1,\tag{74}$$

$$\boldsymbol{\lambda}=\boldsymbol{r}_4-\boldsymbol{r}_3,\tag{75}$$

$$\boldsymbol{\sigma}=\frac{1}{2}\,(\boldsymbol{r}_3+\boldsymbol{r}_4)-\frac{1}{2}\,(\boldsymbol{r}_1+\boldsymbol{r}_2)\,,\tag{76}$$

$$\boldsymbol{\zeta}=\boldsymbol{r}_5-\frac{m_1\,(\boldsymbol{r}_1+\boldsymbol{r}_2)+m_3\,(\boldsymbol{r}_3+\boldsymbol{r}_4)}{2m_1+2m_3}.\tag{77}$$

The corresponding momenta are

$$\boldsymbol{p}_\rho=\frac{1}{2}\,(\boldsymbol{p}_2-\boldsymbol{p}_1)\,,\tag{78}$$

$$\boldsymbol{p}_\lambda=\frac{1}{2}\,(\boldsymbol{p}_4-\boldsymbol{p}_3)\,,\tag{79}$$

$$\boldsymbol{p}_\sigma=\frac{m_1}{m_1+m_3}\,(\boldsymbol{p}_3+\boldsymbol{p}_4)-\frac{m_3}{m_1+m_3}\,(\boldsymbol{p}_1+\boldsymbol{p}_2)\,,\tag{80}$$

$$\boldsymbol{p}_\zeta=\frac{2(m_1+m_3)}{2m_1+2m_3+m_5}\boldsymbol{p}_5-\frac{m_5}{2m_1+2m_3+m_5}\,(\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3+\boldsymbol{p}_4)\,.\tag{81}$$

The centre of mass coordinate **R** and the total momentum **P** are given respectively by

$$\mathbf{R}=\frac{m_1(\boldsymbol{p}_1+\boldsymbol{p}_2)+m_3(\boldsymbol{p}_3+\boldsymbol{p}_4)+m_5\boldsymbol{p}_5}{2m_1+2m_3+m_5},\tag{82}$$

$$\mathbf{P}=\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3+\boldsymbol{p}_4+\boldsymbol{p}_5.\tag{83}$$

The relations (74–77) and (82) can be inverted to express the individual coordinates in terms of the Jacobi and the centre of mass coordinates. One obtains

$$\boldsymbol{r}_1 = -\tfrac{1}{2}\boldsymbol{\rho} - \tfrac{m_3}{m_1+m_3}\boldsymbol{\sigma} - \frac{m_5}{2m_1+2m_3+m_5}\boldsymbol{\zeta} + \mathbf{R}, \tag{84}$$

$$\boldsymbol{r}_2 = \tfrac{1}{2}\boldsymbol{\rho} - \tfrac{m_3}{m_1+m_3}\boldsymbol{\sigma} - \frac{m_5}{2m_1+2m_3+m_5}\boldsymbol{\zeta} + \mathbf{R}, \tag{85}$$

$$\boldsymbol{r}_3 = -\tfrac{1}{2}\boldsymbol{\rho} - \tfrac{m_1}{m_1+m_3}\boldsymbol{\sigma} - \frac{m_5}{2m_1+2m_3+m_5}\boldsymbol{\zeta} + \mathbf{R}, \tag{86}$$

$$\boldsymbol{r}_4 = \tfrac{1}{2}\boldsymbol{\rho} - \tfrac{m_1}{m_1+m_3}\boldsymbol{\sigma} - \frac{m_5}{2m_1+2m_3+m_5}\boldsymbol{\zeta} + \mathbf{R}, \tag{87}$$

$$\boldsymbol{r}_5 = \frac{2m_1+2m_3}{2m_1+2m_3+m_5}\boldsymbol{\zeta} + \mathbf{R}. \tag{88}$$

In the same manner, the relations (78–81) and (83) can be inverted to express the individual momenta in terms of the Jacobi momenta and the total momentum, with the result

$$\boldsymbol{p}_1 = -\boldsymbol{p}_\rho - \tfrac{1}{2}\boldsymbol{p}_\sigma - \tfrac{m_1}{2m_1+2m_3}\,\boldsymbol{p}_\zeta + \frac{m_1}{2m_1+2m_3+m_5}\mathbf{R}, \tag{89}$$

$$\boldsymbol{p}_2 = \boldsymbol{p}_\rho - \tfrac{1}{2}\boldsymbol{p}_\sigma - \tfrac{m_1}{2m_1+2m_3}\,\boldsymbol{p}_\zeta + \frac{m_1}{2m_1+2m_3+m_5}\mathbf{R}, \tag{90}$$

$$\boldsymbol{p}_3 = -\boldsymbol{p}_\lambda - \tfrac{1}{2}\boldsymbol{p}_\sigma + \tfrac{m_1}{2m_1+2m_3}\,\boldsymbol{p}_\zeta + \frac{m_3}{2m_1+2m_3+m_5}\mathbf{R}, \tag{91}$$

$$\boldsymbol{p}_4 = \boldsymbol{p}_\lambda - \tfrac{1}{2}\boldsymbol{p}_\sigma + \tfrac{m_3}{2m_1+2m_3}\,\boldsymbol{p}_\zeta + \frac{m_3}{2m_1+2m_3+m_5}\mathbf{R}, \tag{92}$$

$$\boldsymbol{p}_5 = \boldsymbol{p}_\zeta + \frac{m_5}{2m_1+2m_3+m_5}\mathbf{R}. \tag{93}$$

Then the potential energy $V$,

$$V = k_{12}\boldsymbol{r}_{12}^2 + k_{34}\boldsymbol{r}_{34}^2 + k_{13}\left(\boldsymbol{r}_{13}^2 + \boldsymbol{r}_{14}^2 + \boldsymbol{r}_{23}^2 + \boldsymbol{r}_{24}^2\right) + k_{15}\left(\boldsymbol{r}_{15}^2 + \boldsymbol{r}_{25}^2\right) + k_{35}\left(\boldsymbol{r}_{35}^2 + \boldsymbol{r}_{45}^2\right), \tag{94}$$

can be expressed in terms of the Jacobi coordinates. One ends with

$$\begin{aligned} V = {} & \left(k_{12} + k_{13} + \frac{1}{2}k_{15}\right)\boldsymbol{\rho}^2 + \left(k_{34} + k_{13} + \frac{1}{2}k_{35}\right)\boldsymbol{\lambda}^2 \\ & + \left(4k_{13} + \frac{2m_3^2}{(m_1+m_3)^2}k_{15} + \frac{2m_1^2}{(m_1+m_3)^2}k_{35}\right)\boldsymbol{\sigma}^2 \\ & + 2\left(k_{15} + k_{35}\right)\boldsymbol{\zeta}^2 + 4\frac{m_3k_{15} - m_1k_{35}}{m_1+m_3}\,\boldsymbol{\sigma}.\boldsymbol{\zeta}. \end{aligned} \tag{95}$$

The kinetic energy $T$,

$$T = \frac{\boldsymbol{p}_1^2}{2m_1} + \frac{\boldsymbol{p}_2^2}{2m_1} + \frac{\boldsymbol{p}_3^2}{2m_3} + \frac{\boldsymbol{p}_4^2}{2m_3} + \frac{\boldsymbol{p}_5^2}{2m_5}, \tag{96}$$

can be expressed in terms of the Jacobi momenta and the total momentum with the result

$$T = \frac{\mathbf{P}^2}{2(2m_1+2m_3+m_5)} + \frac{1}{m_1}\boldsymbol{p}_\rho^2 + \frac{1}{m_3}\boldsymbol{p}_\lambda^2 + \frac{m_1+m_3}{4m_1m_3}\boldsymbol{p}_\sigma^2 + \frac{2m_1+2m_3+m_5}{4m_5(m_1+m_3)}\boldsymbol{p}_\zeta^2. \tag{97}$$

Subtracting the kinetic energy of the centre of mass, $\mathbf{P}^2/[2(2m_1+2m_3+m_5)]$, from the Hamiltonian $H$, one ends with the Hamiltonian of relative motion $H_R$

$$\begin{aligned} H_R = {} & \frac{1}{m_1}\boldsymbol{p}_\rho^2 + \left(k_{12}+k_{13}+\frac{1}{2}k_{15}\right)\boldsymbol{\rho}^2 + \frac{1}{m_3}\boldsymbol{p}_\lambda^2 + \left(k_{34}+k_{13}+\frac{1}{2}k_{35}\right)\boldsymbol{\lambda}^2 \\ & + \frac{m_1+m_3}{4m_1m_3}\boldsymbol{p}_\sigma^2 + \frac{2m_1+2m_3+m_5}{4m_5(m_1+m_3)}\boldsymbol{p}_\zeta^2 \\ & + \left(4k_{13} + \frac{2m_3^2}{(m_1+m_3)^2}k_{15} + \frac{2m_1^2}{(m_1+m_3)^2}k_{35}\right)\boldsymbol{\sigma}^2 + 2\left(k_{15}+k_{35}\right)\boldsymbol{\zeta}^2 \\ & + 4\frac{m_3k_{15}-m_1k_{35}}{m_1+m_3}\,\boldsymbol{\sigma}.\boldsymbol{\zeta}. \end{aligned} \tag{98}$$

$H_R$ shows as a system of 4 harmonic oscillators, two decoupled: the $\rho$- and $\lambda$-oscillators, and two coupled: the $\sigma$- and $\zeta$-oscillators. To decouple them one proceeds as for the previous configuration. One performs a dilatation followed by a rotation. An appropriate dilatation on the Jacobi coordinate $\boldsymbol{\sigma}$ is

$$\boldsymbol{\sigma} \to \boldsymbol{\sigma}' = \left(\frac{m_1m_3(2m_1+2m_3+m_5)}{m_5}\right)^{1/2}\frac{1}{m_1+m_3}\boldsymbol{\sigma}.$$

$H_R$ then reduces to

$$\begin{aligned} H_R = {} & \frac{1}{m_1}\boldsymbol{p}_\rho^2 + \left(k_{12}+k_{13}+\frac{1}{2}k_{15}\right)\boldsymbol{\rho}^2 + \frac{1}{m_3}\boldsymbol{p}_\lambda^2 + \left(k_{34}+k_{13}+\frac{1}{2}k_{35}\right)\boldsymbol{\lambda}^2 \\ & + \frac{2m_1+2m_3+m_5}{4m_5(m_1+m_3)}\left(\boldsymbol{p}_{\sigma'}^2+\boldsymbol{p}_\zeta^2\right) + 2m_5\frac{2(m_1+m_3)^2k_{13}+m_3^2k_{15}+m_1^2k_{35}}{m_1m_3(2m_1+2m_3+m_5)}\boldsymbol{\sigma}'^2 \\ & + 2(k_{15}+k_{35})\boldsymbol{\zeta}^2 + 4\left(m_3k_{15}-m_1k_{45}\right)\left(\frac{m_5}{m_1m_3(2m_1+2m_3+m_5)}\right)^{1/2}\boldsymbol{\sigma}'.\boldsymbol{\zeta}. \end{aligned} \tag{99}$$

Now, let us make use of rotated Jacobi coordinates $\widetilde{\boldsymbol{\sigma}}$ and $\widetilde{\boldsymbol{\zeta}}$

$$\begin{aligned} \widetilde{\boldsymbol{\sigma}} &= \cos(\theta')\boldsymbol{\sigma}' + \sin(\theta')\boldsymbol{\zeta}, \\ \widetilde{\boldsymbol{\zeta}} &= -\sin(\theta')\boldsymbol{\sigma}' + \cos(\theta')\boldsymbol{\zeta}. \end{aligned}$$

$H_R$ can then be cast in the form

$$\begin{aligned} H_R = {} & \frac{1}{m_1}\boldsymbol{p}_\rho^2 + \left(k_{12}+k_{13}+\frac{1}{2}k_{15}\right)\boldsymbol{\rho}^2 + \frac{1}{m_3}\boldsymbol{p}_\lambda^2 + \left(k_{34}+k_{13}+\frac{1}{2}k_{35}\right)\boldsymbol{\lambda}^2 \\ & + \frac{2m_1+2m_3+m_5}{4m_5(m_1+m_3)}\left(\boldsymbol{p}_{\widetilde{\sigma}}^2+\boldsymbol{p}_{\widetilde{\zeta}}^2\right) \\ & + \left(A'\cos^2(\theta') + B'\sin^2(\theta') + 2C'\cos(\theta')\sin(\theta')\right)\widetilde{\boldsymbol{\sigma}}^2 \\ & + \left(A'\sin^2(\theta') + B'\cos^2(\theta') - 2C'\cos(\theta')\sin(\theta')\right)\widetilde{\boldsymbol{\zeta}}^2 \\ & + \left(-2A'\sin(\theta')\cos(\theta') + 2B'\sin(\theta')\cos(\theta') + C'[\cos^2(\theta') - \sin^2(\theta')]\right)\widetilde{\boldsymbol{\sigma}}.\widetilde{\boldsymbol{\zeta}}, \end{aligned} \tag{100}$$

with $A'$, $B'$ and $C'$ defined by

$$\begin{aligned} A' &:= \left(2(m_1+m_4)^2k_{13} + m_3^2k_{15} + m_1^2k_{35}\right)\frac{2m_5}{m_1m_3(2m_1+2m_3+m_5)}, \\ B' &:= 2\left(k_{15}+k_{35}\right), \\ C' &:= 2\left(\frac{m_5}{m_1m_3(2m_1+2m_3+m_5)}\right)^{1/2}\left(m_3k_{15}-m_1k_{35}\right). \end{aligned}$$

The rest of the procedure is almost identical of the one followed in the case of the previous mass configuration. We choose a value of $\theta'$, $\theta'_0$ such that to decouple the $\widetilde{\sigma}$- and the $\widetilde{\zeta}$-oscillators:

$$\tan(2\theta_0) = \frac{2C'}{A' - B'}. \tag{101}$$

The ground state energy is then given by:

$$\begin{aligned} E = 3\Bigg[ &\left(\frac{1}{m_1}\,(2k_{12} + 2k_{13} + k_{15})\right)^{1/2} + \left(\frac{1}{m_3}\,(2k_{34} + 2k_{13} + k_{35})\right)^{1/2} \\ &+ \left(\frac{2m_1 + 2m_3 + m_5}{4m_5(m_1 + m_3)}\right)^{1/2} \sqrt{A' + B' + 2\sqrt{A'B' - C'^2}}\ \Bigg]. \end{aligned} \tag{102}$$

Here

$$\begin{aligned} A' + B' &= 2\frac{2m_5\,(m_1 + m_3)^2}{m_1 m_3\,(2m_1 + 2m_3 + m_5)}k_{13} + \frac{(2m_1 + m_5)\,(m_1 + m_3)}{m_1\,(2m_1 + 2m_3 + m_5)}k_{15} + \frac{(2m_3 + m_5)\,(m_1 + m_3)}{m_3\,(2m_1 + 2m_3 + m_5)}k_{35}, \\ A'B' - C'^2 &= \frac{4m_5\,(m_1 + m_3)^2}{m_1 m_3\,(2m_1 + 2m_3 + m_5)}\,(2k_{13}k_{15} + 2k_{13}k_{35} + k_{15}k_{35}) \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{2m_1 + 2m_3 + m_5}{4m_5(m_1 + m_3)}\right)^{1/2} \sqrt{A' + B' + 2\sqrt{A'B' - C'^2}} \\ &= \frac{2m_1+2m_3}{2m_1m_3}k_{13} + \frac{2m_1+m_5}{2m_1m_5}k_{15} + \frac{2m_3+m_5}{2m_3m_5}k_{35} + \sqrt{\frac{2m_1+2m_3+m_5}{m_1m_3m_5}\,(2k_{13}k_{15}+2k_{13}k_{35}+k_{15}k_{35})}\ \Bigg]^{1/2}. \end{aligned}$$

Finally the ground state energy of the system can be put in the form

$$\begin{aligned} E = 3\Bigg[ &\sqrt{\frac{1}{2m_1}\,(2k_{12} + 2k_{13} + k_{15})} + \sqrt{\frac{1}{2m_3}\,(2k_{13} + 2k_{34} + k_{35})} \\ &+ \sqrt{\frac{2m_1+2m_3}{2m_1m_3}k_{13} + \frac{2m_1+m_5}{2m_1m_5}k_{15} + \frac{2m_3+m_5}{2m_3m_5}k_{35} + \sqrt{\frac{2m_1+2m_3+m_5}{m_1m_3m_5}\,(2k_{13}k_{15}+2k_{13}k_{35}+k_{15}k_{35})}}\ \Bigg]. \end{aligned} \tag{103}$$

## 6 Numerical Results

Beside the harmonic forces, where the optimized lower bound is saturated, we have also considered five-body systems with other types of potentials, especially power-law potentials, that is five-body Hamiltonians of the form

$$H = \sum_{i=1}^{5} \frac{\boldsymbol{p}_i^2}{2m_i} + \sum_{i<j=1}^{5} k_{ij}\boldsymbol{r}_{ij}^{\nu_{ij}}, \tag{104}$$

where the coupling constant $k_{ij}$ and the power $\nu_{ij}$ must be of the same sign, i.e., $k_{ij}\nu_{ij} > 0$. The Tables 1, 2, 3 and 4 illustrate our numerical results for mass configurations $(m_1, m_1, m_1, m_4, m_5)$ and $(m_1, m_1, m_3, m_3, m_5)$. Four representative powers $\nu_{ij}$ are considered $\nu = 2$, $\nu = 1$, $\nu = 0.1$ and $\nu = -1$. We have also included here, the results of a variational calculation using a trial wave function of Gaussian form [5]

**Table 1** Lower and upper bounds for the ground state energy of five-body systems for the harmonic potential $V^{(ij)} = r_{ij}^2$

| $m_1, ..., m_5$ | $E_{\text{olb}}$ | $E_{\text{var}}$ |
|---|---|---|
| 1,1,1,1,1 | 18.9737 | 18.9737 |
| 1,1,1,1,5 | 17.0763 | 17.0763 |
| 1,1,1,2,3 | 16.3598 | 16.3598 |
| 2,2,2,1,3 | 14.0893 | 14.0893 |
| 3,3,3,1,2 | 13.1232 | 13.1232 |
| 5,5,5,5,1 | 10.7114 | 10.7114 |

**Table 2** Lower and upper bounds for the ground state energy of five-body systems for the linear potential $V^{(ij)} = r_{ij}$

| $m_1, ..., m_5$ | $E_{\text{olb}}$ | $E_{\text{var}}$ |
|---|---|---|
| 1,1,1,1,1 | 17.2273 | 17.2765 |
| 1,1,1,1,5 | 16.0304 | 16.0761 |
| 1,1,1,2,3 | 15.5828 | 15.6272 |
| 2,2,2,1,3 | 14.1081 | 14.1483 |
| 3,3,3,1,2 | 13.4442 | 13.4825 |
| 5,5,5,5,1 | 11.6940 | 11.7274 |

**Table 3** Lower and upper bounds for the ground state energy of five-body systems for the Martin potential $V^{(ij)} = r_{ij}^{0.1}$

| $m_1, ..., m_5$ | $E_{\text{olb}}$ | $E_{\text{var}}$ |
|---|---|---|
| 1,1,1,1,1 | 11.8297 | 11.8525 |
| 1,1,1,1,5 | 11.7053 | 11.7279 |
| 1,1,1,2,3 | 11.6584 | 11.6808 |
| 2,2,2,1,3 | 11.4943 | 11.5164 |
| 3,3,3,1,2 | 11.4136 | 11.4356 |
| 5,5,5,5,1 | 11.1726 | 11.2013 |

**Table 4** Lower and upper bounds for the ground state energy of five-body systems for the Coulomb potential $V^{(ij)} = -r_{ij}^{-1}$

| $m_1, ..., m_5$ | $E_{\text{olb}}$ | $E_{\text{var}}$ |
|---|---|---|
| 1,1,1,1,1 | −06.2500 | −05.3052 |
| 1,1,1,1,5 | −07.8752 | −06.6847 |
| 1,1,1,2,3 | −08.5715 | −07.2757 |
| 2,2,2,1,3 | −11.5390 | −09.7946 |
| 3,3,3,1,2 | −13.4680 | −11.4316 |
| 5,5,5,5,1 | −21.5805 | −18.3181 |

$$\Psi(\boldsymbol{\rho}, \boldsymbol{\lambda}, \boldsymbol{\sigma}, \boldsymbol{\zeta}) = B \exp(-A_{11}\boldsymbol{\rho}^2 - A_{22}\boldsymbol{\lambda}^2 - A_{33}\boldsymbol{\sigma}^2 - A_{44}\boldsymbol{\zeta}^2 - 2A_{34}\boldsymbol{\sigma}.\boldsymbol{\zeta}), \tag{105}$$

where the Jacobi coordinates $\boldsymbol{\rho}$, $\boldsymbol{\lambda}$, $\boldsymbol{\sigma}$, $\boldsymbol{\zeta}$ are defined by (46–49) or by (74–77) according to which configuration we consider $(m_1, m_1, m_1, m_4, m_5)$ or $(m_1, m_1, m_3, m_3, m_5)$, respectively. $B$ is a normalized constant and $A_{11}$, $A_{22}$, $A_{33}$, $A_{44}$ and $A_{34}$ are variational parameters determined by minimizing the expectation value of the Hamiltonian for the trial wave function (105). The coupling $\boldsymbol{\sigma}.\boldsymbol{\zeta}$ in the Gaussian has been included to recover the exact ground state energy for both the configurations considered in the case of harmonic interactions (analytical expressions for the ground state energy of a five-body Hamiltonian for mass configurations $(m_1, m_1, m_1, m_4, m_5)$ and $(m_1, m_1, m_3, m_3, m_5)$ have been given in the preceding section). Before pursuing, it is worthwhile to notice that (105) is not the first term on an expansion on correlated Gaussians. The first term of such an expansion is of the form

$$\begin{aligned}\Psi_{CGE}(\boldsymbol{\rho}, \boldsymbol{\lambda}, \boldsymbol{\sigma}, \boldsymbol{\zeta}) = B \big[&\exp(-A_{11}\boldsymbol{\rho}^2 - A_{22}\boldsymbol{\lambda}^2 - A_{33}\boldsymbol{\sigma}^2 - A_{44}\boldsymbol{\zeta}^2 \\ &-2A_{12}\boldsymbol{\rho}.\boldsymbol{\lambda} - 2A_{13}\boldsymbol{\rho}.\boldsymbol{\sigma} - 2A_{14}\boldsymbol{\rho}.\boldsymbol{\zeta} - 2A_{23}\boldsymbol{\lambda}.\boldsymbol{\sigma} - 2A_{24}\boldsymbol{\lambda}.\boldsymbol{\zeta} - 2A_{34}\boldsymbol{\sigma}.\boldsymbol{\zeta}) + \cdots\big],\end{aligned} \tag{106}$$

where the dots stand for counter terms to restore the permutation symmetry and are obtained from the first one between brakets by appropriate permutations. For mass configurations $(m_1, m_1, m_1, m_4, m_5)$, one has six terms between the brakets corresponding to the $3! = 6$ permutations of the three particles of mass $m_1$, whereas for mass configurations $(m_1, m_1, m_3, m_3, m_5)$, one has 4 such terms corresponding to the two permutations of the two particles of mass $m_1$ and to the two permutations of the two particles of mass $m_3$. But we prefer to use $\Psi$, (105), instead of $\Psi_{CGE}$, (106), as a trial wave function because of its simplicity. It is worthwhile to stress that our main concern in this paper is the optimized lower bound. Our inclusion of a simple variational calculation is for illustrative purposes only. A variational calculation allows us to determine an upper bound for the ground state energy of a five-body system, which when combined with the optimized lower bound determine a frame for the ground state energy, which lies between. The results furnished by the variational calculation do not differ too much from those given by the optimized lower bound for regular power-law potential, i.e., those with positive power $\nu$. This is not the case for singular power-law potentials, i.e., those with negative power $\nu$. We can expect that the optimized lower bound is closer to the exact ground state energy than the result of the variational calculation. We can argue that the optimized lower bound possesses a higher degree of flexibility than the variational calculation using the partially correlated Gaussian trial wave function. In both cases, we optimize over a set of free parameters. But in the case of the optimized lower bound, we use the ground state energies of two-body Hamiltonians with the same interaction type as in the five-body case. While in the variational calculation, we use a trial wave function appropriate to harmonic interactions even in the case where the interactions are not harmonic.

To obtain the entries of Tables 1, 2, 3 and 4 we have used the following values for the ground state energies of two-body Hamiltonians $E^{(2)}(1, 1, \nu)$

$$\begin{aligned} E^{(2)}(1,1,2) &= 3.00000, \quad & E^{(2)}(1,1,1) &= 2.33811, \\ E^{(2)}(1,1,0.1) &= 1.23573, \quad & E^{(2)}(1,1,-1) &= -0.25000. \end{aligned}$$

## 7 Conclusion

An optimized lower bound has been previously derived for the ground state energy of a three-body Hamiltonian [1], and then for that of a four-body Hamiltonian [2], under the two assumptions of non-relativistic kinematics and translationally invariant two-body forces. Recently [3], the methodology has been generalized to obtain an optimized lower bound for the ground state energy of an $N$-body Hamiltonian, with arbitrary $N$ under the same assumptions as for the three- and the four-body cases. In this paper, the procedure is applied to a five-body system, the next system in the degree of complexity after the three- and the four-body cases. Particular attention is devoted to the universal dynamical constraints, relations among the values of the parameters corresponding to the optimized lower bound, which have the properties to be of dynamical nature and to be independent of the particular form of the potential part of the Hamiltonian. The universal dynamical constraints which, in the five-body case, are in number of 21, to be compared with 1 relation and 7 relations in the three- and the four-body cases, respectively, have been derived and great calculational details have been given. These relations are very important, since they permit to reduce the number of parameters over which one has to optimize from 30 to 9, a considerable simplification, if we have in mind that we deal with a non-linear optimization problem. Our interest in the five-body problem is mainly motivated by the fact that the five-body problem is somewhat a borderline. The five-body system with harmonic interactions is exactly solvable in the most general case, i.e., arbitrary masses and arbitrary coupling constants, which is not the case for more complex systems: the six-, seven-body systems and so on. Indeed, one can show that solving an $N$-body problem with harmonic forces is equivalent to solve an algebraic equation of degree $N - 1$. But as is well known, an algebraic equation is solvable in the most general case only for a degree lower or equal to 4. Hence the most general $N$-body system with harmonic forces can be solved only for $N - 1 \leq 4$, that is $N \leq 5$. The saturability of the optimized lower bound, i.e., the equality of the optimized lower bound and the exact ground state energy, for a five-body system with harmonic forces is satisfied numerically. By this we mean that this property of saturability has not been taken in default for all mass configurations and coupling constants we have considered. Moreover, an analytical proof of saturability is available for the particular mass configurations $(m_1, m_1, m_1, m_1, m_1)$, $(m_1, m_1, m_1, m_1, m_5)$ and $(m_1, m_1, m_1, m_3, m_4)$. For other types of interaction, we suspect that the optimized lower bound is more accurate than the result of a one Gaussian calculation (105) (see the argument of flexibility given in Sect. 6). To obtain the same degree of accuracy as that reached by the optimized lower bound, one has to include more than one Gaussian, i.e., a superposition of partially correlated Gaussians

$$\Psi(\boldsymbol{\rho}, \boldsymbol{\lambda}, \boldsymbol{\sigma}, \boldsymbol{\zeta}) = \sum_i B_i \exp(-A_{11}^{(i)}\boldsymbol{\rho}^2 - A_{22}^{(i)}\boldsymbol{\lambda}^2 - A_{33}^{(i)}\boldsymbol{\sigma}^2 - A_{44}^{(i)}\boldsymbol{\zeta}^2 - 2A_{34}^{(i)}\boldsymbol{\sigma}.\boldsymbol{\zeta}). \tag{107}$$

The number of Gaussians to include increases as the potential departs from the harmonic case. For instance, Coulomb potential ($\nu_{ij} = -1$) would require more Gaussians than the linear potential ($\nu_{ij} = 1$) to obtain an approximation to the ground state energy of the same degree of accuracy as that given by the optimized lower bound. Furthermore, our investigations show that the optimized lower bound for five-body Hamiltonians is superior to the previously derived naïve [6–11] and improved lower bounds [12–14]. These conclusions are also valid in the case of an $N$-body system, with arbitrary $N$.

## A Proof of the Universal Dynamical Constraints in the Five-Body Case

Let us make here the same change of notations as in Sect. 3, (17), and apply the procedure given in [3]. The $(10 \times 10)$ matrix $\widetilde{\mathbf{D}}$ is given by Eq. (18). Each column of the matrix $\widetilde{\mathbf{D}}$ gives rise to three columns of the $(10 \times 30)$ matrix $\widetilde{\mathbf{M}}$ which thus shows under the form of blocks, each of them consisting of three columns. More explicitly:

$$2\widetilde{\mathbf{M}} = \left(\mathbf{C}\ \mathbf{D}\ \mathbf{E}\ \mathbf{F}\ \mathbf{G}\ \mathbf{H}\ \mathbf{J}\ \mathbf{K}\ \mathbf{L}\ \mathbf{N}\right), \tag{108}$$

where the $(10 \times 3)$ matrices **C**, **D**, **E**, **F**, **G**, **H**, **J**, **K**, **L** and **N** are given by

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 \\ c_3 - 1 & 0 & 0 \\ 0 & c_4 - 1 & 0 \\ 0 & 0 & c_5 - 1 \\ c_3 + 1 & 0 & 0 \\ 0 & c_4 + 1 & 0 \\ 0 & 0 & c_5 + 1 \\ c_3 - c_4 & c_4 - c_3 & 0 \\ c_3 - c_5 & 0 & c_5 - c_3 \\ 0 & c_4 - c_5 & c_5 - c_4 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} d_2 - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & d_4 - 1 & 0 \\ 0 & 0 & d_5 - 1 \\ d_2 + 1 & 0 & 0 \\ d_2 - d_4 & d_4 - d_2 & 0 \\ d_2 - d_5 & 0 & d_5 - d_2 \\ 0 & d_4 + 1 & 0 \\ 0 & 0 & d_5 + 1 \\ 0 & d_4 - d_5 & d_5 - d_4 \end{pmatrix},$$

$$\mathbf{E} = \begin{pmatrix} e_2 - 1 & 0 & 0 \\ 0 & e_3 - 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_5 - 1 \\ e_2 - e_3 & e_3 - e_2 & 0 \\ e_2 + 1 & 0 & 0 \\ e_2 - e_5 & 0 & e_5 - e_2 \\ 0 & e_3 + 1 & 0 \\ 0 & e_3 - e_5 & e_5 - e_3 \\ 0 & 0 & e_5 + 1 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} f_2 - 1 & 0 & 0 \\ 0 & f_3 - 1 & 0 \\ 0 & 0 & f_4 - 1 \\ 0 & 0 & 0 \\ f_2 - f_3 & f_3 - f_2 & 0 \\ f_2 - f_4 & 0 & f_4 - f_2 \\ f_2 + 1 & 0 & 0 \\ 0 & f_3 - f_4 & f_4 - f_3 \\ 0 & f_3 + 1 & 0 \\ 0 & 0 & f_4 + 1 \end{pmatrix},$$

$$\mathbf{G} = \begin{pmatrix} g_1 - 1 & 0 & 0 \\ g_1 + 1 & 0 & 0 \\ g_1 - g_4 & g_4 - g_1 & 0 \\ g_1 - g_5 & 0 & g_5 - g_1 \\ 0 & 0 & 0 \\ 0 & g_4 - 1 & 0 \\ 0 & 0 & g_5 - 1 \\ 0 & g_4 + 1 & 0 \\ 0 & 0 & g_5 + 1 \\ 0 & g_4 - g_5 & g_5 - g_4 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} h_1 - 1 & 0 & 0 \\ h_1 - h_3 & h_3 - h_1 & 0 \\ h_1 + 1 & 0 & 0 \\ h_1 - h_5 & 0 & h_5 - h_1 \\ 0 & h_3 - 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_5 - 1 \\ 0 & h_3 + 1 & 0 \\ 0 & h_3 - h_5 & h_5 - h_3 \\ 0 & 0 & h_5 + 1 \end{pmatrix},$$

$$\mathbf{J}=\begin{pmatrix} j_1-1 & 0 & 0\\ j_1-j_3 & j_3-j_1 & 0\\ j_1-j_4 & 0 & j_4-j_1\\ j_1+1 & 0 & 0\\ 0 & j_3-1 & 0\\ 0 & 0 & j_4-1\\ 0 & 0 & 0\\ 0 & j_3-j_4 & j_4-j_3\\ 0 & j_3+1 & 0\\ 0 & 0 & j_4+1 \end{pmatrix},\quad \mathbf{K}=\begin{pmatrix} k_1-k_2 & k_2-k_1 & 0\\ k_1-1 & 0 & 0\\ k_1+1 & 0 & 0\\ k_1-k_5 & 0 & k_5-k_1\\ 0 & k_2-1 & 0\\ 0 & k_2+1 & 0\\ 0 & k_2-k_5 & k_5-k_2\\ 0 & 0 & 0\\ 0 & 0 & k_5-1\\ 0 & 0 & k_5+1 \end{pmatrix},$$

$$\mathbf{L}=\begin{pmatrix} l_1-l_2 & l_2-l_1 & 0\\ l_1-1 & 0 & 0\\ l_1-l_4 & 0 & l_4-l_1\\ l_1+1 & 0 & 0\\ 0 & l_2-1 & 0\\ 0 & l_2-l_4 & l_4-l_2\\ 0 & l_2+1 & 0\\ 0 & 0 & l_4-1\\ 0 & 0 & 0\\ 0 & 0 & l_4+1 \end{pmatrix},\quad \mathbf{N}=\begin{pmatrix} n_1-n_2 & n_2-n_1 & 0\\ n_1-n_3 & 0 & n_3-n_1\\ n_1-1 & 0 & 0\\ n_1+1 & 0 & 0\\ 0 & n_2-n_3 & n_3-n_2\\ 0 & n_2-1 & 0\\ 0 & n_2+1 & 0\\ 0 & 0 & n_3-1\\ 0 & 0 & n_3+1\\ 0 & 0 & 0 \end{pmatrix}.$$

Let us consider the first three blocks of the matrix $2\widetilde{\mathbf{M}}$, namely the blocks **C**, **D** and **E**. They involve 9 parameters. We may choose up to 8 among them as independent parameters, for example $c_3$, $c_4$, $c_5$, $d_2$, $d_4$, $d_5$, $e_2$ and $e_5$. But since we know that there are 9 independent parameters, we must complete by a further parameter taken out of the blocks **C**, **D** and **E**. One possibility is to adjoin $j_1$ to the above chosen parameters. To summarize, we have taken $c_3$, $c_4$, $c_5$, $d_2$, $d_4$, $d_5$, $e_2$, $e_5$ and $j_1$ as independent parameters and we will try to express the remaining 21 parameters in terms of these, by imposing the rank condition on the matrix $2\widetilde{\mathbf{M}}$ resulting in 21 universal dynamical constraints. To determine the universal dynamical constraints we shall use the third step of three-step procedure of reference [3]. In particular, to select the correct solution, we will apply the two criteria of reference [3], which we recall here:

- The solution must correspond to the correct limits when the system exhibits symmetries; for instance, all the parameters must be equal to zero in the equal mass case.
- The parameters chosen as independent parameters must be treated as independent parameters in the sense that an expression containing only independent parameters can not be set to zero.

These two criteria will be refered hereafter as symmetry and self-consistency criteria respectively.

To obtain $e_3$, let us take the blocks **C**, **D**, **E** and adjoin to them the first column of the **F** block. We obtain in this way a $(10\times 10)$ matrix, namely

$$\begin{pmatrix} 0 & 0 & 0 & d_2-1 & 0 & 0 & e_2-1 & 0 & 0 & f_2-1\\ c_3-1 & 0 & 0 & 0 & 0 & 0 & 0 & e_3-1 & 0 & 0\\ 0 & c_4-1 & 0 & 0 & d_4-1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & c_5-1 & 0 & 0 & d_5-1 & 0 & 0 & e_5-1 & 0\\ c_3+1 & 0 & 0 & d_2+1 & 0 & 0 & e_2-e_3 & e_3-e_2 & 0 & f_2-f_3\\ 0 & c_4+1 & 0 & d_2-d_4 & d_4-d_2 & 0 & e_2+1 & 0 & 0 & f_2-f_4\\ 0 & 0 & c_5+1 & d_2-d_5 & 0 & d_5-d_2 & e_2-e_5 & 0 & e_5-e_2 & f_2+1\\ c_3-c_4 & c_4-c_3 & 0 & 0 & d_4+1 & 0 & 0 & e_3+1 & 0 & 0\\ c_3-c_5 & 0 & c_5-c_3 & 0 & 0 & d_5+1 & 0 & e_3-e_5 & e_5-e_3 & 0\\ 0 & c_4-c_5 & c_5-c_4 & 0 & d_4-d_5 & d_5-d_4 & 0 & 0 & e_5+1 & 0 \end{pmatrix}. \qquad (109)$$

Computing the determinant of the above matrix, which we will denote hereafter $(CDE1f)$, one can factorize the result in the form of a product of three factors:

$$\begin{aligned}
&(\mathbf{5} + c_4d_2e_5 + c_4d_5e_3 - c_5e_3f_2 - 2c_4e_5f_2 - 2c_3d_4e_2 + c_3d_4e_5 + c_4e_3f_2 + c_5e_3f_4 + c_5e_2f_3\\
&\quad -c_4e_2f_3 - c_4d_2e_5f_3 + c_4d_5f_2 + c_4e_5f_3 - c_4d_5f_3 + e_2f_4 + e_3f_4 + d_2e_5f_3 + c_4d_2f_3 + d_5f_2\\
&\quad +d_4 + c_5 + d_5 + e_5 + f_3 + f_4 - c_5d_4f_2 + d_4e_5f_2 - d_4e_3f_2 - 2d_4e_5f_3 + d_4e_2f_3 + e_5f_2\\
&\quad +c_5d_2e_3 + c_3d_4f_2 + c_5d_4e_2 + c_3e_2f_4 - c_4d_5e_2 + c_3e_5f_2 + d_2 + e_2 + f_2 + c_3 + e_3 + c_4\\
&\quad -2c_3d_5f_2 + e_5f_3 + d_5e_3f_2 + c_5d_4f_3 - c_3d_2e_5 + d_5e_2f_4 + c_3d_5e_2 - d_5e_2f_3\\
&\quad -3c_5f_2 + d_5e_3 - c_5d_4e_3 + c_3f_2 + c_3d_5 + d_4e_2 - d_2e_5f_4 - 3d_5f_3 - 3d_4e_3\\
&\quad +d_2e_3 + d_4e_5 - 3e_5f_4 - 2d_5e_3f_4 + d_2e_3f_4 + d_4f_3 + d_2f_3 - 2c_5d_2f_3\\
&\quad +d_5f_4 + c_4d_5e_2f_3 + c_4f_2 + c_4e_5 - c_3d_5e_2f_4 - 2c_5e_2f_4 + c_5f_3\\
&\quad +c_5f_4 + c_5d_2 + c_5e_2 - c_5d_4e_2f_3 - c_5d_2e_3f_4 - 3c_3d_2\\
&\quad +c_3e_2 + c_3d_4 + c_5d_4e_3f_2 - c_3d_4e_5f_2 + c_4d_2 + c_4e_3\\
&\quad -3c_4e_2 - 2c_4d_2e_3 + c_5d_2f_4 - c_3e_5f_4 + c_3d_5f_4\\
&\quad -c_3d_2f_4 + c_3d_2e_5f_4 - c_4d_5e_3f_2)\,,\\
&(2c_3d_5 - \mathbf{1} - d_5 - d_5e_3 + c_4d_5e_3 - c_4d_5 - e_5 - c_3e_5 + 2c_4e_5 - c_3d_4 + d_4e_3 + 3c_5 + c_5e_3\\
&\quad +c_5d_4 - c_5d_4e_3 + c_3d_4e_5 - d_4e_5 - c_3 - c_4 - c_4e_3)\,,
\end{aligned}$$

and

$$(c_3d_4 - c_3d_4e_2 + c_3e_2 - c_3 + d_4e_2 - d_4 - d_2e_3 - e_2 + d_2 + c_4 + e_3 - c_4d_2 - c_4e_3 + c_4d_2e_3)$$

Then putting det $(CDE1f)$ to zero, at least one of these three factors must vanish. However the application of the criterium of symmetry forbids the vanishing of the two first factors. It follows that it is the third factor that must vanish, i.e.,

$$c_3d_4 - c_3d_4e_2 + c_3e_2 - c_3 + d_4e_2 - d_4 - d_2e_3 - e_2 + d_2 + c_4 + e_3 - c_4d_2 - c_4e_3 + c_4d_2e_3 = 0, \quad (110)$$

which gives the first universal dynamical constraint

$$e_3 = \frac{c_3d_4e_2 - c_4 - d_2 + c_4d_2 + c_3 + d_4 - c_3e_2 + e_2 - c_3d_4 - d_4e_2}{1 - c_4 - d_2 + c_4d_2}. \quad (111)$$

To determine $j_3$ and $j_4$, let us consider the two matrices $(CDJ1e)$ and $(CEJ1g)$. The determinant of each of these two matrices may be put in the form of product of factors. In each case, the symmetry requirements imply that only one of the factors may vanish. More explicitly $\det(CDJ1e) = 0$ and $\det(CEJ1g) = 0$ imply, respectively,

$$c_5j_1 + j_1 - c_3d_2j_1 - c_5d_2j_1 + c_3d_5j_1 - d_5j_1 + d_5 - c_3d_5 - j_3 - c_5j_3 + d_2j_3 - d_2 + c_3d_2 + c_5d_2j_3 = 0$$

and

$$e_5 - e_5c_4 + c_4e_5j_1 - e_5j_1 + e_2j_4 - e_2 + e_2c_4 + e_2j_4c_5 - c_4e_2j_1 - c_5e_2j_1 - j_4 - j_4c_5 + c_5j_1 + j_1 = 0,$$

from which follow the second and third universal dynamical constraints

$$j_3 = \frac{c_5j_1 + j_1 - c_3d_2j_1 - c_5d_2j_1 + c_3d_5j_1 - d_5j_1 + d_5 - c_3d_5 + c_3d_2 - d_2}{1 + c_5 - d_2 - c_5d_2}, \quad (112)$$

$$j_4 = \frac{e_5 - e_5c_4 + c_4e_5j_1 - e_5j_1 - c_4e_2j_1 - e_2 + e_2c_4 + c_5j_1 - c_5e_2j_1 + j_1}{1 + c_5 - e_2 - c_5e_2}. \quad (113)$$

To determine $f_2$, $f_3$ and $f_4$, it is appropriate to consider the three matrices $(CDF1e)$, $(CEF1d)$ and $(DJF1e)$. For each of these matrices, the determinant shows as a product of factors. Rejecting the solutions which do not satisfy to the symmetry requirements, $\det(CDF1e) = 0$ imply

$$c_5 - c_3 + c_3f_2 - c_3d_5f_2 + c_3d_5 + d_5f_5 - d_5 - c_5f_3 + d_2 - f_2 + f_3 + c_5d_2f_3 - d_2f_3 - c_5d_2 = 0, \quad (114)$$

which gives

$$f_3 = \frac{c_3 - f_2 c_3 + f_2 c_3 d_5 - c_3 d_5 - d_5 f_2 + d_5 - d_2 - c_5 + f_2 + c_5 d_2}{1 - c_5 - d_2 + c_5 d_2}, \tag{115}$$

and $\det(CEF1d) = 0$ imply

$$f_4 - e_2 f_4 - c_5 f_4 + c_5 e_2 f_4 - e_5 - c_4 e_5 f_2 + e_5 f_2 + c_4 e_5 - f_2 + c_4 f_2 - c_5 e_2 + c_5 - c_4 + e_2 = 0. \tag{116}$$

Using the relations (112) and (115), $\det(DJF1e) = 0$ reduces to

$$2\frac{(d_5 - 1)(d_2 - d_5)(f_2 - c_5 - j_1 + f_2 j_1 c_5)(c_3 - 1)}{(d_2 - 1)(c_5 - 1)(c_5 + 1)} = 0. \tag{117}$$

Since $c_3$, $d_2$ and $d_5$ have been chosen as independent parameters, neither of the factors $d_5 - 1$, $d_2 - d_5$ and $c_3 - 1$ in the numerator of (117) can be put to zero. Thus, it is the factor $f_2 - c_5 - j_1 + f_2 j_1 c_5$, which must vanish, giving rise to a fourth universal dynamical constraint

$$f_2 = \frac{c_5 + j_1}{1 + c_5 j_1}. \tag{118}$$

Replacing $f_2$ by its expression in (115) and in (116), one gets two universal dynamical constraints

$$f_3 = \frac{j_1 - d_2 + c_3 - j_1 c_3 - d_5 j_1 + d_5 + d_5 j_1 c_3 - c_3 d_5 - j_1 c_5 d_2 + j_1 c_5}{(1 - d_2)(1 + j_1 c_5)} \tag{119}$$

and

$$f_4 = \frac{j_1 c_4 e_5 - c_5 e_2 j_1 + c_4 - e_5 c_4 - c_4 j_1 - e_5 j_1 + j_1 c_5 - e_2 + e_5 + j_1}{(1 - e_2)(1 + j_1 c_5)}. \tag{120}$$

To determine the $g$: $g_1$, $g_4$ and $g_5$, one may consider the matrices $(CEG1f)$, $(CFG1h)$ and $(DFG1e)$. Equating the determinants of the previous matrices to zero and making use of the symmetry conditions, one obtains for $g_1$, $g_4$ and $g_5$ the following relations:

$$g_1 = \frac{d_2 - c_3}{1 - c_3 d_2}. \tag{121}$$

$$g_4 = \frac{c_4 d_2 - c_4 - c_3 d_4 + d_4}{1 - c_3 d_2}, \tag{122}$$

$$g_5 = \frac{c_5 d_2 - c_5 - c_3 d_5 + d_5}{1 - c_3 d_2}. \tag{123}$$

The expressions of $h_1$, $h_3$ and $h_5$ in terms of the independent parameters may be obtained by equating the determinants of $(CDH2j)$, $(FGH1e)$ and $(EGH1f)$ to zero. Making use of the expressions of $g_1$, (121) and $g_4$, (122), and applying the symmetry and self-consistency conditions one gets

$$h_1 = \frac{c_4 - e_2}{e_2 c_4 - 1}, \tag{124}$$

$$h_3 = \frac{c_4 d_2 - c_4 - d_4 e_2 + c_3 d_2 + d_4 - e_2 c_3 d_2 + e_2 - c_3 d_4 - d_2 + c_3 d_4 e_2}{(d_2 - 1)(e_2 c_4 - 1)}, \tag{125}$$

$$h_5 = \frac{c_5 - c_5 e_2 + c_4 e_5 - e_5}{e_2 c_4 - 1}. \tag{126}$$

By equating the determinants of $(CDK1f)$, $(FHK1d)$ and $(EHK1f)$ to zero, making use of the conditions of symmetry and self-consistency and taking into account of the expressions for $e_3$, (111), $f_2$, (118), $f_3$, (119), $f_4$, (120), $h_1$, (124), $h_3$, (125) and $h_5$, (126), one gets for $k_1$, $k_2$ and $k_5$ the following expressions

$$k_1 = -\frac{c_3 e_2 - c_3 - e_2 + c_4 + d_2 - c_4 d_2}{c_3 d_4 e_2 - c_3 d_4 - d_4 e_2 + d_4 - c_4 - d_2 + c_4 d_2 + 1}, \tag{127}$$

$$k_2 = -\frac{-c_3 d_2 + e_2 c_3 d_2 - e_2 c_4 d_2 - e_2 + e_2 c_4 + d_2}{c_3 d_4 e_2 - c_3 d_4 - d_4 e_2 + d_4 - c_4 - d_2 + c_4 d_2 + 1}, \tag{128}$$

$$k_5 = -\frac{c_3 d_5 e_2 - c_3 d_5 - d_5 e_2 + d_5 + d_2 e_5 - d_2 c_4 e_5 - e_5 + c_4 e_5}{c_3 d_4 e_2 - c_3 d_4 - d_4 e_2 + d_4 - c_4 - d_2 + c_4 d_2 + 1}. \tag{129}$$

To obtain the expressions of $l_1$, $l_2$ and $l_4$, a convenient choice is that of the matrices $(CDL1e)$, $(CFL3k)$ and $(EGL1f)$. Equating the determinants of these three matrices to zero, making again of the two criteria of symmetry and self-consistency and substituting for $e_3$, $g_1$, $g_4$ and $g_5$ their respective expressions (111), (121), (122) and (123), one ends with

$$l_1 = -\frac{c_5 d_2 j_1 - j_1 - c_5 j_1 - c_3 + c_3 j_1 + d_2}{c_5 j_1 + c_3 d_5 j_1 - d_5 j_1 + 1 - c_5 d_2 j_1 + d_5 - d_2 - c_3 d_5}, \tag{130}$$

$$l_2 = -\frac{j_1 d_2 c_3 - c_3 d_2 - c_5 - j_1 + c_5 d_2 + d_2}{c_5 j_1 + c_3 d_5 j_1 - d_5 j_1 + 1 - c_5 d_2 j_1 + d_5 - d_2 - c_3 d_5}, \tag{131}$$

$$l_4 = \frac{d_4 (c_3 - 1)(e_2 - 1)(j_1 - 1) + (1 - d_2)\left[(c_4 - 1)(e_5 - 1)(j_1 - 1) - c_5 (e_2 - 1)(j_1 - 1) - (1 + c_5)(e_2 - 1)\right]}{(1 - e_2)\left[j_1 (c_5 + 1)(1 - d_2) + d_5 (1 - c_3)(1 - j_1) + (1 - d_2)(1 - j_1)\right]}. \tag{132}$$

The matrices $(CDN1e)$, $(DJN1e)$ and $(EJN3d)$ are appropriate choice in order to obtain the three last universal dynamical constraints. Putting the determinants of the three chosen matrices to zero, making once more again of the two criteria of symmetry and self-consistency, along with the expressions of $j_3$, (112), and $j_4$, (113), one obtains for $n_1$, $n_2$ and $n_3$:

$$n_1 = -\frac{-c_4 - j_1 + c_4 j_1 + c_5 e_2 j_1 + e_2 - c_5 j_1}{-e_2 - c_5 e_2 j_1 - e_5 c_4 + c_4 e_5 j_1 + 1 - e_5 j_1 + c_5 j_1 + e_5}, \tag{133}$$

$$n_2 = -\frac{c_4 e_2 j_1 - c_4 e_2 - j_1 + e_2 - c_5 + e_2 c_5}{-e_2 - c_5 e_2 j_1 - e_5 c_4 + c_4 e_5 j_1 + 1 - e_5 j_1 + c_5 j_1 + e_5}, \tag{134}$$

$$n_3 = \frac{(e_2 - 1)\left[(c_3 - 1)(d_4 - d_5)(j_1 - 1) + j_1 (c_5 - 1)(d_2 - 1) + (d_2 - 1)(1 + j_1)\right] + (c_4 - 1)(d_2 - 1)(j_1 - 1)}{(1 - d_2)(1 - e_2)(1 + c_5 j_1) + e_5 (1 - c_4)(1 - d_2)(1 - j_1)}. \tag{135}$$

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## References

1. Basdevant, J.-L., Martin, A., Richard, J.-M., Wu, T.T.: Optimized lower bounds in the three-body problem. Nucl. Phys. B **393**, 111 (1993)
2. Benslama, A., Metatla, A., Bachkhaznadji, A., Zouzou, S.R., Krikeb, A., Basdevant, J.-L., Richard, J.-M., Wu, T.T.: Optimized lower bound for four-body Hamiltonians. Few Body Syst. **24**, 39 (1998)
3. Boudjemaa, Kh., Zouzou, S.R.: Optimized lower bound on $N$-body Hamiltonians. J. Phys. A: Math. Gen. **39**, 7383–7409 (2006)
4. Boudjemaa, Kh., Zouzou, S.R.: An analytical proof of saturability of an optimized lower bound for $N$-body Hamiltonians, for some mass configurations, with arbitrary $N$. J. Phys. A: Math. Gen. **39**, 5857–5872 (2006)
5. Fleck, S., Richard, J.-M.: Small exotic molecules with bosonic constituents. Few Body Syst. **19**, 19 (1995)
6. Fisher, M.E., Ruelle, D.: The stability of many-particle systems. J. Math. Phys. **7**, 260 (1966)
7. Dyson, F.J., Lenard, A.: Stability of matter, I. J. Math. Phys. **8**, 423 (1967)
8. Lévy-Leblond, J.-M.: Nonsaturation of gravitational forces. J. Math. Phys. **10**, 806 (1969)
9. Ader, J.-P., Richard, J.-M., Taxil, P.: Do narrow heavy multiquark states exist? Phys. Rev. D **25**, 2370 (1982)
10. Nussinov, S.: Baryon-Meson mass inequality. Phys. Rev. Lett. **51**, 2081 (1983)
11. Richard, J.-M.: On the masses of mesons and baryons in potential models. Phys. Lett. B **139**, 408 (1984)
12. Hall, R.L., Post, H.R.: Many-particle systems: IV. Short-range interactions. Proc. Phys. Soc. **90**, 381 (1967)
13. Basdevant, J.-L., Martin, A., Richard, J.-M.: Improved bounds on many-body Hamitonians, I. Self-graviting bosons. Nucl. Phys. B **343**, 60 (1990)
14. Basdevant, J.-L., Martin, A., Richard, J.-M.: Improved bounds on many-body Hamitonians, II. Baryons from mesons in the quark model. Nucl. Phys. B **343**, 69 (1990)