

Exercise 1 (6pts)

Let f_1^*, f_2^* be two linear forms on $V = \mathbb{R}^2$, defined by:

$$f_1^*(x, y) = x - y, f_2^*(x, y) = 3x + 2y$$

1. Show that (f_1^*, f_2^*) forms a basis for V^* (the dual space of V).
2. Express $h(x, y) = -7x - 8y$ in terms of the basis (f_1^*, f_2^*) .
3. Let (f_1, f_2) be the **Predual** (antedual) basis of (f_1^*, f_2^*) . Deduce (without calculations) the values of:

$$h(f_1), h(f_2).$$

Exercise 2 (4pts)

Determine the real value a for which the following quadratic form defines an inner (scalar) product:

$$q(X) = x_1^2 + (1+a)x_2^2 + (1+a+a^2)x_3^2 + 2x_1x_2 - 2ax_2x_3.$$

Exercise 3 (10pts)

Consider the quadratic form defined on $E = \mathbb{R}^3$ by:

$$q(X) = 2x_1^2 + 2x_1x_2 + x_2^2 - 2x_2x_3 + 2x_3^2.$$

1. Determine the polar form (the bilinear form associated with q) and the associated matrix A of q .
2. Using Gaussian elimination, diagonalize and deduce its rank and signature.
3. Find an orthogonal basis of $E = \mathbb{R}^3$ with respect to q .
4. Determine an invertible matrix P and a diagonal matrix D such that $D = {}^tPAP$.

Ex 01 = Model Solution

(1) f_1^*, f_2^* form a basis of V^* if and only if they are independent:

$$\alpha f_1^* + \beta f_2^* = 0 \Leftrightarrow \begin{cases} \alpha(x-y) + \beta(3x+2y) = 0 \\ \forall x, y \in \mathbb{R} \end{cases}$$

$$\Leftrightarrow \begin{cases} x(\alpha + 3\beta) + y(-\alpha + 2\beta) = 0 \\ \forall x, y \in \mathbb{R} \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha + 3\beta = 0 \\ -\alpha + 2\beta = 0 \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \end{cases}$$

$\Leftrightarrow f_1^*, f_2^*$ are independent.

(2) Find $\alpha, \beta \in \mathbb{R}$ such that: $h = \alpha f_1^* + \beta f_2^*$

This means:

$$\alpha(x-y) + \beta(3x+2y) = -7x - 8y, \forall x, y \in \mathbb{R}$$

$$\Leftrightarrow x(\alpha + 3\beta + 7) + y(-\alpha + 2\beta + 8) = 0, \forall x, y \in \mathbb{R}$$

$$\Leftrightarrow \begin{cases} \alpha + 3\beta + 7 = 0 \\ -\alpha + 2\beta + 8 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha + 3\beta + 7 = 0 \\ 5\beta + 15 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha = 2 \\ \beta = -3 \end{cases}$$

Then $h = 2f_1^* - 3f_2^*$

(3) From this expression of h , deduce that

$$h(f_1) = 2 \text{ and } h(f_2) = -3$$

exercise 2 =

* Express q in orthogonal form =

Reduction of x_1 :

$$x_1^2 + (1+a+a^2) 2x_1x_2 = (x_1 + x_2)^2 - x_2^2$$

$$\text{then } q(x) = l_1^2 - x_2^2 + (1+a)x_1^2 + (1+a+a^2)x_3^2 - 2ax_2x_3$$

$$= l_1^2 + a x_2^2 + (1+a+a^2)x_3^2 - 2ax_2x_3$$

Reduction of x_2 :

$$a x_2^2 - 2ax_2x_3 = a(x_2^2 - 2x_2x_3)$$

$$= a[(x_2 - x_3)^2 - x_3^2] = a \underbrace{(x_2 - x_3)^2}_{l_2^2} - a x_3^2$$

then

$$q(x) = l_1^2 + a l_2^2 - a x_3^2 + (1+a+a^2)x_3^2$$

$$= l_1^2 + a l_2^2 + (1+a^2)x_3^2$$

* A scalar product is a symmetric, positive definite bilinear form. Therefore, a must be strictly greater than zero ($a > 0$)

Exo 3 =

$$(1) \quad \frac{1}{2} \frac{\partial g}{\partial x_1} = \frac{1}{2} [4x_1 - 2x_2] = 2x_1 - x_2$$

$$\frac{1}{2} \frac{\partial g}{\partial x_2} = \frac{1}{2} [2x_1 + 2x_2 - 2x_3] = x_1 + x_2 - x_3$$

$$\frac{1}{2} \frac{\partial g}{\partial x_3} = \frac{1}{2} [-2x_2 + 4x_3] = -x_2 + 2x_3$$

Then $A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

$$f_g(x, y) = {}^t x A y$$

(2) Reduction of x_1

$$2x_1^2 + 2x_1x_2 = 2 \left(x_1^2 + x_1x_2 \right) = 2 \left[\left(x_1 + \frac{x_2}{2} \right)^2 - \frac{x_2^2}{4} \right]$$

$$= 2 \left(x_1 + \frac{x_2}{2} \right)^2 - \frac{x_2^2}{2}$$

Then $q(x) = 2 \ell_1^2 - \frac{x_2^2}{2} + x_2^2 - 2x_2x_3 + 2x_3^2$

$$= 2 \ell_1^2 + \frac{1}{2} x_2^2 - 2x_2x_3 + 2x_3^2$$

Reduction of x_2

$$\frac{1}{2} x_2^2 - 2x_2x_3 = \frac{1}{2} [x_2^2 - 4x_2x_3]$$

$$= \frac{1}{2} [(x_2 - 2x_3)^2 - 4x_3^2]$$

$$= \frac{1}{2} (x_2 - 2x_3)^2 - 2x_3^2$$

$$q(x) = 2 \ell_1^2 + \frac{1}{2} \ell_2^2 - 2x_3^2 + 2x_3^2$$

$$= 2 \ell_1^2 + \frac{1}{2} \ell_2^2$$

$$\text{Rank}(a) = 2$$

$$\text{Sign}(a) = (2, 0)$$

$$(3) \quad \mathcal{B} = P \begin{matrix} (e_i^*) \rightarrow l_i^* \end{matrix} =$$

$$P = P \begin{matrix} (e_i) \rightarrow l_i \end{matrix} = {}^t \mathcal{B}^{-1}$$

$$\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \quad l_3^*(x) = x_3$$

$$\text{Then } P = \begin{pmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{finally } l_1 = (1, 0, 0), l_2 = \left(-\frac{1}{2}, 1, 0\right), l_3 = (1, -2, 1)$$

$$(4) \quad P = \begin{pmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$