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A correction of the first exam

<u>Exercise01</u> consider $A=\left\{\frac{2n}{3n+1}:n\in\mathbb{N}\right\}$

1. <u>finding the real numbers a,b so that</u> $\frac{2n}{3n+1}=a+\frac{b}{3n+1}$.

   Using long division gives $\frac{2n}{3n+1}=\frac{2}{3}-\frac{\frac{2}{3}}{3n+1}$

2. Showing that the set A is bounded.

   Let $n\geq 0$. Then $3n+1\geq 1$ it follows $-\frac{2}{3}\leq-\frac{\frac{2}{3}}{3n+1}<0$.

   Adding 1 to each sides of the last inequality gives $0\leq\frac{2}{3}-\frac{\frac{2}{3}}{3n+1}<\frac{2}{3}$.

   Therefore $A\subset\left[0;\frac{2}{3}\right[$. Thus A is bounded.

3. <u>Determining $\min(A)$, $\max(A)$, $\inf(A)$ *and* $\sup(A)$ if there exist.</u>

   For $n=0$ we have $\frac{2n}{3n+1}=\frac{2\times 0}{3\times 0+1}=0$ which means $0\in A$.
   Using the fact that 0 is a lower bound of A yields $\mathbf{min}\,\boldsymbol{A}=\mathbf{0}.$
   Hence $\mathbf{inf}\,\boldsymbol{A}=\mathbf{0}$.

   As $\lim_{n\to\infty}\frac{2n}{3n+1}=\frac{3}{2}$ *with* $\frac{3}{2}$ *is an upper bound of* A, *we suppose that* $\sup(A)=\frac{3}{2}$.

   Prove that $\sup(A)=\frac{3}{2}$.

   $sup(A)=\frac{3}{2}\Leftrightarrow\forall\varepsilon>0,\exists n_\varepsilon\epsilon\mathbb{N}\ s.t\ \frac{3}{2}-\varepsilon<\frac{2n_\varepsilon}{3n_\varepsilon+1}$.

   Let $\varepsilon>0$. *By Archimedean property there exist* $n_\varepsilon>\frac{\varepsilon}{2}-\frac{1}{3}$.

   So $3n_\varepsilon+1>\frac{3\varepsilon}{2}$. *It follows* $-\frac{\frac{2}{3}}{3n_\varepsilon+1}>-\varepsilon$.

   Adding $\frac{2}{3}$ *gives* $\frac{3}{2}-\varepsilon<\frac{2n_\varepsilon}{3n_\varepsilon+1}$.

   We thus conclude $\mathbf{sup}(\boldsymbol{A})=\frac{\mathbf{3}}{\mathbf{2}}$.

Since $\frac{2n}{3n+1}=\frac{3}{2}$ has no solution in $\mathbb{N}$, we deduce that *max A* does not exist.

<u>Exercise02</u>

let $u_n=1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}$ *and* $v_n=1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\ /\ n\in\mathbb{N}$.

1. <u>Showing that that</u>*:* $\forall n\in\mathbb{N}^*;\ \frac{1}{n!}\leq\frac{1}{2^{n-1}}$
   - We denote by $p(n)$: $\frac{1}{n!}\leq\frac{1}{2^{n-1}}\ /\ n\in\mathbb{N}^*$.
   - <u>Base step</u>**:** For $n=1$. We get $\frac{1}{1!}\leq\frac{1}{2^{1-1}}$. So $p(1)$ holds.

- Induction step**:** suppose that $p(n): \frac{1}{n!} \leq \frac{1}{2^{n-1}} / n \in \mathbb{N}^*$ holds.

  Then prove that $p(n+1): \frac{1}{(n+1)!} \leq \frac{1}{2^n} / n \in \mathbb{N}$ holds.

  Let $n \in \mathbb{N}^*$.

  We have $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$. Multiplying by $\frac{1}{2}$ yields $\in \mathbb{N}$: $\frac{1}{2\times n!} \leq \frac{1}{2^n}$.

  For $n \geq 1$, we have $n+1 \geq 2$. Then. It follows $\frac{1}{n+1}\frac{1}{n!} \leq \frac{1}{2}\frac{1}{n!} \leq \frac{1}{2^n}$. which implies $\frac{1}{(n+1)!} \leq \frac{1}{2^n}$.

  Therefore $p(n+1)$ holds.
- By principle of mathematical induction $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$ for all $n \in \mathbb{N}^*$.

2. Prove that for all $n \in \mathbb{N}$: $u_n \leq 2$.

   Let $n \in \mathbb{N}$

   It is easy to that u_n is a sum of n+1-terms of geometric sequence with common ratio $\frac{1}{2}$.

   So $u_n = 1\frac{1-\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}}$. Then $u_n = 2 - \left(\frac{1}{2}\right)^n$.

   Since $-\left(\frac{1}{2}\right)^n \leq 0$, It is obvious that $u_n \leq 2$.

3. Showing that (v_n) is monotone.

   $v_{n+1} - v_n = \left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \frac{1}{n+1!}\right) - \left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) = \frac{1}{n+1!}$.

   Then $v_{n+1} - v_n > 0$. Therefore (v_n) is increasing. Which implies (v_n) is monotone.

4. deducing that the sequence(v_n) is convergent .

   As (v_n) is increasing, it suffices to prove that (v_n) is bounded to show that it is convergent.

   From the first question we have found $\frac{1}{k!} \leq \frac{1}{2^{k-1}} \quad \forall k \in \mathbb{N}^*$.

   Hence $\sum_{k=1}^{n} \frac{1}{k!} \leq \sum_{k=1}^{n} \frac{1}{2^{k-1}}$.

   Furthermore $\sum_{k=1}^{n} \frac{1}{k!} \leq \left(\sum_{k=1}^{n} \frac{1}{2^{k-1}}\right) + \frac{1}{2^n}$.

   Which means $v_n \leq u_n$. Since $u_n \leq 2 \quad \forall n \in \mathbb{N}$, we obtain $v_n \leq 2 \quad \forall n \in \mathbb{N}$.

   The sequence (v_n) is bounded and increasing then it is convergent.

Exercise03

$f: \mathbb{R} \to \mathbb{R} \qquad f(x) = \begin{cases} \sin(x^2) & if\ x \geq 0 \\ xe^x \sin(x^2) & if\ x < 0 \end{cases}$

1. Showing that: $\forall \alpha \in \mathbb{R}$: $|\sin \alpha| \leq |\alpha|$

   Let α be positive real.

   The function f is continuous on $[0, \alpha]$ and differentiable on $]0, \alpha[$.

   By mean value theorem there exist $\theta \in]0, \alpha[$ such that

   $\sin \alpha - \sin 0 = \cos \theta \ (\alpha - 0)$. Hence $\sin \alpha = \alpha \cos \theta$.

   Since $-\alpha \leq \alpha \cos \theta \leq \alpha$, we get $-\alpha \leq \sin \alpha \leq \alpha$.

   So $|\sin \alpha| \leq \alpha = |\alpha|$.

Let $\beta = -\alpha$ be negative real.
We have $|\sin\alpha| \leq |\alpha|$. Then $|-\sin\alpha| \leq |-\alpha|$. It follows $|\sin(-\alpha)| \leq |-\alpha|$.
Therefore $|\sin(\beta)| \leq |\beta|$ for all negative number.
In conclusion $\forall\alpha \in \mathbb{R}$: $|\sin\alpha| \leq |\alpha|$

2. Using the definition of limit, Prove that $lim_{x\to 0}(\,sin(x^2)) = 0.$
   We want: $\forall\varepsilon > 0, \exists\delta > 0\ s.t\ \forall x \in \mathbb{R}\ 0 < |x - 0| < \delta \Rightarrow |\,sin(x^2) -\ sin(0^2)| < \varepsilon$
   In other words $\forall\varepsilon > 0, \exists\delta > 0\ s.t\ \forall x \in \mathbb{R}\ 0 < |x| < \delta \Rightarrow |\,sin(x^2)| < \varepsilon.$
   - Let $\varepsilon > 0$. Choose $\delta = \sqrt{\varepsilon}$. Let $x \in \mathbb{R}$. we have $|x| < \sqrt{\varepsilon}.$

     $|\,sin(x^2)| \leq |x^2| \leq |x|^2 \leq \sqrt{\varepsilon}^2 = \varepsilon.$

     Therefore $lim_{x\to 0}(\,sin(x^2)) = 0.$
3. Prove that f is continuous over $\mathbb{R}$.
   The function f is a combination (composition and product) of continuous functions on $\mathbb{R}^*$ then f is continuous over$\mathbb{R}^*$.
   At the point $x_0 = 0$.
   - $f(0) = \sin(0) = 0$
   - $\lim\limits_{x\to 0^-}\big(f(x)\big) = \lim\limits_{x\to 0^-}[\sin(x^2)] = 0$
   - $\lim\limits_{x\to 0^+}\big(f(x)\big) = \lim\limits_{x\to 0^+}[xe^x\sin(x^2)] = 0$

   From above we have $\lim\limits_{x\to 0}\big(f(x)\big) = f(0).$
   Therefore f is continuous at 0.
   Conclusion: the function f is continuous over $\mathbb{R}$.
4. Show that f is differentiable on $\mathbb{R}$
   The function f is a combination (composition and product) of differentiable functions on $\mathbb{R}^*$ then f is differentiable on $\mathbb{R}^*$.

   We have $[\sin(x^2)]' = 2x\cos x^2$ over neighborhood of 0. Then $f_l'(x) = 2x\cos x^2$.
   Hence $f_l'(0) = 0$.
   We have $[xe^x\sin(x^2)]' = (x+1)e^x\sin(x^2) + 2x^2e^x\cos x^2$ over neighborhood of 0. Then $f_r'(x) = (x+1)e^x + 2x^2e^x\cos x^2$.
   Hence $f_r'(0) = 0$.
   Therefore $f_r'(0) = f_l'(0)$. We thus conclude that f is differentiable at 0.
   Conclusion: the function f is differentiable over $\mathbb{R}$.
5. Calculate f' the derivative of the function f over $\mathbb{R}$.

$$f'(x) = \begin{cases} 2x\cos x^2 & if\ x > 0 \\ 0 & if\ x = 0 \\ (x+1)e^x\sin(x^2) + 2x^2e^x\cos x^2 & if\ x < 0. \end{cases}.$$

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