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Measure of noncompactness and applications to fractional differential equations

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Dedication

With our deepest feelings of gratitude ,

We dedicate this modest work

To my parents,

To my family and freinds,

To my colleagues.

Acknowledgment

We thank God for giving us courage, health,

patience and courage to finish this dissertation .

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Contents

Introduction

About the fractional calculus, as we know it today, the point of beginning of this area can be traced back to the end of the seventeenth century, the time when the famous mathematicians Newton and Leibniz improved the formation of differential and integral calculus. In particular, Leibniz introduced the notation $\frac{d^n f(x)}{dx^n}$ to stand for the nth derivative of a function f. When he announced this in a letter to de l'Hospital (seemingly with the implicit assumption that $n \in \mathbb{N}$), de l'Hospital replied: "What does $\frac{d^n f(x)}{dx^n}$ mean if $n = 1/2$?" This letter from de l'Hospital, written in 1695, is nowadays generally accepted as the first incidence of what we today call a fractional derivative, and the fact that de l'Hospital specifically asked for $n = 1/2$, i.e., a fraction (rational number), actually gave rise to the name of this part of mathematics.

Therefore, Fractional calculus (FC) generalizes integrals and derivatives to non-integer orders. During the last decades, FC was found to play a fundamental role in the model-ing of a considerable number of phenomena; the motivation for studying fractional differential equations comes from the fact that the theory of fractional differential equations has fundamentally been attracted by the enormous numbers of interesting and novel applications arising in physics, chemistry, biology, engineering, finance, and other areas which have been developed in the last few decades. control theory, relaxation in filled polymer networks, modeling of viscoelastic materials, heat propagation, modeling of the behavior of viscoelastic and viscoplastic materials under external influences, image processing, description of mechanical systems subject to damping. Also, it is worth pointing out that a completely different and very novel applicable field is the area of mathematical psychology, where fractional-order systems may be used to model the behavior of human beings, more precisely, using fractional operators, a model of memory-dependent phenomena is prepared which is based on human reaction and the external influences depending on the backgrounds that has been made in the past. Hence FC emerged as an important and efficient tool for the study of dynamical systems where classical methods reveal strong limitations.

One of the powerful tools used in the theory of FC is the so-called Measures of noncompactness that constitutes a very important branch of nonlinear functional analysis and are widely applied in fixed point theory and are especially useful in investigations connected with the theories of differential equations, integral equations, functional integral equations of arbitrary order . It finds a lot of applications in operator theory. First of all, it allows us to select very significant class of operators being generalizations of compact operators. Those operators are known as operators satisfying the Darbo condition or contractions with respect to a measure of noncompactness as well as condensing operators.

We will present one of the application of the MNC in FC theory In this thesis, organised as follows :

- **Chapter 1 :** In section 1, We give some preliminaries about functional spaces that we will be using. In section 2, we introduce the notion of a measure of non-compactness and explore some of its properties. In section 3, we review some special function that will be useful. In section 4, we define the Riemann-Liouville fractional integral and derivative.

- **Chapter 2 :** In this chapter we will apply the technique of MNC to investigate the existence of solutions of an infinite system of FDE, than using a semi-analytic method (a modified homotopy perturbation method), we approximate the system's solutions. Providing an illustrative example of the used techniques.

Chapter 1

Preliminaries

1.1 Functional analysis

In this section we give a brief review of some basic definitions, classical theorems and inequalities in functional analysis.

1.1.1 Metric spaces

Definition 1. *a **metric space** is an ordered pair (E, d) where E is a set and d is a metric on E, i.e., a function $d : E \times E \to \mathbb{R}_+$ satisfying the following axioms for all points $x, y, z \in E$:*

(i) $d(x, y) = 0 \iff x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$

If (E, d) is a metric space, then : the open ball with center at x and radius r is denoted by $B(x, r)$: $B(x, r) = \{Y \in E;\ d(x, Y) < r\}$. The closed ball of radius r around x is defined as :

$$\bar{B}(a, r) = \{x \in E;\ d(x, a) \leqslant r\}.$$

A set $A \subset (E, d)$ is said to be **bounded** if there exists $x \in E$ and $M > 0$ such that : $A \subset B(x, M)$ If $A \subset (E, d)$ and $x \in E$, we call **the distance of x from** A- and we denote $d(x, A)$- the real :

$$d(x, A) = \inf\{d(x, a);\ a \in A\}.$$

If $A \subset (E, d)$ is a bounded set, we call **diameter of A** the real :

$$\text{diam}(A) = \sup\{d(a, b);\ a, b \in A\}.$$

ϵ-net : If M and S are subsets of a metric space (X, d) and $\epsilon > 0$, then the set S is called an ϵ-net of M if for any $x \in M$ there exists $s \in S$, such that $d(x, s) < \epsilon$. If the set S is finite, then the ϵ-net S of M is called finite ϵ-net.

Sequences in metric spaces

A sequence (u_n) of (E,d) is **convergent to** $\ell \in E$ if :

$$\forall\, \varepsilon > 0,\, \exists N \in \mathbb{N},\, \forall n \geqslant N,\ d(u_n, \ell) \leqslant \varepsilon.$$

A **subsequence** of a given sequence (x_n) is any other sequence (y_k) that is of the form $y_k = x_{(n_k)}$ where (n_k) is an increasing sequence of natural numbers, i.e. $n_1 < n_2 < n_3 < \cdots$.

A sequence of points x_n in a metric space X **converges** to a point $a \in X$, if for every $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that for all $n \geqslant N_\varepsilon$ we have $x_n \in B_\varepsilon(a)$ We write $\lim_{n\to\infty} x_n = a$ or $x_n \to a$.

Continuity

Let X, Y be two metric spaces, f a map from X to Y and a a point of X. We say that f is continuous in a if :

$$\forall \varepsilon > 0,\, \exists \delta > 0,\, \forall x \in X,\ d(x,a) < \delta \Rightarrow d(f(x), f(a)) < \varepsilon.$$

f is continuous on X if it is continuous at every point of X.

Definition 2. *Let (X,d) and (Y,δ) be two metric spaces and F a family of functions from X to Y. The family $F \in \mathscr{C}(X,Y)$ is* ***equicontinuous*** *if :*

$$\forall x \in X,\ \forall\, \varepsilon > 0,\, \exists \eta > 0,\, \forall f \in F,\ \forall y \in X,\ d(x,y) < \eta \Rightarrow \delta(f(x), f(y)) < \varepsilon.$$

Let X and Y be metric spaces. A family F of functions from X to Y is said to be **equibounded** if there exists a bounded subset B of Y such that for all $f \in F$ and all $x \in X$ it holds: $f(x) \in B$. Notice that if $F \subset \mathscr{C}_b(X,Y)$ (continuous bounded functions) then is F is equibounded if and only if F is bounded (with respect to the metric of uniform convergence).

Let A be a subset of a metric space E. A point a in A is is a **limit point** (or cluster point or accumulation point) of the set A if : every neighbourhood of a contains at least one point of A different from a itself .i.e :

$$\forall r > 0,\ B(a,r) \cap A \neq \emptyset.$$

The **closure** of a subset A of points in a metric space -denoted $\overline{A}$ consists of all points in A together with all limit points of A. The closure of A may equivalently be defined as the union of A and its boundary, and also as the intersection of all closed sets containing A.

We say that l is a **valeur d'adhérence** of the sequence (u_n) if there exists a subsequence of (u_n) that converges to l.

We say that a sequence (x_n) in a metric space (X,d) is a **Cauchy sequence** if :

$$\forall\ \varepsilon > 0, \exists N \in \mathbb{N}, \forall p,q \geqslant N,\ d(u_p,u_q) < \varepsilon$$

Completeness

A metric space E is **complete** if every Cauchy sequence on E converges.

Lim sup and Lim inf

A sequence of real numbers -even bounded- doesn't have to be convergent, for instance : the sequence $(u_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$. The concept of **limit superior** and **limit inferior** makes it look like it's real i.e : $(u_n)_{n\in\mathbb{N}}$ converges !

Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of reals. We set for every $n \in \mathbb{N}$: $v_n = \sup\{u_k : k \geq n\}$. Then $(u_n)_{n\in\mathbb{N}}$ is a decreasing sequence bounded from below, hence it converges to a limit l. The real l is called the **limit superior** of the sequence $(u_n)_{n\in\mathbb{N}}$ and we denote it by $\limsup\limits_{n\to+\infty} u_n$.. Hence we could define the limit superior and the limit inferior of $(u_n)_{n\in\mathbb{N}}$, respectively, as :

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} (\sup_{m\geqslant n} x_m)$$

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} (\inf_{m\geqslant n} x_m)$$

Remark 1. *If $(u_n)_{n\in\mathbb{N}}$ is not bounded, then for all $n \in \mathbb{N}$, we have :* $\sup\{u_k : \ k \geqslant n\} = +\infty$ *and we put :* $\limsup_n u_n = +\infty$.

We define similarly, the limit inferior replacing sup by inf. The limit superior of a sequence (u_n) is the greatest value of the limit points of (u_n) and the limit inferior is the lowest one.

Compactness

A subset M of a metric space X is **compact** if every sequence (x_n) in M has a convergent subsequence, and in this case the limit of that subsequence is in M.

The set M is said to be **relatively compact** if the closure $\overline{M}$ of M is a compact set.

A subset M of a metric space X is **relatively compact** if and only if every sequence (x_n) in M has a convergent subsequence; in that case, the limit of that subsequence need not be in M.

Let X and Y be infinite-dimensional complex Banach spaces. A linear operator L from X to Y is called **compact** (or completely continuous) if $D(L) = X$ for the domain of L, and for every sequence $(x_n) \in X$ such that $\|x_n\| \leqslant c$, the sequence $(L(x_n))$ has a subsequence which converges in Y.

Convexity

Let X be a vector space over the field $\mathbb{R}$. A subset E of X is said to be **convex** if $\lambda x + (1-\lambda)\, y \in E$ for all x, y ∈ E and for all $\lambda \in (0,1)$. The intersection of any family of convex sets is a convex set. If F is a subset of X, then the intersection of all convex sets that contain F is called **convex hull** (or convex cover) of F denoted by $co(F)$.

Definition 3 (Schauder basis)**.** *A Schauder basis in a Banach space X is a sequence $\{e_n\}_{n\geqslant 0}$ of vectors in X with the property that for every vector $x \in X$, there exist uniquely defined scalars $\{x_n\}_{n\geqslant 0}$ depending on x, such that:* $x = \sum_{n=0}^{\infty} x_n e_n$, *i.e.,* $x = \lim_n P_n(x)$, $P_n(x) := \sum_{k=0}^{n} x_k e_k$.

1.1.2 Normed vector spaces

A map $\|\cdot\| : E \to \mathbb{R}_+$ is called a **norm** if it satisfies :

(i) $\forall x \in E,\ \|x\| = 0 \Leftrightarrow x = 0$

(ii) $\forall x \in E,\ \forall \lambda \in \mathbb{F},\ \|\lambda x\| = |\lambda| \cdot \|x\|$

(iii) $\forall x, y \in E, \left\|x + y\right\| \leqslant \|x\| + \left\|y\right\|$

Then we say that $(E, \|.\|)$ is a **normed vector space**. If $(E, \|.\|)$ is a normed vector space, then the map $d : E \times E \to \mathbb{R}_+$ defined by : $d(x, y) = \|x - y\|$ is called the **associated distance** to the norm $\|.\|$.

Banach space

A complete normed vector space is said to be a **Banach space**.

Theorem 1 (Banach–Steinhaus theorem/ Uniform Boundedness Principle)**.** *Let X be a Banach space, Y a normed vector space and $B(X,Y)$ the space of all continuous linear operators from X into Y. Suppose that F is a collection of continuous linear operators from X to Y. If* $\sup_{T\in F} \|T(x)\|_Y < \infty$ *for all $x \in X$,, then :* $\sup_{\substack{T\in F\\ \|x\|=1}} \|T(x)\|_Y = \sup_{T\in F} \|T\|_{B(X,Y)} < \infty$

Minkowski inequality

Let $p \in [1, +\infty[$ and $u_1, \ldots, u_n,\ v_1, \ldots, v_n$ complex numbers. Then :

$$\left(\sum_{k=1}^{n} |u_k + v_k|^p\right)^{1/p} \leqslant \left(\sum_{k=1}^{n} |u_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |v_k|^p\right)^{1/p}.$$

Hölder's inequality

Let $p, q \in [1, +\infty[$ such that: $1/p + 1/q = 1$.. If $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are complex numbers, then :

$$\sum_{k=1}^{n} |u_k||v_k| \leqslant \left(\sum_{k=1}^{n} |u_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |v_k|^q\right)^{1/q}.$$

1.1.3 Sequence spaces

A **sequence space** is a vector space whose elements are infinite sequences of real or complex numbers. Equivalently, it is a function space whose elements are functions from the natural numbers to the field K of real or complex numbers. Some of the classical sequence spaces are (which are Banach spaces with their respective norms) :

- The space of all bounded sequences:

$$\ell_\infty := \left\{ x \in w : \sup_k |x_k| < \infty ;\ \|x\|_\infty = \sup_k |x_k| \right\}$$

- the space of all absolutely p-summable series :

$$\ell_p := \left\{ x \in w : \sum_k |x_k|^p < \infty ;\ \|x\|_p = \left(\sum_k |x_k|^p\right)^{\frac{1}{p}},\ 1 \leqslant p < \infty \right\}$$

Where w is the space of all real or complex sequences $x = (x_k)_{k=1}^{\infty}$

- Let X be a linear space and d a metric on X. Then d is called **translation invariant** if: $d(x+z, y+z) = d(x, y)$ for all x,y,z in X.

- Let X be a linear space which is also a metric space with the translation invariant metric d on X. Then (X,d) or X for short, is said to be a **linear metric space**, if the algebraic operations on X are continuous functions. That is, that X is a linear metric space if and only if it is both a linear and a metric space such that

(i) the distance between any two points are preserved under the identical translations of these points.

(ii) the vector addition map (x,y) $\mapsto x+y$ is a continuous function from $X \times X$ into X, and

(iii) the scalar multiplication map (λ , x) $\mapsto \lambda x$ is a continuous function from $\mathbb{R} \times X$ into X.

- A sequence space X with linear topology is called a **K space** if each of the maps $p_n : X \to \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous, for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$.

- A complete linear metric space is called a **Fréchet space**.

- K space X is called an **FK space** if X is a complete linear metric space, that is, X is an FK space if X is Fréchet space with continuous coordinates (i.e., if $x^{(n)} \xrightarrow[n\to\infty]{} x$ in the metric of X then ${x_k}^{(n)} \xrightarrow[n\to\infty]{} x_k$ for each k).

- A normed FK space is called a **BK space**, that is, a BK space is a Banach sequence space with continuous coordinates .

- An FK space X is said to **have AK** (The notation AK arises from the German words Abschnittskonvergenz (for Sectional Convergence) if every sequence $x = (x_k) \in X$ has a unique representation $x = \sum_{k=1}^{\infty} {x_k}^{(k)} e^{(k)}$, that is, $\lim_{n\to\infty} \sum_{k=1}^{n} {x_k}^{(k)} e^{(k)} = x$ .This means that $\{e^{(k)}\}_{k=1}^{\infty}$ is a Schauder basis for any FK space with AK such that every sequence, in an FK space with AK, coincides with its sequence of coefficients with respect to this basis. Example: ℓ_p $(1 \leq p < \infty)$ have AK, but ℓ_∞ do not have AK.

- **monotone norm** We say that a norm ;$\|.\|$on a sequence space is monotone if $x, \tilde{x} \in X$ with $|x_k| \leq |\tilde{x}_k|$ for all k, implies: $\|x\| \leq \|\tilde{x}\|$. Example: The standard unit vector bases of ℓ_p for $1 \leq p < \infty$, are monotone Schauder bases. In this unit vector basis $\{x_n\}_{n\geq 0}$,the vector x_n in ℓ_p is the scalar sequence $\left[x_{n,i}\right]_i$ where all coordinates $x_{n,i}$ are 0 except the n-th coordinate: $x_n = \{x_{n,i}\}_{i=0}^{\infty} \in \ell_p,\ x_{n,i} = \delta_{n,i}, where \delta_{n,i} is the Kronec$ker $delta$.

The space ℓ_∞ is not separable, and therefore has no Schauder basis.

1.2 Measure of non-compactness

1.2.1 Basic concept

The measure of non-compactness (MNC) is a positive real valued function defined on the set of all bounded subsets of a metric space that measures the distance between a set and the set of relatively compact sets, in other words it measures the degree of non-compactness.**?** i.e : how far is the set from being compact. The notion of measure of non compactness was first introduced by Kuratowski in 1930.

Definition 4 (1)**.** *Let $d_H : \mathcal{M}_X \times \mathcal{M}_X \to \mathbb{R}$ be the function defined by :*

$$d_H(S,\ Q) = \max\left\{ \sup_{s\in S} d(s,Q), \sup_{q\in Q} d(q,S) \right\}; S,Q \in \mathcal{M}_X.$$

*The function d_H is called the **Hausdorff distance**, and $d_H(S,\ Q)$ is the Hausdorff distance between S and Q. If $\phi \neq F \subset X$ and $B(F,r) = \bigcup_{x\in F} B(x,r) = \{ y \in X : d(y,F) < r\,; r > 0\}$ is the open ball with center in F and radius r, then :*

$$d_H(S,\ Q) = \inf\{ \varepsilon > 0 : S \subset B(Q,\varepsilon)\ and\ Q \subset B(S,\varepsilon)\}$$

Definition 5 (The Kuratowski Measure of Non-compactness : KMNC)**.** *[1] Let (X,d) be a metric space and Q a bounded subset of X. Then the Kuratowski (set) measure of non-compactness of Q, denoted by $\alpha(Q)$,is the infimum of the set of all numbers $\epsilon > 0$ such that Q can be covered by a finite number of sets with diameters $\epsilon > 0$, that is :*

$$\alpha(Q) = \inf\left\{ \varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} S_i : S_i \subset X, diam(S_i) < \varepsilon,\ (i = 1,..,n); n \in \mathbb{N} \right\}$$

In general the computation of α is difficult, another MNC which is more convenient is the The Hausdorff Measure of Non-compactness (HMNC).

Definition 6 (The Hausdorff Measure of Non-compactness)**.** *[1,6] Let (X,d) be a metric space* *and Q a bounded subset of X. Then the Hausdorff measure of non-compactness (or ball measure of non-compactness) of the set Q ,denoted by $\chi(Q)$, is defined to be the infimum of the set of all reals $\epsilon > 0$ such that Q can be covered by a finite number of balls of radii $< \epsilon$, that is:*

$$\chi(Q) = \inf\left\{ \varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i) : x_i \in X, r_i < \varepsilon,\ (i = 1,..,n); n \in \mathbb{N} \right\}$$

Remark 2. *From the definition of the HMNC we have :*

$$\forall\ q\ \in\ Q, \exists i \in \mathbb{N} : q \in B(x_i, r_i) \Rightarrow \forall\ q\ \in\ Q, \exists x_i \in X : d\left(x_i, q\right) \leqslant r_i \leqslant \varepsilon.$$

Hence: $\{x_i, i \in \mathbb{N}\}$ $is\ a\ finite\ \varepsilon-net\ of\ Q$, since the x_i's were not supposed to belong to Q, we can state the definition of the HMNC equivalently :

$$\chi(Q) = \inf\left\{ \varepsilon > 0 : Q\ has\ a\ finite\ \varepsilon - net\ in\ X \right\}$$

Proposition 1 (1)**.** *Let (X, d) be a metric space.$Q, Q_1,\ Q_2 \in \mathcal{M}_X$ and $\mathcal{N}_X^c$ be the set of all non-empty and compact subsets of (X, d). Then :*

$$\left|\chi\left(Q_1\right) - \chi\left(Q_2\right)\right| \leqslant d_H\left(Q_1,\ Q_2\right)$$

$$\chi\left(Q\right) = d_H\left(Q,\ \mathcal{N}_X^c\right)$$

Where d_H is Hausdorff distance.

The HMNC shares similar properties with the KMNC with some additional properties suitable in applications.

Proprieties 1. *.[1,6] Let Q, Q_1 and Q_2 be bounded subsets of the metric space (X, d), Then :*

1. *The family* $\ker\chi = \left\{ Q \in \mathcal{M}_X : \chi\left(Q\right) = 0\right\}$ *is non-empty and* $\ker\chi \subset \mathcal{N}_X$.
2. *Regularity :* $\chi(Q) = 0 \Leftrightarrow$ *Q is a precompact set.*
3. *nonsingularity :* χ *is equal to zero on every single-element set.*

4. *Invariant under closure* : .$\chi(Q) = \chi(\bar{Q})$

5. *monotonicity:* $Q_1 \subset Q_2 \Rightarrow \chi(Q_1) \leq \chi(Q_2)$.

6. *Semi-additivity:* $\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}$.

7. $\chi(Q_1 \cap Q_2) \leqslant \min\{\chi(Q_1), \chi(Q_2)\}$.

8. *Kuratowski's intersection theorem (is a generalisation of Cantor's intersection theorem. Whereas Cantor's result requires that the sets involved be compact, Kuratowski's result allows them to be non-compact, but insists that their non-compactness "tends to zero" in an appropriate sense.): If* (X_n) *is a sequence of closed sets from* M_E *such that* $X_{n+1} \subset X_n$ *for* $n = 1, 2, ...$ *and if* $lim_{n\to\infty}\mu(X_n) = 0$ *, then the set* $X = \bigcap_{n=1}^{\infty} X_n$ *is a non-empty compact set.*

■ *If X is normed, then :*

9. *(algebraic subadditivity):* $\chi(Q_1 + Q_2) \leqslant \chi(Q_1) + \chi(Q_2)$

10. *(invariance under translations):* $\chi(Q + x) = \chi(Q), \forall x \in X$.

11. *(semihomogeneity):* $\chi(\lambda Q) = |\lambda|\,\chi(Q), \forall \lambda \in \mathbb{F}$.

12. *invariant under passage to the convex hull:* $\chi(Q) = \chi(co(Q))$.

13. *Convexity* : $\chi(\lambda Q_1 + (1-\lambda) Q_2) \leqslant \lambda\chi(Q_1) + (1-\lambda)\chi(Q_2)$; $\lambda \in [0,1]$.

14. *Lipschitzianity:* $\left|\chi(Q_1) - \chi(Q_2)\right| \leqslant \rho(Q_1, Q_2)$, *where* ρ *denotes the Hausdorff semimetric defined by:* $\rho(Q_1, Q_2) = \inf\{\varepsilon > 0 : Q_1 \subset Q_2 + \varepsilon\bar{B}, Q_2 \subset Q_1 + \varepsilon\bar{B}\}$.

15. $\chi(Q) = \rho(Q, \mathcal{N}_X^c)$.

16. *Continuity:* $\forall\varepsilon > 0, \exists\delta > 0 : \rho(Q_1, Q_2) < \delta \Rightarrow \left|\chi(Q_1) - \chi(Q_2)\right| < \varepsilon$.

17. *If X is infinite-dimensional:* $\chi(B_X) = 1$. *(We note that these measures of non-compactness are useless for subsets of finite-dimensional spaces like the Euclidean space* $\mathbb{R}^n$*: by the*

Heine–Borel theorem, every bounded closed set is compact there, which means that $\chi(Q) = 0$ *if Q is bounded ,*∞ *otherwise.Measures of non-compactness are however useful in the study of infinite-dimensional Banach spaces.*

18. $\chi(B(A,r)) = \chi(A+r)$, *where* $B(A,r) = \bigcup_{x\in A} B(x,r)$.

1.2.2 Computation of the HMNC

Let X be a Banach space with a Schauder basis $\{e_1, e_2, ...\}$.Then each element $x \in X$ has a unique representation $x = \sum_{i=1}^{\infty} \phi_i(x)\, e_i$ where the functions ϕ_i are the basis functionals. Let $p_n : X \to X$ be the projector onto the linear span of $\{e_1, e_2, .., e_n\}$,that is $p_n(x) = \sum_{i=1}^{n} \phi_i(x)\, e_i$ .Then, by Banach–Steinhaus theorem, all operators p_n and $I - p_n$ are equibounded.

Theorem 2. *(2) Let X be a BK space with Schauder basis* (b_n), $Q \in \mathcal{M}_X$, $P_n : X \to X\,(n \in \mathbb{N}))$ *be the projector onto the linear span of* $\{e_1, e_2, .., e_n\}$ *and I be the identity operator on X.Then:*

$$\frac{1}{a} \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I-P_n)(x)\| \right) \leq \chi(Q) \leq \inf_n \left(\sup_{x\in Q} \|(I-P_n)(x)\| \right) \leq \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I-P_n)(x)\| \right)$$

Where: $a = \limsup_{n\to\infty} \|I - P\|$.

Proof. For all $n \in \mathbb{N}$ we have $Q \subset P_n Q + (I - P_n) Q$ .It follows that :

$$\chi(Q) \leq \chi(P_n Q) + \chi((I-P_n)Q) = \chi((I-P_n)Q) \leq \sup_{x\in Q} \|(I-P_n)(x)\|.$$

We obtain :

$$\chi(Q) \leq \inf_n \left(\sup_{x\in Q} \|(I-P_n)(x)\| \right) \leq \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I-P_n)(x)\| \right).$$

Now let $\varepsilon > 0$ *and* $\{z_1, ...z_k\}$ *be a* $[\chi(Q)+\varepsilon]$ *− net of* Q. *Since* $Q \subset \{z_1, ...z_k\} + [\chi(Q)+\varepsilon]B_X$,

this implies that for any x ∈ Q , there exist z ∈ $\{z_1, ...z_k\}$ *and* $s \in B_X$ such that

$x = z + [\chi(Q)+\varepsilon]\, s$, *and so* :

$$\sup_{x\in Q} \|(I-P_n)(x)\| \leqslant \sup_{1\leqslant i\leqslant k} \|(I-P_n)(z_i)\| + [\chi(Q)+\varepsilon]\,\|(I-P_n)\|$$

This yields : $\limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right) \leqslant \left(\chi(Q)+\varepsilon\right)\limsup_{n\to\infty}\|(I-P_n)\|$.

hence $\frac{1}{a}\limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right) \leqslant \chi(Q)+\varepsilon$.

□

The next theorem gives an explicit expression of the Hausdorff measure of non-compactness for any bounded subset of an arbitrary monotone BK space with AK. This indicates why the Hausdorff measure of non compactness is the most suitable in applications.

Theorem 3. *[1] Let* X *be a BK space with AK and monotone norm,* $Q \in \mathcal{M}_X$ *and* $p_{n(n\in\mathbb{N})}$: $X \to X$ *be the operator (projection) defined by :* $p_n(x_1, x_2, ..) = x^{[n]} = (x_1, x_2, .., x_n, 0, 0, ..)$ *for all* $x = (x_1, x_2, ..) \in X$. *Then:*

$$\chi(Q) = \lim_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right)$$

Proof. For $Q \in \mathcal{M}_X$ and a fixed $n \in \mathbb{N}$, we put $\mu_n = \mu_n(Q) = \sup_{x\in Q}\|(I-P_n)(x)\|$. Since X is a monotone BK space with Ak, it follows that :

$$\|(I-P_n)(x)\| = \left\|x - x^{[n]}\right\| \geqslant \left\|x - x^{[n+1]}\right\| = \|(I-P_{n+1})(x)\|.$$

Let $n \in \mathbb{N}_0$ and $\epsilon > 0$ be given.Then, there is a sequence $x^0 \in Q$ such that :

$$\left\|(I-P_{n+1})(x^0)\right\| \geqslant \mu_{n+1} - \varepsilon$$

Hence :

$$\mu_n(Q) \geqslant \left\|(I-P_n)(x^0)\right\| \geqslant \left\|(I-P_{n+1})(x^0)\right\| \geqslant \mu_{n+1} - \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we have: $\mu_n(Q) \geqslant \mu_{n+1}(Q) \geqslant 0$, for all $n \in \mathbb{N}$ and so $\lim_{n\to\infty} \mu_n(Q)$ exists for all $Q \in \mathcal{M}_X$ and we can replace $limsup$ by lim in theorem 2. It remains to show that $\limsup_{n\to\infty} \|I - P_n\| = 1$. Since $\|.\|$ is monotone, $\|(I - P_n)(x)\| = \left\|x - x^{[n]}\right\| \leqslant \|x\|$ for all x ∈ X and all n ∈ ℕ, hence $\|I - P_n\| \leqslant 1$ for all n. To show the converse inequality, let $n \in \mathbb{N}$ be given. We obtain $\left\|(I - P_n)(e^{n+1})\right\| = \left\|e^{n+1}\right\| \neq 0$, which implies : $\|I - P_n\| \geqslant 1$ for all n. □

Since $\ell_p \left(1 \leqslant p < \infty\right)$ is a BK space with AK and monotone norm $\|.\|_p$, the HMNC in ℓ_p is given by :

Theorem 4. *[9] Let Q be a bounded subset of* $X = \ell_p \; for \; \left(1 \leqslant p < \infty\right)$. *Since* $\left\{ e^{(1)}, e^{(2)}, ...\right\}$ *is a Schauder basis for* $\ell_p \; \left(1 \leqslant p < \infty\right)$ *then :*

$$\chi_{\ell_p}(Q) = \lim_{n\to\infty} \sup_{x\in Q} \left(\sum_{k\geqslant n} |x_k|^p\right)^{\frac{1}{p}}$$

Proof. By theorem 3 :

$$\chi_{\ell_p}(Q) = \lim_{n\to\infty} \left(\sup_{x\in Q} \|(Id - P_n)(x)\|_{\ell_p}\right) = \lim_{n\to\infty} \left(\sup_{x\in Q} \|(0,0,..,x_{n+1}..)\|_{\ell_p}\right) = \lim_{n\to\infty} \left(\sup_{x\in Q} (\sum_{k>n} |x_k|^p)^{\frac{1}{p}}\right).$$

□

1.2.3 Measure of Non-compactness of Operators

Definition 7 (1,6)**.** *Let* μ_1 *and* μ_2 *be measures of non-compactness defined on Banach spaces E and F,respectively. Let* $L : E \to F$ *be an operator. Then :*

- *L is called* $\left(\mu_1, \mu_2\right)$*−contractive operator with constant* $k > 0$ *(or simply* $k - \left(\mu_1, \mu_2\right)$*-contractive) if L is continuous and*

$$\mu_2\left(L(Q)\right) \leqslant k\mu_1\left(Q\right) \; for\, each\, Q \in \mathcal{M}_E.$$

In particular, if $E = F$ *and* $\mu_1 = \mu_2 = \mu$ *then we say that L is a k-μ-contractive operator.*

- *L is called (μ_1, μ_2)-condensing operator with constant $k > 0$ (or simply $k-(\mu_1,\mu_2)$–condensing) if L is continuous and*

$$\mu_2\,(L(Q)) < k\mu_1\,(Q) \text{ for each non-precompact } Q \in \mathcal{M}_E.$$

In particular, if $E = F$ and $\mu_1 = \mu_2 = \mu$ then we say that L is a k-μ-condensing operator. Moreover, if $k = 1$, we say that L is a μ-condensing operator.

1.2.4 Darbo's fixed point theorem

One of the most used fixed point theorems in proving existence results for functional equations is Schauder fixed point theorem which asserts that every continuous self-mapping on a nonempty, convex and compact subset of a Banach spaceEhas at least one fixed point. The main difficulty in applying this theorem lies in finding a convex and compact subset of E, which is transformed into itself by a continuous operator that depends on the considered equation. In order to overcome these difficulties, one of the possible strategies is the use of techniques associated with the concept of the measure of noncompactness

Darbo formulated his celebrated fixed point theorem in 1955 for the case of the Kuratowski measure of non-compactness. This was the first theorem that involves the notion of measure of non-compactness.

Theorem 5. *[1,6] Let E be a Banach space, $M \subset E$ a non-empty, closed, convex, bounded subset, and $T : M \to M$ a μ-condensing operator. Then T has at least one fixed point and the set of fixed points of T belongs to $\ker \mu$, where μ is an arbitrary measure of non-compactness.*

Remark 3. *Schauder's theorem extends Brouwer's theorem (from $\mathbb{R}^n$ to infinite-dimensional Banach spaces) and Darbo's theorem extends Schauder's theorem(from compact to condensing operators).*

1.3 special functions

1.3.1 Gamma function

Definition 8 (3)**.** *If the real part of the complex number z is strictly positive* $(Re(z) > 0)$*, then the integral* $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$ *converges absolutely and is known as the Euler integral of the second kind or* Γ *function.*

Proprieties 2.

1. $\Gamma(1) = 1,\ \Gamma\left(0^+\right) = +\infty$.
2. $\Gamma(z+1) = z\Gamma(z), z \in \mathbb{C}$
3. $\Gamma(n+1) = n!, n \in \mathbb{N}$

1.3.2 Incomplete gamma function

Definition 9 (3)**.** *Let s be a complex parameter, such that the real part of s is positive.*

The upper incomplete gamma function is defined as:

$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}dt,$$

The lower incomplete gamma function is defined as:

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt.$$

1.3.3 Beta function

Definition 10 (3)**.** *the beta function, also called the Euler integral of the first kind, is a special function that is closely related to the gamma function and to binomial coefficients. It is defined by the integral:*

$$B(z_1,z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}dt$$

for complex number inputs z_1, z_2 *such that* $\mathrm{Re}(z_1), \mathrm{Re}(z_2) > 0$.

Proprieties 3 (3)**.**

1. $B\left(\alpha,\beta\right) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
2. $B\left(\alpha,\beta\right) = B\left(\beta,\alpha\right)$
3. $\alpha B\left(\alpha,\beta+1\right) = \beta B\left(\alpha+1,\beta\right)$

1.4 Fractional calculus

Let $f:[a,b[\rightarrow\mathbb{R}$ be a continuous function, then a primitive of f is given by :$(If)(x)=\int\limits_{0}^{x}f(t)dt$. For a sec ond primitive we have : $I^2f(x)=\int_a^x dt\int_a^t f(u)du$, using Fubini : $I^2f(x)=\int_a^x(x-t)f(t)dt$. By induction we get the Cauchy formula for repeted integration : $\left(I^nf\right)(x)=\frac{1}{(n-1)!}\int\limits_{a}^{x}(x-t)^{n-1}f(t)dt$, which leads to a generalization for real n. Using the gamma function to remove the discrete nature of the factorial function gives us a natural candidate for fractional applications of the integral operator :

$$\left(I^nf\right)(x)=\frac{1}{(\alpha-1)}\int\limits_{a}^{x}(x-t)^{\alpha-1}f(t)dt.$$

1.4.1 Riemann-Liouville Fractional Integrals and Fractional Derivatives

Definition 11 (3)**.** *Let $f\in\mathscr{C}([a,b]),(-\infty<a<b<+\infty)$ and $\alpha\in\mathbb{R}_+^*$. The Riemann Liouville left-sided and right-sided integral are defined respectively by :*

$$_{a^+}I_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-\tau)^{\alpha-1}f(\tau)d\tau\,;\quad t>a.$$

$$_{b^-}I_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_t^b(\tau-t)^{\alpha-1}f(\tau)d\tau\,;\quad t<b.$$

Definition 12 (3)**.** *The Riemann Liouville left-sided and right-sided derivatives of order $\alpha\in\mathbb{C}$ $(\mathrm{Re}(\alpha)\geqslant0)$ are defined respectively by :*

$$(D_{a^+}^\alpha y)(x):=\left(\frac{d}{dx}\right)^n(I_{a^+}^{n-\alpha}y)(x)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int_a^x\frac{y(t)dt}{(x-t)^{\alpha-n+1}}\,;x>a.$$

$$(D_{b^-}^\alpha y)(x):=\left(-\frac{d}{dx}\right)^n(I_{b^-}^{n-\alpha}y)(x)=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n\int_x^b\frac{y(t)dt}{(t-x)^{\alpha-n+1}}\,;x<b.$$

Proprieties 4 (3). 1. *Let* $f \in L^p([a,b])$, $p \geqslant 1, \alpha > 0, \beta > 0$. *Then :*

$$I^{\alpha}_{a^+} I^{\beta}_{a^+} f(x) = I^{\alpha+\beta}_{a^+} f(x)$$

2. ${}^{RL}D^{\alpha}_{a^+}$ *is a linear operator.*

3. .$\left({}^{RL}D^{\alpha}_{a^+}\right)\left({}^{RL}I^{\alpha}_{a^+}\right)(t) = f(t)$

Example 1. *Riemann - Liouville right - sided integral of* $f(t) = (t-a)^{\beta}$:

${}^{RL}I^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}(s-a)^{\beta} ds,\ via\ change\ of\ \text{var}iable\ s = a + (t-a)\ s :$
${}^{RL}I^{\alpha}_{a^+}(t-a)^{\beta} = \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} t^{\beta} dt = \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)}\ B\left(\alpha, \beta+1\right)$
$= \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)} \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\ (t-a)^{\alpha+\beta}.$

Example 2. *The Riemann - Liouville derivative of* $the\ function\ f(x) = (x-a)^{\beta}$, $\left(\alpha > 0\,, \beta \geqslant 0\right)$:

${}^{RL}D^{\alpha}_{a^+} f(x) = {}^{RL}D^{\alpha}_{a^+}(x-a)^{\beta} = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{(t-a)^{\beta} dt}{(x-t)^{\alpha-n+1}}$,
$via\ change\ of\ \text{var}iable\ t = a + (x-a)\ s\ we\ get :$
${}^{RL}D^{\alpha}_{a^+}(x-a)^{\beta} = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n (x-a)^{n+\beta-\alpha} \int_a^x (1-s)^{n-\alpha+1}\ s^{\beta} ds$
$= \frac{\Gamma(n+\beta-\alpha+1) B(n-\alpha,\beta+1)}{\Gamma(n-\alpha)}\ \left(x-\beta\right)^{\beta-\alpha}$
$= \frac{\Gamma(n+\beta-\alpha+1)\Gamma(n-\alpha)\Gamma(\beta+1)}{\Gamma(n-\alpha)\,\Gamma(\beta-\alpha+1)\,\Gamma(n+\beta-\alpha+1)}\ \left(x-\beta\right)^{\beta-\alpha}$
$= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\ (x-a)^{\beta-\alpha}, (a \leqslant x \leqslant b).$
$In\ the\ same\ way\ we\ obtain :\ {}^{RL}D^{\alpha}_{b^-}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\ (b-x)^{\beta-\alpha}.$

Remark 4. *In particular, if* $\beta = 0$ and $\text{Re}(\alpha) \geqslant 0$, *then the Riemann-Liouville fractional derivatives of a constant are, in general, not equal to zero:* $(D^{\alpha}_{a^+}\ 1)(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$.

lemma 1 (3). *For* $\alpha > 0$, *the general solution of the fractional differential equation :* ${}^{RL}D^{\alpha}_0 f(t) = 0$ *with* $f \in \mathscr{C}(0,t) \cap L^1_{loc}(0,\infty)$ *is given by :*

$$f(t) = c_1 t^{\alpha-1} + \ldots + c_n t^{\alpha-n}.\ \left(c_{i=1,\ldots,n} \in \mathbb{R}\right)$$

Proof.

$$If\ {}^{RL}D^{\alpha}_{a^+}f(t)=0\ then\ f(t)=\sum_{k=0}^{n-1}c_k\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1-n)}\ (t-a)^{k+\alpha-n},(c_k)_{k=0,\dots m-1}\in\mathbb{R}.$$

${}^{RL}D^{\alpha}_{a^+}f(t)=\left(\frac{d}{dt}\right)^n\left({}^{RL}I^{n-\alpha}_{a^+}f\right)(t)=0$, $this\ means\ the\ usual\ derivative\ of\ order\ n\ of$ ${}^{RL}I^{n-\alpha}_{a^+}f\ is\ null, hence: \left({}^{RL}I^{n-\alpha}_{a^+}f\right)(t)=\sum_{k=0}^{n-1}c_k(t-a)^k, applying\ {}^{RL}I^{\alpha}_{a^+}\ on\ both\ sides\ gives:$

$\left({}^{RL}I^{n}_{a^+}f\right)(t)=\left({}^{RL}I^{\alpha+(n-\alpha)}_{a^+}f\right)(t)={}^{RL}I^{\alpha}_{a^+}\left(\sum_{k=0}^{n-1}c_k(t-a)^k\right)=\sum_{k=0}^{n-1}c_k\left({}^{RL}I^{\alpha}_{a^+}\left((t-a)^k\right)\right)=$

$\sum_{k=0}^{n-1}c_k\left(\frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}\ (t-a)^{\alpha+k}\right), applying \left(\frac{d}{dt}\right)^n\ on\ both\ sides: f(t)=\left(\frac{d}{dt}\right)^n\left({}^{RL}I^{n}_{a^+}f\right)(t)$

$=\sum_{k=0}^{n-1}c_k\left(\frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}\left(\frac{d}{dt}\right)^n(t-a)^{\alpha+k}\right)=\sum_{k=0}^{n-1}c_k\left(\frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}\ \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+1-n)}\ (t-a)^{\alpha+k-n}\right)$

$=\sum_{k=0}^{n-1}c_k\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1-n)}\ (t-a)^{\alpha+k-n}.$

$$for\ a=0\ and\ i=n-k\ we\ have: f(t)=\sum_{i=1}^{n}c_i\frac{\Gamma(n-i+1)}{\Gamma(-i+\alpha+1)}\ t^{\alpha-i}=\sum_{i=1}^{n}C_i\ t^{\alpha-i}.$$

□

Theorem 6. *[8] Let $f(t)\in\mathscr{C}[0,a]$ and $1<\alpha<2$, then the unique solution of* $\begin{cases} D^{\alpha}z(t)=f(t),;t\in[0,a] \\ z(0)=z(a)=0\end{cases}$ *is given by : $z(t)=\int_0^a U(t,s)f(s)\,ds$, where*

$$U(t,s)=\begin{cases}\frac{-1}{a^{\alpha-1}\Gamma(\alpha)}\left(t^{\alpha-1}(a-s)^{\alpha-1}-a^{\alpha-1}(t-s)^{\alpha-1}\right),\ 0\leqslant s\leqslant t.\\ \frac{-1}{a^{\alpha-1}\Gamma(\alpha)}(a-s)^{\alpha-1},\quad t\leqslant s\leqslant a.\end{cases}$$

is the green function.

Proof. $\mathrm{D}^{\alpha}z(t)=f(t)\Leftrightarrow \mathrm{D}^{\alpha}z(t)-f(t)=0\Leftrightarrow \mathrm{D}^{\alpha}z(t)-\mathrm{D}^{\alpha}I^{\alpha}f(t)=0\Leftrightarrow \mathrm{D}^{\alpha}\left(z(t)-I^{\alpha}f(t)\right)=0.$

$By\ the\ previous\ lemma\ : z(t)-I^{\alpha}f(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}$; $c_1,c_2\in\mathbb{R}$, $then\ z(t)=I^{\alpha}f(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}.$

$The\ boundary\ conditions\ give\ c_1=\frac{-1}{a^{\alpha-1}\Gamma(\alpha)}\int_0^a(a-s)^{\alpha-1}f(s)\,ds\ and\ c_2=0. Hence:$

$z(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\,ds+\frac{-t^{\alpha-1}}{a^{\alpha-1}\Gamma(\alpha)}\int_0^a(a-s)^{\alpha-1}f(s)\,ds$

$$= \int_0^t \left(\frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} - \frac{t^{\alpha-1}}{a^{\alpha-1}\Gamma(\alpha)} (a-s)^{\alpha-1} \right) f(s)\, ds + \int_t^a \frac{-t^{\alpha-1}}{a^{\alpha-1}\Gamma(\alpha)} (a-s)^{\alpha-1} f(s)\, ds$$

$$= \int_0^t \left(\left(\frac{1}{a^{\alpha-1}\Gamma(\alpha)} \right) \left(-a^{\alpha-1}(t-s)^{\alpha-1} + t^{\alpha-1}(a-s)^{\alpha-1} \right) \right) f(s)\, ds + \int_t^a \frac{-t^{\alpha-1}}{a^{\alpha-1}\Gamma(\alpha)} (a-s)^{\alpha-1} f(s)\, ds.$$

□

Chapter 2

MNC on tempered sequence spaces and its application to FDE

Infinite systems of differential equations have many applications in real world problems: the theory of neural nets, the theory of branching processes, the theory of dissociation of polymers among many others. Moreover, when we consider some problems of partial differential equations, we can use the process of semi-discretization to transform those problems into infinite systems of differential equations.

2.1 Tempered sequence spaces

[8] Consider the infinite system of ordinary differential equations which can be written in the general form :

$$x_n' = f_n(t, x_1, x_2, ...) \; ; \; t \in I = [0, T], \; n = 1, 2, ... \tag{2.1.1}$$

The Cauchy problem for the above system can be formulated as the initial value conditions :

$$x_n(0) = x_n^0 \; ; \; n = 1, 2, ... \tag{2.1.2}$$

The solutions of (2.1.1,2.1.2) has the form of a function sequence :

$$x(t) = (x_n(t)) = (x_1(t), x_2(t), ...)$$

Thus, for each fixed $t \in I$ the sequence $(x_n(t))$ presents certain sequence of real numbers. Therefore, we consider the solvability of problem (2.1.1,2.1.2) in some sequence space. But even in simple situations, the classical sequence spaces fail to be suitable for the investigation of our problem .

Example 3. *Taking $x_n(0) = n$ in 2.1.2, then the solution of (2.1.1,2.1.2) is given by : $x(t) = (x_n(t)) = (ne^t) = (e^t, 2e^t, ...)$. This means: $x(t) \notin \ell_\infty$ for every $t \in I$. Here the situation seems quite naturally since the initial value $x_n^0 \notin \ell_\infty$.*

The above example suggests that we have to enlarge the spaces under considerations to ensure that solutions of infinite systems of differential equations starting from a point in such a space remain in the space in question when t runs over some interval I. It seems that a natural way to realize the enlargement is to consider **tempered sequence spaces**. Those spaces can be obtained from classical sequence spaces with the help of a tempering sequence. For example, if we take the tempering sequence $(\beta_n) = \left(\frac{1}{n}\right)$, $n = 1, 2, ..$ then the space ℓ_∞^β is the space of all sequences (x_n) such that the sequence $(\beta_n x_n) = \left(\frac{1}{n} x_n\right)$ is bounded.

Definition 13 (9)**.** *For* $1 \leqslant p < \infty$*, the tempered space* ℓ_p^α *is the space of all real or complex sequences* $x = (x_i)_{i\in\mathbb{N}^*}$ *such that* $\sum_{i=1}^{\infty} \alpha_i^p |x_i|^p < \infty$*, where* $\alpha = (\alpha_i)_{i\in\mathbb{N}^*}$ *is a fixed positive non-increasing real sequence called tempering sequence.*
On ℓ_p^α *we define the map* $\|.\|_{\ell_p^\alpha} : x \mapsto \|x\|_{\ell_p^\alpha} = \left(\sum_{i=1}^{\infty} \alpha_i^p |x_i|^p\right)^{\frac{1}{p}}$.

Proposition 2. $(\ell_p^\alpha, \|.\|_{\ell_p^\alpha})$ *is a banach space for* $1 \leqslant p < \infty$.

Proof. For x, y in ℓ_p^α we have :

$$\|x+y\|_{\ell_p^\alpha} = \left(\sum_{i=1}^{\infty} \alpha_i^p |x_i + y_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{\infty} (|\alpha_i|\,|x_i + y_i|)^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{\infty} |\alpha_i (x_i + y_i)|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^{\infty} |\alpha_i x_i + \alpha_i y_i|^p\right)^{\frac{1}{p}} \underbrace{\leqslant}_{Minkowski\ inequality} \left(\sum_{i=1}^{\infty} |\alpha_i x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |\alpha_i y_i|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^{\infty} \alpha_i^p |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} \alpha_i^p |y_i|^p\right)^{\frac{1}{p}} = \|x\|_{\ell_p^\alpha} + \|y\|_{\ell_p^\alpha}.$$

Hence $\|.\|_{\ell_p^\alpha}$ is a norm.

Now let $\left(x^{(k)}\right)_{k\in\mathbb{N}}$ be a Cauchy sequence in ℓ_p^α then:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} :\ \forall\ k, l \geqslant N :\ \left\|x^{(k)} - x^{(l)}\right\|_{\ell_p^\alpha} < \varepsilon.$$

Hence :

$$\alpha_i{}^p \left|x_i^{(k)} - x_i^{(l)}\right|^p \leqslant \sum_{i=1}^{\infty} \left|\alpha_i x_i^{(k)} - \alpha_i x_i^{(l)}\right|^p = \sum_{i=1}^{\infty} \alpha_i^p \left|x_i^{(k)} - x_i^{(l)}\right|^p = \left\|x^{(k)} - x^{(l)}\right\|^p_{\ell_p^\alpha} < \varepsilon^p.$$

So for fixed i $\left(x_i^{(l)}\right)_{l\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{F},|.|)$ complete, so: $\lim_{l\to\infty} x_i^{(l)} = \tilde{x}_i \in \mathbb{F}$. Set $(\tilde{x}_i)_{i\in\mathbb{N}^*} = \tilde{x}$ then : $\forall\varepsilon>0, \exists N\in\mathbb{N} : \forall k,l \geqslant N$:

$$\left\|x^{(k)}-\tilde{x}\right\|_{\ell_p^\alpha}^p = \sum_{i=1}^{\infty}\alpha_i^p\left|x_i^{(k)}-\tilde{x}_i\right| = \sum_{i=1}^{\infty}\alpha_i^p\left|x_i^{(k)}-\lim_{l\to\infty}x_i^{(l)}\right|^p = \lim_{l\to\infty}\sum_{i=1}^{\infty}\alpha_i^p\left|x_i^{(k)}-x_i^{(l)}\right|^p < \varepsilon^p.$$

This means : $\left(x^{(k)}\right)_{k\in\mathbb{N}}$ converges to $\tilde{x}$ in ℓ_p^α . It follows that : $\tilde{x} = \underbrace{\left(\tilde{x}-x^{(k)}\right)}_{\in\ell_p^\alpha} + \underbrace{x^{(k)}}_{\in\ell_p^\alpha} \in \ell_p^\alpha$. Thus ℓ_p^α is complete. □

2.2 HMNC on ℓ_p^α

lemma 2. *[9] ℓ_p is isometric to ℓ_p^α.*

Proof. Let $Is : \ell_p^\alpha \to \ell_p$ be the mapping defined by : $Is\ (x) = Is\left((x_i)_{i=1}^\infty\right) = (\alpha_i x_i)_{i=1}^\infty$. Then for fixed x,y in ℓ_p^α we have :

$$\|Is(x)-Is(y)\|_{\ell_p} = \|(\alpha_i x_i)_{i=1}^\infty - (\alpha_i y_i)_{i=1}^\infty\|_{\ell_p^\alpha} = \left(\sum_{i=1}^{\infty}\alpha_i^p\left|x_i-y_i\right|^p\right)^{\frac{1}{p}} = \|x-y\|_{\ell_p^\alpha}.$$

□

Theorem 7. *[9] The HMNC $\chi_{\ell_p^\alpha}$ of a non-empty bounded subset B^α of ℓ_p^α $(p\geqslant 1)$ is given by :*

$$\chi_{\ell_p^\alpha}(B^\alpha) = \lim_{n\to\infty}\left(\sup_{x\in B^\alpha}\ (\sum_{k\geqslant n}\ \alpha_k^p|x_k|^p)^{\frac{1}{p}}\right)$$

Proof. By theorem 4 the HMNC on ℓ_p is given by :

$$\chi_{\ell_p}(Q) = \lim_{n\to\infty}\left(\sup_{x\in Q}\ (\sum_{k>n}\ |x_k|^p)^{\frac{1}{p}}\right).$$

Now by lemma 2 we have :

$$\chi_{\ell_p^\alpha}(Q) = \lim_{n\to\infty}\left(\sup_{x\in Q}\ \|(Id-P_n)(x)\|_{\ell_p^\alpha}\right) = \lim_{n\to\infty}\left(\sup_{x\in Q}\ \|I((Id-P_n)(x))\|_{\ell_p}\right)$$

$$= \lim_{n\to\infty}\left(\sup_{x\in Q} \| I((0,0,..,x_{n+1},..)) \|_{\ell_p}\right) = \lim_{n\to\infty}\left(\sup_{x\in Q} \| (\alpha_k x_k)_{k\geqslant 1} \|_{\ell_p}\right)$$

$$= \lim_{n\to\infty}\left(\sup_{x\in Q} \Big(\sum_{k>n} |\alpha_k x_k|^p\Big)^{\frac{1}{p}}\right) = \lim_{n\to\infty}\left(\sup_{x\in Q} \Big(\sum_{k>n} \alpha_k{}^p |x_k|^p\Big)^{\frac{1}{p}}\right).$$

□

2.3 Existence of solutions of infinite system of FDEs in $\mathscr{C}\left([0,\mathrm{a}],\ell_p^\alpha\right)$ for $p \geqslant 1$

Proposition 3 (9)**.** *Let $\mathscr{C}\left([0,a],\ell_p^\alpha\right)$ be the space of all continuous functions on $[0,a]$ $(a>0)$, with values in $\ell_p^\alpha\,(1 \leqslant p < \infty)$. Then $\left(\mathscr{C}\left([0,a],\ell_p^\alpha\right), \|.\|_{\mathscr{C}\left([0,a],\ell_p^\alpha\right)}\right)$ is a Banach space , where :* $\|y\|_{\mathscr{C}\left([0,a],\ell_p^\alpha\right)} = \sup_{t\in[0,a]} \|y(t)\|_{\ell_p^\alpha}$; $y(t) = (y_i(t))_{i=1}^\infty \in \mathscr{C}\left([0,a],\ell_p^\alpha\right)$.

Proof. Let $(y^{(k)})_k$ be a Cauchy sequence in $\mathscr{C}\left([0,\mathrm{a}],\ell_p^\alpha\right)$, then: $\forall \varepsilon>0, \exists N \in \mathbb{N} : \forall k,l \geqslant N$:

$$\left\|y^{(k)}(t) - y^{(l)}(t)\right\|_{\ell_p^\alpha} \leqslant \sup_{t\in[0,\mathrm{a}]} \left\|y^{(k)}(t) - y^{(l)}(t)\right\|_{\ell_p^\alpha} = \left\|y^{(k)} - y^{(l)}\right\|_{\mathscr{C}\left([0,\mathrm{a}],\ell_p^\alpha\right)} < \varepsilon.$$

and the proof is similar to Proposition 2. □

Proposition 4 (9)**.** *Let B^α be a nonempty bounded subset of $\mathscr{C}\left([0,a],\ell_p^\alpha\right)$ and for $t \in [0,a]$ define: $B^\alpha(t) = \left\{x(t) : x \in B^\alpha\right\}$. Then :*

$$\chi_{\mathscr{C}\left([0,a],\ell_p^\alpha\right)}(B^\alpha) = \sup_{t\in[0,a]} \chi_{\ell_p^\alpha}(B^\alpha(t)).$$

Proof. By theorems 3 and 7 :

$$\chi_{\ell_p^\alpha}(B^\alpha) = \lim_{n\to\infty}\left(\sup_{x\in B^\alpha} \|(Id-P_n)(x)\|_{\ell_p^\alpha}\right).$$

Taking supremum over t :

$$\sup_{t\in[0,\mathrm{a}]} \chi_{\ell_p^\alpha}(B^\alpha(t)) = \lim_{n\to\infty}\left(\sup_{x\in B^\alpha} \underbrace{\sup_{t\in[0,\mathrm{a}]} \|(Id-P_n)(x(t))\|_{\ell_p^\alpha}}_{\|(Id-P_n)(x)\|_{\mathscr{C}\left([0,\mathrm{a}],\ell_p^\alpha\right)}}\right) = \chi_{\mathscr{C}\left([0,\mathrm{a}],\ell_p^\alpha\right)}(B^\alpha).$$

□

Now that we have an explicit formula for the HMNC, we are ready to investigate the following infinite system of FDEs with boundary conditions :

$$\begin{cases} \mathscr{D}^\gamma z_i(t) = \mathscr{H}_i(t, z(t)) \\ z_i(0) = z_i(a) = 0 \end{cases} ; z_i(0), z_i(a), z_i(t) \in \mathscr{C}^{\lfloor\gamma\rfloor+1}; t \in]0, a[; i = 1, 2, .. \tag{2.3.1}$$

Where $z(t) = (z_i(t))_{i=1}^\infty$ *and* $(\mathscr{H}_i(t, z(t)))_{i=1}^\infty$ are elements of a Banach space . For $J = [0, a]$, we define the functions $\mathscr{H}_i$ and the operator Θ as follows :

$$\begin{aligned} \mathscr{H}_i : J \times \mathscr{C}\left(j, \ell_p^\alpha\right) &\to \mathbb{R} & \qquad \Theta : \mathscr{C}\left(j, \ell_p^\alpha\right) &\to \mathscr{C}\left(j, \ell_p^\alpha\right) \\ (t, z(t) &\longmapsto \mathscr{H}_i(t, z(t) & z(t) &\longmapsto (\Theta z)(t) = (\mathscr{H}_i(t, z(t))_{i\geqslant 1} \end{aligned}$$

The unique solution of 2.3.1 is given by :

$$z_i(t) = \int_0^a U(t,s)\mathscr{H}_i(s, z(s)\, ds$$

where $U(t,s)$ is the green function (theorem 6). Now, define the operator $\Im : \left(j, \ell_p^\alpha\right) \to \left(j, \ell_p^\alpha\right)$ by :

$$(\Im z)(t) = ((\Im z)_i(t))_{i=1}^\infty = \underbrace{\left(\underbrace{\int_0^a U(t,s)\mathscr{H}_i(s, z(s)\, ds}_{=z_i(t)}\right)_{i=1}^\infty}_{=z(t)} .$$

Then, proving the existence of solutions for system 2.3.1 is equivalent to prove the existence of a fixed point for $\Im$. Which is the main aim of this section.

lemma 3. *Let* $1 < \gamma < 2$, *and :*

$$\begin{cases} (\Im z)_i(t) = z_i(t) \\ z_i(0) = z_i(a) = 0 \end{cases} ; z_i(0), z_i(a), z_i(t) \in \mathscr{C}^{\lfloor\gamma\rfloor+1}; t \in]0, a[; i = 1, 2, .. \tag{2.3.2}$$

Then, z(t) is a solution of 2.3.1 if and only if z(t) is a solution of 2.3.2.

Theorem 8. *[9] Assume :*

(i) *The family* $((\Theta z)(t))_{t\in J}$ *is equi-continuous at every point of* $\mathscr{C}\left([0,a],\ell_p^\alpha\right)$,

(ii) *There exists non-negative functions* $Q_i(t)$ *and* $R_i(t)$, *continuous on* J *for all* $i = 1,2,..$ *that satisfy the inequality :* $|\mathscr{H}_i(t,z(t)|^p \leqslant Q_i(t) + R_i(t)|z_i(t)|^p$.

(iii) $\tilde{Q}(t) = \sum_{i=1}^{\infty} \alpha_i^p Q_i(t)$ *is uniformly convergent on* J.

(iv) $a^{\frac{2-p}{p}}\tilde{U}\,\tilde{R}^{\frac{1}{p}} < 1$.

$$(\tilde{U} = \sup_{s,t\in J}|U(t,s)|,\ \tilde{Q} = \sup_{t\in J}\tilde{Q}(t),\ \tilde{R} = \sup_{t\in J} R_i(t)).$$

Then : the system 2.3.1 has at least one solution $z(t) \in \mathscr{C}\left(J,\ell_p^\alpha\right)$.

Proof. We will use Darbo fixed point theorem 5.

- $\Im$ **is well defined** For a fixed $t \in J$, we have :

$$\sum_{i\geqslant 1}\left(\alpha_i^p\left|\int_0^a U(t,s)\mathscr{H}_i(s,z(s)\,ds\right|^p\right)$$

$$\leqslant \underbrace{\left(\sup_{t,s\in[0,a]}|U(t,s)|\right)^p}_{=\tilde{U}^P}\sum_{i\geqslant 1}\left(\alpha_i^p\left|\int_0^a \mathscr{H}_i(s,z(s))\,ds\right|^p\right)$$

$$\leqslant \tilde{U}^P\sum_{i\geqslant 1}\left(\alpha_i^p\right|\,\|1\|_{p'}\,\|\mathscr{H}_i(s,z(s))\|_p\,\left|^p ds\right) \quad \text{" By Holder inequality}\left(\tfrac{1}{p}+\tfrac{1}{p'}=1\right)\text{"}$$

$$= \tilde{U}^P\underbrace{\left|\left(\int_0^a |1|^{p'}\right)^{\frac{1}{p'}}\right|^p}_{=a^{p-1}}\sum_{i\geqslant 1}\left(\alpha_i^p\int_0^a |\mathscr{H}_i(s,z(s))|^p\,ds\right)$$

$$\leqslant \tilde{U}^P a^{p-1}\sum_{i\geqslant 1}\left(\alpha_i^p\int_0^a (Q_I(s)+R_i(s)|z_i(s)|)^p ds\right) \quad \text{"By assumption (ii)"}$$

$$\leqslant a^{p-1}\tilde{U}^P\left(\int_0^a \underbrace{\sum_{i\geqslant 1}\alpha_i^p Q_I(s)}_{\tilde{Q}(s)}\,ds + \underbrace{\sup_{s\in[0,a]} R_i(s)}_{\tilde{R}}\int_0^a \underbrace{\sum_{i\geqslant 1}\alpha_i^p|z_i(s)|^p}_{=\|z(t)\|^p_{\ell_p^\alpha}}\,ds\right)$$

$$\leqslant a^p \tilde{U}^P \left(\underbrace{\sup_{s\in[0,a]} \tilde{Q}(s)}_{\tilde{Q}} + \tilde{R} \underbrace{\sup_{t\in[0,a]} \|z(t)\|_{\ell_p^\alpha}^p}_{=\|z\|^p_{\mathscr{C}\left([0,a],\ell_p^\alpha\right)}} \right)$$

Let d_0 be the optimal solution (the smallest value of $d \in \mathbb{R}$) of the inequality :

$$a^p \tilde{U}^P \left(\tilde{Q} + \tilde{R}\,\mathrm{d}^p\right) \leqslant d^p$$

Then :

$$\|\Im z\|_{\mathscr{C}([0,a],\ell_p^\alpha)} \leqslant \mathrm{d}_0 \tag{2.3.3}$$

Let $\mathrm{B}^\alpha = \left\{z \in \mathscr{C}\left(J,\ell_p^\alpha\right) : \|z\|_{\mathscr{C}(J,\ell_p^\alpha)} \leqslant \mathrm{d}_0,\ z_i(0) = z_i(a) = 0\right\}$ and $\Im(\mathrm{B}^\alpha) = \left\{\Im z \in \mathscr{C}\left(J,\ell_p^\alpha\right) : z \in \mathrm{B}^\alpha\right\}$. . Since $U(0,s) = U(a,s) = 0$ for all $s \in J$, we have for all i = 1,2,..: $(\Im z)_i(0) = (\Im z)_i(a) = 0$..

Hence, by 2.3.3 :

$\Im(\mathrm{B}^\alpha) \subset \mathrm{B}^\alpha$ and $\Im$ is well defined.

- **B^α is bounded, closed and convex**

  B^α is bounded by definition.

  For $(z_n)_n$ in B^α such that $\lim\limits_{n\to\infty} z_n = z$ we have :

$$\|z\|_{\mathscr{C}([0,a],\ell_p^\alpha)} = \left\|\lim_{n\to\infty} z_n\right\|_{\mathscr{C}([0,a],\ell_p^\alpha)} = \lim_{n\to\infty} \|z_n\|_{\mathscr{C}([0,a],\ell_p^\alpha)} \leqslant \mathrm{d}_0.$$

  Hence $z \in \mathrm{B}^\alpha$ and B^α is closed in $\mathscr{C}\left(J,\ell_p^\alpha\right)$.

  Let $x, y \in \mathrm{B}^\alpha$ and $\lambda \in [0,1]$, then :

$$\left\|\lambda x + (1-\lambda) y\right\|_{\mathscr{C}([0,a],\ell_p^\alpha)} \leqslant \|\lambda x\|_{\mathscr{C}(J,\ell_p^\alpha)} + \left\|(1-\lambda) y\right\|_{\mathscr{C}(J,\ell_p^\alpha)} \leqslant (\lambda + (1-\lambda))\,\mathrm{d}_0 = \mathrm{d}_0.$$

  Hence B^α is convex.

- **$\Im$ is continuous with respect to z on B^α**

For a fixed $\hat{z} \in B^\alpha$ and for all $t \in J$ we have :

$$\sum_{i\geqslant 1} \alpha_i^p |(\Im z)_i(t) - (\Im\hat{z})_i(t)|^p = \sum_{i\geqslant 1} \alpha_i^p \left| \int_0^a U(t,s)\mathscr{H}_i(s,z(s)\,ds - \int_0^a U(t,s)\mathscr{H}_i(s,\hat{z}(s)\,ds \right|^p$$

$$= \sum_{i\geqslant 1} \alpha_i^p \left| \int_0^a U(t,s)\,(\mathscr{H}_i(s,z(s) - \mathscr{H}_i(s,\hat{z}(s))\,ds \right|^p$$

$$\leqslant \tilde{U}^p \sum_{i\geqslant 1} \alpha_i^p \left(\int_0^a |\mathscr{H}_i(s,z(s) - \mathscr{H}_i(s,\hat{z}(s)|\,ds \right)^p$$

$$\leqslant \tilde{U}^P \sum_{i\geqslant 1} \left(\alpha_i^p \big| \, \|1\|_{p'} \, \|\mathscr{H}_i(s,z(s) - \mathscr{H}_i(s,\hat{z}(s)\|_p \, \big|^p ds \right)$$

$$= \tilde{U}^P \underbrace{\left| \left(\int_0^a |1|^{p'} \right)^{\frac{1}{p'}} \right|^p}_{=a^{p-1}} \sum_{i\geqslant 1} \left(\alpha_i^p \int_0^a |\mathscr{H}_i(s,z(s) - \mathscr{H}_i(s,\hat{z}(s)|^p\,ds \right)$$

By assumption (i) we have (F is a family of operators Θ) :

$$\forall \hat{z} \in B^\alpha,\ \forall\, \varepsilon > 0,\ \exists \delta > 0,\ \forall \Theta \in F,\ \forall z \in B^\alpha,\ \|z - \hat{z}\|_{\mathscr{C}(J,\ell_p^\alpha)} < \delta \Rightarrow \|\Theta z - \Theta \hat{z}\|_{\mathscr{C}(J,\ell_p^\alpha)} < \varepsilon.$$

Since :

$$\|\Theta z - \Theta \hat{z}\|_{\mathscr{C}(J,\ell_p^\alpha)} = \sup_{t\in J} \|(\Theta z)(t) - (\Theta \hat{z})(t)\|_{\ell_p^\alpha} = \sup_{t\in J} \|(\mathscr{H}_i(t,z(t))_{i\geqslant 1} - (\mathscr{H}_i(t,\hat{z}(t))_{i\geqslant 1}\|_{\ell_p^\alpha}$$

$$= \sup_{t\in J} \left(\sum_{i=1}^{\infty} \alpha_i^p |\mathscr{H}_i(t,z(t) - \mathscr{H}_i(t,\hat{z}(t)|^p \right)^{\frac{1}{p}}$$

Taking $\varepsilon = \frac{\hat{\varepsilon}}{a\tilde{U}}$ gives :

$$\left(\sum_{i\geqslant 1} \alpha_i^p |(\Im z)_i(t) - (\Im\hat{z})_i(t)|^p \right)^{\frac{1}{p}} \leqslant a^{\frac{p-1}{p}} \tilde{U} \left(\sum_{i\geqslant 1} \left(\alpha_i^p \int_0^a |\mathscr{H}_i(s,z(s) - \mathscr{H}_i(s,\hat{z}(s)|^p\,ds \right) \right)^{\frac{1}{p}}$$

$$= a^{\frac{p-1}{p}} \tilde{U} \left(\left(\int_0^a \sum_{i\geqslant 1} \alpha_i^p |\mathscr{H}_i(s,z(s) - \mathscr{H}_i(s,\hat{z}(s)|^p\,ds \right) \right)^{\frac{1}{p}}$$

$$= a^{\frac{p-1}{p}} \tilde{U} \left(\left(\int_0^a \left(\underbrace{\sum_{i\geqslant 1} \alpha_i^p |\mathscr{H}_i(s,z(s) - \mathscr{H}_i(s,\hat{z}(s)|^p}_{\leqslant \varepsilon^p = \frac{\hat{\varepsilon}^p}{a^p \tilde{U}^p}} \right) ds \right) \right)^{\frac{1}{p}}$$

$$\leqslant a^{\frac{p-1}{p}} \tilde{U} \left(\left(\int_0^a \frac{\hat{\varepsilon}^p}{a^p \tilde{U}^p} \, ds \right) \right)^{\frac{1}{p}} = a^{\frac{p-1}{p}} \tilde{U} \left(\frac{\hat{\varepsilon}^p}{a^p \tilde{U}^p} \left(\underbrace{\int_0^a 1 \, ds}_{=a} \right) \right)^{\frac{1}{p}}$$

$$= a^{\frac{p-1}{p}} \tilde{U} \frac{\hat{\varepsilon}}{a\tilde{U}} a^{\frac{1}{p}} = a^{\frac{p-1}{p}} \, a^{-1+\frac{1}{p}} \, \hat{\varepsilon} = a^{\frac{p-1-p+1}{p}} \, \hat{\varepsilon} = \hat{\varepsilon}.$$

Therefor :

$$\|(\Im z)\,(t) - (\Im \hat{z})\,(t)\|_{\ell_p^\alpha} = \left(\sum_{i\geqslant 1} \alpha_i^p |(\Im z)_i\,(t) - (\Im \hat{z})_i\,(t)|^p \right)^{\frac{1}{p}} \leqslant \hat{\varepsilon}$$

Taking the supremum over t in J yields :

$$\|\Im z - \Im \hat{z}\|_{\mathscr{C}(J,\ell_p^\alpha)} \leqslant \hat{\varepsilon}$$

Hence, $\Im$ is continuous with respect to z on B^α.

- **$\Im$ is continuous for each t in J**

  By continuity of $U(t,s)$ for $\gamma > 1$, we have for fixed $\tilde{t} \in J$:

$$\forall \tilde{t} \in J,\ \forall\ \hat{\varepsilon} > 0,\ \exists \hat{\delta} > 0,\ \forall \tilde{t} \in J:\ \left|t - \tilde{t}\right| < \hat{\delta} \Rightarrow \left|U(t,s) - U(\tilde{t},s)\right| < \frac{a^{1-p}\hat{\varepsilon}^p}{a\tilde{Q} + \tilde{R}}.$$

Now :

$$\left\|(\Im z)(\tilde{t}) - (\Im z)(t)\right\|_{\ell_p^\alpha}^p = \sum_{i\geqslant 1} \alpha_i^p \left|(\Im z)_i(\tilde{t}) - (\Im z)_i(t)\right|^p$$

$$\leqslant \sum_{i\geqslant 1} \left(\alpha_i^p \left| \int_0^a U(\tilde{t},s)\mathscr{H}_i(s,z(s)\, ds - \int_0^a U(t,s)\mathscr{H}_i(s,z(s)\, ds \right|^p \right)$$

$$\leqslant a^{p-1} \int_0^a \left|U(t,s) - U(\tilde{t},s)\right|^p \sum_{i\geqslant 1} \alpha_i^p |\mathscr{H}_i(s,z(s)|^p ds$$

$$\leqslant a^{p-1} \int \left|U(t,s) - U(\tilde{t},s)\right|^p \sum_{i\geqslant 1} \alpha_i^p \left(Q_i\,(t) + R_i\,(t)\,|z_i(t)|^p\right) ds$$

$$\leqslant a^{p-1}\left(\frac{a^{1-p}\hat{\varepsilon}^p}{a\tilde{Q}+\tilde{R}d_0}\right)\int_0^a\left(\tilde{Q}+\tilde{R}\sum_{i\geqslant 1}\alpha_i^p|z_i(t)|^p ds\right)$$

$$\leqslant\left(\frac{\hat{\varepsilon}^p}{a\tilde{Q}+\tilde{R}d_0}\right)\left(a\tilde{Q}+\tilde{R}d_0\right)=\hat{\varepsilon}^p$$

Takin supremum over t ∈ J :

$$\left\|(\Im z)(\tilde{t})-(\Im z)(t)\right\|_{\mathscr{C}\left(J,\ell_p^\alpha\right)}<\hat{\varepsilon}$$

Hence, $\Im$ is continuous for each $t\in J$.

- **$\Im$ is a condensing operator**

  We already proved that $\Im$ is continuous on B^α, now for fixed $t\in J$ and $\mathrm{B}^\alpha(t)=\left\{z(t):z\in\mathrm{B}^\alpha\right\}$, we have :

$$\chi_{\ell_p^\alpha}\left(\Im\mathrm{B}^\alpha(t)\right)=\lim_{n\to\infty}\sup_{z(t)\in\mathrm{B}^\alpha(t)}\left(\sum_{i\geqslant n}\alpha_i^p\left|\int_0^a U(t,s)\mathscr{H}_i(s,z(s)\,ds\right|^p\right)^{\frac{1}{p}}$$

$$\leqslant a^{\frac{1-p}{p}}\lim_{n\to\infty}\sup_{z(t)\in\mathrm{B}^\alpha(t)}\left(\sum_{i\geqslant n}\alpha_i^p\int_0^a|U(t,s)|^p|\mathscr{H}_i(s,z(s)|^p\right)^{\frac{1}{p}}\quad\text{"By Hölder's inequality"}$$

$$\leqslant a^{\frac{1-p}{p}}\tilde{U}\lim_{n\to\infty}\sup_{z(t)\in\mathrm{B}^\alpha(t)}\left(\sum_{i\geqslant n}\alpha_i^p\int_0^a\left(Q_i(s)+R_i(s)|z_i(s)|^p\right)ds\right)^{\frac{1}{p}}$$

$$=a^{\frac{1-p}{p}}\tilde{U}\lim_{n\to\infty}\sup_{z(t)\in\mathrm{B}^\alpha(t)}\left(\int_0^a\underbrace{\sum_{i\geqslant n}\alpha_i^pQ_i(s)\,ds}_{\tilde{Q}(t)}+\underbrace{\sup_{s\in J}R_i(s)}_{\tilde{R}}\int_0^a\left(\sum_{i\geqslant n}\alpha_i^p|z_i(s)|^p\right)ds\right)^{\frac{1}{p}}$$

$$\leqslant a^{\frac{1-p}{p}}\tilde{U}\left(a\left(\lim_{n\to\infty}\sum_{i\geqslant n}\alpha_i^pQ_i(s)\right)+\tilde{R}\,a\left(\underbrace{\lim_{n\to\infty}\sup_{z(t)\in\mathrm{B}^\alpha(t)}\sum_{i\geqslant n}\alpha_i^p|z_i(t)|^p}_{\chi_{\ell_p^\alpha}(\mathrm{B}^\alpha(t))}\right)\right)^{\frac{1}{p}}$$

By assumption (iii) : $\lim_{n\to\infty}\sum_{i\geqslant n}\alpha_i^p Q_i(s) = 0$, hence:

$$\chi_{\ell_p^\alpha}\left(\Im \mathrm{B}^\alpha(t)\right) \leqslant a^{\frac{1-p}{p}}\ a^{\frac{1}{p}}\ \tilde{U}\ \tilde{R}^{\frac{1}{p}}\ \chi_{\ell_p^\alpha}\left(\mathrm{B}^\alpha(t)\right) = a^{\frac{2-p}{p}}\ \tilde{U}\ \tilde{R}^{\frac{1}{p}}\ \chi_{\ell_p^\alpha}\left(\mathrm{B}^\alpha(t)\right).$$

take sup over $t \in J$ yields :

$$\chi_{\mathscr{C}\left([0,a],\ell_p^\alpha\right)}\left(\Im \mathrm{B}^\alpha\right) \leqslant a^{\frac{2-p}{p}}\,\tilde{U}\,\tilde{R}^{\frac{1}{p}}\ \chi_{\mathscr{C}\left([0,a],\ell_p^\alpha\right).}\left(\mathrm{B}^\alpha\right)$$

By assumption (iv) : $\Im$ is condensing.

- All conditions of theorem 5 are verified. Hence, by lemma 3 : the system 2.3.1 has a solution in $\mathscr{C}\left(J,\ell_p^\alpha\right)$.

□

2.4 Example

[9] Taking $p=2, \gamma=\frac{5}{4}$, $J=[0,1]$ and $\mathscr{H}_i\,(t,z(t)) = \frac{t^{\frac{1}{2}}\cos(it)}{i^2} + \sum_{j\geqslant i} \frac{z_i(t)\ln(1+t)}{2j^2(1+i)^3}$ in system 2.3.1, we get :

$$\begin{cases} \mathscr{D}^{\frac{5}{4}} z_i(t) = \frac{t^{\frac{1}{2}}\cos(it)}{i^2} + \sum_{j\geqslant i} \frac{z_i(t)\ln(1+t)}{2j^2(1+i)^3} \\ z_i(0) = z_i(1) = 0 \end{cases} \quad ; \; i = 1,2,.. \tag{2.4.1}$$

We will study the existence of solutions for the above system in $\mathscr{C}\left([0,1]\,,\ell_2^\alpha\right)$ choosing as a tempering sequence $\alpha = (\alpha_i)_{i\in\mathbb{N}} = \left(\frac{1}{i}\right)_{i\in\mathbb{N}}$. Now let's verify the assumptions of theorem 8 :

- **Assumption (i)** :

  Let $z,\tilde{z} \in \mathscr{C}\left([0,1]\,,\ell_2^\alpha\right)$ and $t\in[0,1]$ been fixed, then we have :

$$\|(\Theta z)(t) - (\Theta\hat{z})(t)\|_{\ell_2^\alpha}^2 = \|(\mathscr{H}_i(t,z(t))_{i\geqslant 1} - (\mathscr{H}_i(t,\hat{z}(t))_{i\geqslant 1}\|_{\ell_2^\alpha}^2 = \sum_{i=1}^{\infty} \alpha_i^2 |\mathscr{H}_i(t,z(t) - \mathscr{H}_i(t,\hat{z}(t)|^2$$

$$= \sum_{i\geqslant 1}\frac{1}{i}\left|\frac{t^{\frac{1}{2}}\cos(it)}{i^2} + \sum_{j\geqslant i}\frac{z_i(t)\ln(1+t)}{2j^2(1+i)^3} - \frac{t^{\frac{1}{2}}\cos(it)}{i^2} - \sum_{j\geqslant i}\frac{\tilde{z}_i(t)\ln(1+t)}{2j^2(1+i)^3}\right|^2$$

$$= \sum_{i\geqslant 1}\frac{1}{i^2}\left|\sum_{j\geqslant i}\frac{(z_i(t)-\tilde{z}_i(t))\ln(1+t)}{2j^2(1+i)^3}\right|^2 \leqslant \left(\frac{1}{2}\right)^2 \sum_{i\geqslant 1}\frac{1}{i^2}\frac{|z_i(t)-\tilde{z}_i(t)|^2|\ln(1+t)|^2}{(1+i)^6}\left|\sum_{j\geqslant i}\frac{1}{j^2}\right|^2$$

$$= \frac{1}{4}|\ln(1+t)|^2\left(\underbrace{\sum_{i\geqslant 1}\frac{1}{i^2}|z_i(t)-\tilde{z}_i(t)|^2}_{\|z(t)-\tilde{z}(t)\|_{\ell_2^\alpha}^2}\right)\left|\sum_{j\geqslant i}\frac{1}{j^2}\right|^2 \leqslant \frac{1}{4}\,|\ln(1+t)|^2\left(\|z(t)-\tilde{z}(t)\|_{\ell_2^\alpha}^2\right)\left|\underbrace{\sum_{j\geqslant 1}\frac{1}{j^2}}_{=\frac{\pi^2}{6}}\right|^2$$

$$= \frac{1}{4}|\ln(1+t)|^2\left(\frac{\pi^2}{6}\right)^2 \|z(t)-\tilde{z}(t)\|_{\ell_2^\alpha}^2 \leqslant \frac{\pi^4}{144}|\ln(1+1)|^2\,\|z(t)-\tilde{z}(t)\|_{\ell_2^\alpha}^2$$

  Hence :

$$\|(\Theta z)(t) - (\Theta\hat{z})(t)\|_{\ell_p^\alpha} \leqslant \frac{\pi\,(\ln(2))^2}{12}\|z_i(t)-\tilde{z}_i(t)\|_{\ell_2^\alpha}$$

Taking supremum over $t \in [0,1]$:

$$\|\Theta z - \Theta z\|_{\mathscr{C}([0,1],\ell_2^\alpha)} \leqslant \frac{\pi\,(\ln(2))^2}{12}\|z-\tilde{z}\|_{\mathscr{C}([0,1],\ell_2^\alpha)}$$

Now by choosing $\delta = \frac{12}{\pi\,(\ln(2))^2}$, we get :

$$\forall \tilde{z} \in \mathscr{C}\left([0,1],\ell_2^\alpha\right),\ \forall\ \varepsilon>0,\ \exists\delta>0,\ \forall\Theta \in ((\Theta z)\,(t))_{t\in[0,1]},\ \forall z \in \mathscr{C}\left([0,1],\ell_2^\alpha\right):$$

$$\|z-\hat{z}\|_{\mathscr{C}(J,\ell_p^\alpha)} < \delta = \frac{12\,\varepsilon}{\pi\,(\ln(2))^2} \Rightarrow \|\Theta z - \Theta\hat{z}\|_{\mathscr{C}(J,\ell_p^\alpha)} \leqslant \frac{\pi\,(\ln(2))^2}{12}\underbrace{\|z-\tilde{z}\|_{\mathscr{C}([0,1],\ell_2^\alpha)}}_{\leqslant \frac{12\,\varepsilon}{\pi\,(\ln(2))^2}} \leqslant \varepsilon.$$

Hence, $((\Theta z)\,(t))_{t\in[0,1]}$ is equicontinuous at every point of $\mathscr{C}\left([0,1],\ell_2^\alpha\right)$.

- **assumption (ii)** :

$$|H_i\,(t,z(t))|^2 = \left|\frac{t^{\frac{1}{2}}\cos(i\,t)}{i^2} + \sum_{j\geqslant i}\frac{z_i(t)\ln(1+t)}{2\,j^2(1+i)^3}\right|^2 \leqslant \left(\left|\frac{t^{\frac{1}{2}}\cos(i\,t)}{i^2}\right| + \left|\sum_{j\geqslant i}\frac{z_i(t)\ln(1+t)}{2\,j^2(1+i)^3}\right|\right)^2$$

$$\leqslant 2\left|\frac{t^{\frac{1}{2}}\cos(i\,t)}{i^2}\right|^2 + 2\left|\sum_{j\geqslant1}\frac{z_i(t)\ln(1+t)}{2\,j^2(1+i)^3}\right|^2 \quad \text{"}(|a|+|b|)^2 \leqslant 2|a|^2+2|b|^2\text{"}$$

$$= 2\left|\frac{t^{\frac{1}{2}}}{i^2}\right|^2 + 2\left|\frac{z_i(t)\ln(1+t)}{2\,(1+i)^3}\sum_{j\geqslant1}\frac{1}{j^2}\right|^2 = \frac{2\,|t|}{i^2} + 2\left|\frac{z_i(t)\ln(1+t)}{2\,(1+i)^3}\frac{\pi^2}{6}\right|^2$$

$$\leqslant \frac{2\,|t|}{i^2} + 2\left|\frac{z_i(t)\ln(1+t)}{12\,i^3}\pi^2\right|^2 = \frac{2\,|t|}{i^2} + \frac{\pi^4|\ln(1+t)|^2}{72\,i^6}|z_i(t)|^2.$$

Hence we could take : $Q_i\,(t) = \frac{2|t|}{i^2}$, and $R_i\,(t) = \frac{\pi^4|\ln(1+t)|^2}{72\,i^6}$. Then, $Q_i\,(t)$ and $R_i\,(t)$ are continuous on $[0,1]$ for all $i = 1,2,..$ as composition of continuous functions.

- **Assumption (iii) :**

we have : $\sum\limits_{i\geqslant1}\alpha_i^2 Q_i(t) = \sum\limits_{i\geqslant1}\frac{1}{i^2}\frac{2\,|t|}{i^4} = 2\ |t|\sum\limits_{i\geqslant1}\frac{1}{i^6}$ which converges uniformly to $Q(t) = 2\,|t|\,\frac{\pi^2}{945}$. (Using Fourier series : $\sum\limits_{i\geqslant1}\frac{1}{i^6} = \frac{\pi^2}{945}$)

- **assumption (iv) :**

For $a=1$, $\gamma=\frac{5}{4}$, the green function is given by :

$$U(t,s)=\begin{cases}\frac{1}{\Gamma(\frac{5}{4})}\left(t^{\frac{1}{4}}(1-s)^{\frac{1}{4}}-a^{\alpha-1}(t-s)^{\frac{1}{4}}\right) & ; 0\leqslant s\leqslant t\leqslant 1\,. \\ \frac{1}{\Gamma(\frac{5}{4})}t^{\frac{1}{4}}(1-s)^{\frac{1}{4}} & ; 0\leqslant t\leqslant s\leqslant 1\,.\end{cases}$$

Take $t=1$ and $s=0$ we get : $\tilde{U}=\sup\limits_{s,t\in[0,1]}|U(t,s)|=\frac{1}{\Gamma(\frac{5}{4})}$.

Since the function (ln) is increasing, by taking $t=i=1$:

$$\tilde{R}=\sup_{t\in[0,1]}R_i(t)=\sup_{t\in[0,1]}\frac{\pi^4|\ln(1+t)|^2}{72\,i^6}=\frac{\pi^4|\ln(2)|^2}{72}.$$

Hence :

$$\underbrace{a^{\frac{2-p}{2}}}_{=1^{\frac{2-2}{2}}}\tilde{U}\,\tilde{R}^{\frac{1}{p}}=\frac{1}{\Gamma(\frac{5}{4})}\left[\frac{\pi^4\,|\ln 2|^2}{72}\right]^{\frac{1}{2}}=\frac{\pi^2\,|\ln 2|}{6\sqrt{2}\,\Gamma(\frac{5}{4})}<1.$$

- **All assumptions of theorem 8 are verified, hence: system 2.4.1 has a solution in** $\mathscr{C}\left([0,1],\ell_2^{\alpha}\right)$.

2.5 Semi-analytic method to approximate solutions of infinite system of FDEs

Homotopy perturbation method (HPM) is a semi-analytical technique for solving linear as well as nonlinear ordinary/partial differential equations. The method may also be used to solve a system of coupled linear and nonlinear differential equationsas well as differential equation of fractional order. The idea behind the HPM was proposed by J. He in 1999, by introducing a small parameter p into the differential equation and expanding the solution of the resulting equation $H(v,p) = 0$ in a Taylor series about $p = 0:\ v = v_0 + v_1 p + ...$ This method was developed by making use of artificial parameters and some modifications happened. Next we present a review of the HPM and the modified HPM we will be using here.

2.5.1 Homotopy Perturbation Method(HPM)

[5,8] consider the following differential equation:

$$A(u) - f(r) = 0\,,\ r \in \Omega \tag{2.5.1}$$

subject to the boundary condition :

$$B(u, \frac{\partial u}{\partial r}) = 0\,,\ r \in \Gamma \tag{2.5.2}$$

where A represents a general differential operator, B is a boundary operator, Γ is the boundary of the domain Ω and $f(r)$ is a known analytic function. The operator A can be decomposed into two parts : linear (L) and nonlinear (N). Therefore, equation 2.5.1 may be written in the following form :

$$L(u) + N(u) - f(r) = 0 \tag{2.5.3}$$

An artificial parameter p can be embedded in 2.5.3 as follows :

$$L(u) + p\left(N(u) - f(r)\right) = 0$$

wherep[0, 1] is the embedding parameter (also called as an artificial param-eter). Using homotopy technique proposed by He and Liao, we construct a homotopy $v(r,p) : \Omega \times [0,1] \to \mathbb{R}$ to equation 2.5.1, which satisfies :

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[L(v) + N(v) - f(r)\right] = 0 \tag{2.5.4}$$

and

$$H(v,p) = \left[L(v) - L(u_0)\right] + pL(u_0) + p\left[N(v) - f(r)\right] = 0 \tag{2.5.5}$$

Where u_0 is an initial approximation of equation 2.5.5 which satisfies the given conditions. By substituting $p = 0$ and $p = 1$ in 2.5.5, we get the following equations, respectively :

$$H(v,0) = L(v) - L(u_0)$$

$$H(v,1) = A(v) - f(r) = 0$$

As p changes from zero to unity, $v(r,p)$ changes from $u_0(r))$ to $u(r)$. In topology, this is called **deformation** and $L(v) - L(u_0)$ and $A(v)f(r)$ are said to be **homotopic** to each other. Due to the fact that $p[0,1]$ is a small parameter, we consider the solution of 2.5.4 as a power series in p as shown below :

$$v = v_0 + v_1 p + v_2 p^2 + \dots$$

The approximate solution of equation 2.5.1 may then be obtained as :

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots \tag{2.5.6}$$

The series 2.5.6 is convergent for most cases, and the rate of convergence depends on : $L(u)+N(u)$.

2.5.2 Modified Homotopy Perturbation Method

[9] Consider the general nonlinear non-homogeneous problem (with infinite functions $z(t)=(z_i(t))_{i=1}^{\infty}$ and f a known function (called source term) :

$$\aleph(z(t))-f(i,t)=0,\ \ t\in[0,t]\ ,i\in\mathbb{N} \tag{2.5.7}$$

by decomposition of $\aleph$ into $\aleph_1$ (not necessarily linear), $\aleph_2$ and f into $f_1,\ f_2$ we write :

$$\aleph_1(z(t))-f_1(i,t)+\aleph_2(z(t))-f_2(i,t)=0 \tag{2.5.8}$$

We define the modified homotopy perturbation H of parameter p for v as :

$$\mathrm{H}\big(v(t,s),p\big)=\aleph_1(v(t))-f_1(i,t)+p\left(\aleph_2(v(t))-f_2(i,t)\right)=0 \tag{2.5.9}$$

Let $v(t)=(v_i(t))_{i=1}^{\infty}$ be the approximation of $z(t)=(z_i(t))_{i=1}^{\infty}$, $i\in\mathbb{N}$. Then, we write the components $v_i(t)$ of the solution v as a linear combination of the Adomian polynomials as follows :

$$z_i(t)\approx v_i(t)=\sum_{k=0}^{\infty}p^k v_{i,k}(t) \tag{2.5.10}$$

where $p\in[0,1]$ is the embedding parameter.We get the solution by :

$$z_i(t)=\lim_{p\to 1}\ v_i(t) \tag{2.5.11}$$

We give an illustration of the method through example 2.4.1 :

Step 1 : Transform the differential equation to an integral equation :

Consider the FDE in system 2.4.1 $\left(\gamma = \frac{5}{4},\ m = [\gamma] + 1 = 2\right)$:

$$ {}^{RL}\mathscr{D}^{\frac{5}{4}} z_i(t) + \frac{t^{\frac{1}{2}}\cos(it)}{i^2} + \sum_{j\geqslant i} \frac{z_i(t)\ln(1+t)}{2j^2(1+i)^3} = 0 \tag{2.5.12}$$

By definition of The Riemann-Liouville fractional derivative, we write **??** as :

$$ \frac{1}{\Gamma(2-\frac{5}{4})}\left(\frac{d}{dt}\right)^2 \int_0^t \frac{z_i(s)}{(t-s)^{\frac{5}{4}-2+1}} ds + \frac{t^{\frac{1}{2}}\cos(it)}{i^2} + z_i(t)\frac{\ln(1+t)}{(1+i)^3}\sum_{j\geqslant i}\frac{1}{2j^2} = 0 \tag{2.5.13}$$

Deviding both sides of 2.5.13 by $\frac{\ln(1+t)}{(1+i)^3}\sum_{j\geqslant i}\frac{1}{2j^2}$ gives :

$$ z_i(t) + \left(\frac{(1+i)^3}{\ln(1+t)}\frac{1}{\sum_{j\geqslant i}\frac{1}{2j^2}}\frac{1}{\Gamma(\frac{3}{4})}\right)\left(\left(\frac{d}{dt}\right)^2\int_0^t \frac{z_i(s)}{(t-s)^{\frac{1}{4}}}ds\right) = -\frac{t^{\frac{1}{2}}\cos(it)}{i^2}\frac{(1+i)^3}{\ln(1+t)}\frac{1}{\sum_{j\geqslant i}\frac{1}{2j^2}} \tag{2.5.14}$$

Hence, we write the integral equation 2.5.14

$$ \underbrace{\underbrace{z_i(t)}_{=\aleph_1(z(t))} + \underbrace{\underbrace{\left(\frac{(1+i)^3}{\ln(1+t)}\frac{1}{\sum_{j\geqslant i}\frac{1}{2j^2}}\frac{1}{\Gamma(\frac{3}{4})}\right)}_{=g(i,t)}\left(\left(\frac{d}{dt}\right)^2\int_0^t \frac{z_i(s)}{(t-s)^{\frac{1}{4}}}ds\right)}_{=\aleph_2(z(t))}}_{=\aleph(z(t))} = \underbrace{\underbrace{-\frac{t^{\frac{1}{2}}\cos(it)}{i^2}\frac{(1+i)^3}{\ln(1+t)}\frac{1}{\sum_{j\geqslant i}\frac{1}{2j^2}}}_{=f_1(i,t)+f_2(i,t)}}_{=f(i,t)} $$

Taking :

$$ \aleph(z(t)) = z_i(t) + \left(\frac{(1+i)^3}{\ln(1+t)}\frac{1}{\sum_{j\geqslant i}\frac{1}{2j^2}}\frac{1}{\Gamma(\frac{3}{4})}\right)\left(\left(\frac{d}{dt}\right)^2\int_0^t \frac{z_i(s)}{(t-s)^{\frac{1}{4}}}ds\right) $$

and :

$$ f(i,t) = -\frac{t^{\frac{1}{2}}\cos(it)}{i^2}\frac{(1+i)^3}{\ln(1+t)}\frac{1}{\sum_{j\geqslant i}\frac{1}{2j^2}} $$

The FDE 2.5.12 is written in the general form 2.5.7.

Step 2 : Decomposition :

Now, by taking :

$$\aleph_1(z(t)) = z_i(t) \tag{2.5.15}$$

$$\aleph_2(z(t)) = \left(\frac{(1+i)^3}{\ln(1+t)}\frac{1}{\sum_{j\geqslant i}\frac{1}{2j^2}}\frac{1}{\Gamma(\frac{3}{4})}\right)\left(\left(\frac{d}{dt}\right)^2\int_0^t\frac{z_i(s)}{(t-s)^{\frac{1}{4}}}ds\right) \tag{2.5.16}$$

$$f_1(i,t)+f_2(i,t) = -\frac{t^{\frac{1}{2}}\cos(it)}{i^2}\frac{(1+i)^3}{\ln(1+t)}\frac{1}{\sum_{j\geqslant i}\frac{1}{2j^2}}$$

in the integral equation 2.5.14, we get the desired decomposition 2.5.8.

Step 3 : substituting 2.5.15,2.5.16 and 2.5.10 in 2.5.9, we get :

$$\left(\sum_{k=0}^{\infty}p^k v_{i,k}(t)-f_1(i,t)\right)+p\left(g(i,t)\left(\frac{d}{dt}\right)^2\int_0^t\frac{\sum_{k=0}^{\infty}p^k v_{i,k}(s)}{(t-s)^{\frac{1}{4}}}ds-f_2(i,t)\right)=0$$

Hence :

$$\sum_{k=0}^{\infty}p^k v_{i,k}(t)=f_1(i,t)+p\left(-g(i,t)\left(\frac{d}{dt}\right)^2\int_0^t\frac{\sum_{k=0}^{\infty}p^k v_{i,k}(s)}{(t-s)^{\frac{1}{4}}}ds\right)+p\,f_2(i,t)$$

$$=f_1(i,t)+\sum_{k=0}^{\infty}p^{k+1}\left(-g(i,t)\left(\frac{d}{dt}\right)^2\int_0^t\frac{v_{i,k}(s)}{(t-s)^{\frac{1}{4}}}ds\right)+p\,f_2(i,t)$$

$$=f_1(i,t)+p^1\left(-g(i,t)\left(\frac{d}{dt}\right)^2\int_0^t\frac{v_{i,0}(t)}{(t-s)^{\frac{1}{4}}}ds\right)+..+p^k\left(-g(i,t)\left(\frac{d}{dt}\right)^2\int_0^t\frac{v_{i,k-1}(s)}{(t-s)^{\frac{1}{4}}}ds\right)+..+p\,f_2(i,t)$$

$$=f_1(i,t)+p^1\left(-g(i,t)\left(\frac{d}{dt}\right)^2\int_0^t\frac{v_{i,1}(s)}{(t-s)^{\frac{1}{4}}}ds+f_2(i,t)\right)+..+p^k\left(-g(i,t)\left(\frac{d}{dt}\right)^2\int_0^t\frac{v_{i,k-1}(s)}{(t-s)^{\frac{1}{4}}}ds\right)+..$$

Hence :

$$p^0v_{i,0}(t)+p^1v_{i,1}(t)+\ldots+p^kv_{i,k}(t)+..=$$

$$p^0f_1(i,t)+p^1\left(-g(i,t)\left(\frac{d}{dt}\right)^2\int_0^t\frac{v_{i,1}(s)}{(t-s)^{\frac{1}{4}}}ds+f_2(i,t)\right)+..+p^k\left(-g(i,t)\left(\frac{d}{dt}\right)^2\int_0^t\frac{v_{i,k-1}(s)}{(t-s)^{\frac{1}{4}}}ds\right)+\ldots$$

Comparing the coefficients of same powers of p in the above equation we get the algorithm :

$$\begin{cases} p^0 : v_{i,0}(t) = f_1(i,t) \\ p^1 : v_{i,1}(t) = -g(i,t)\left(\frac{d}{dt}\right)^2 \int_0^t \frac{v_{i,1}(s)}{(t-s)^{\frac{1}{4}}} ds + f_2(i,t) \\ p^k : v_{i,k}(t) = -g(i,t)\left(\frac{d}{dt}\right)^2 \int_0^t \frac{v_{i,k-1}(s)}{(t-s)^{\frac{1}{4}}} ds \end{cases} \tag{2.5.17}$$

Step 4 : solution

Using the boundary conditions : $z_i(0) = z_i(1) = o$ in 2.5.10 gives : $v_{i,k}(0) = v_{i,k}(1) = o$. Then, a simple suggestion for $v_{i,0}(t)$ would be the zero function. Hence we can choose $f_1(i,t) = 0$ in algorithm 2.5.17. Thus, the terms of the sequence $(v_i(t))_{i\in\mathbb{N}}$ can be obtained by induction as follows :

For $i = 1$:

$$\begin{cases} v_{1,0}(t) = 0 \\ v_{1,1}(t) = \frac{-8\sqrt{t}\cos t}{\frac{\pi^2}{6}\ln(1+t)} \\ v_{1,k}(t) = 0\,,\ k = 2,3,.. \end{cases}$$

Hence :

$$z_1(t) \approx v_1(t) = \sum_{k=0}^{\infty} v_{1,k}(t) = 0 + \frac{-8\sqrt{t}\cos t}{\frac{\pi^2}{6}\ln(1+t)} + 0 + ..$$

By iteration :

$$z_i(t) = \begin{cases} \frac{-(1+i)^3\sqrt{t}\cos t}{\left(-\sum\limits_{j=1}^{i-1}\frac{1}{j^2}\right)\frac{\pi^2}{6}\ln(1+t)}\ , & t \in (0,1) \\ 0\,, & t = 0,1 \end{cases}$$

2.5.3 convergence

Theorem 9. *Let $(x_n)_{n\in\mathbb{N}}$ be the sequence of partial sums of the function series of order n:*

$$x_n(t) = \sum_{k=2}^{n} p^k v_{i,k}(t),\ n \geqslant 2$$

in the banach space $\left(\mathscr{C}([0,1]),\ \|.\|_\infty\right)$ then, $(x_n)_{n\in\mathbb{N}}$ is a Cauch sequence.

Proof. we have :

$$\left\| v_{i,k} \right\|_{\infty} = \sup_{t\in[0,1]} \left| v_{i,k}(t) \right| = \sup_{t\in[0,1]} \left| -g(i,t)\left(\frac{d}{dt}\right)^{2} \int_{0}^{t} \frac{v_{i,k-1}(s)}{(t-s)^{\frac{1}{4}}} ds \right|$$

$$\leqslant \underbrace{\sup_{s\in[0,1]} \left| v_{i,k-1}(s) \right|}_{=\| v_{i,k-1}\|_{\infty}} \left(\left| g(i,t) \right| \left| \left(\frac{d}{dt}\right)^{2} \underbrace{\int_{0}^{t} \frac{1}{(t-s)^{\frac{1}{4}}} ds}_{=\frac{4}{3}t^{\frac{3}{4}}} \right| \right) = \left\| v_{i,k-1} \right\|_{\infty} \sup_{t\in[0,1]} \left(\left| g(i,t) \right| \left| -\frac{1}{4} t^{-\frac{5}{4}} \right| \right)$$

$$= \left\| v_{i,k-1} \right\|_{\infty} \underbrace{\sup_{t\in[0,1]} \left(\frac{1}{4} \frac{1}{\sum_{j\geqslant i} \frac{1}{2j^{2}}} \frac{1}{\Gamma(\frac{3}{4})} (1+i)^{3} \right) \frac{1}{t^{\frac{5}{4}}} \frac{1}{|\ln(1+t)|}}_{c=h(i,t)}$$

Hence :

$$\left\| v_{i,k} \right\|_{\infty} \leqslant c \left\| v_{i,k-1} \right\|_{\infty} \leqslant c^{k-1} \left\| v_{i,1} \right\|_{\infty} = c^{k-1} \sup_{t\in[0,1]} \left| v_{i,1}(t) \right|$$

$$= c^{k-1} \sup_{t\in[0,1]} \left| -g(i,t)\left(\frac{d}{dt}\right)^{2} \int_{0}^{t} \frac{\overbrace{v_{i,0}(s)}^{=f_1(i,s)}}{(t-s)^{\frac{1}{4}}} ds + \underbrace{f(i,t) - f_1(i,t)}_{=f_2(i,t)} \right|$$

$$\leqslant c^{k-1} \sup_{t\in[0,1]} \left| \left(\sup_{s\in[0,1]} f_1(i,s) \right) \underbrace{g(i,t)\left(\frac{d}{dt}\right)^{2} \int_{0}^{t} \frac{1}{(t-s)^{\frac{1}{4}}} ds}_{c(t)} + f_2(i,\ t) \right|$$

$$= c^{k-1} \left(c \left\| f_1(i,.) \right\|_{\infty} + \left\| f_2(i,.) \right\|_{\infty} \right)$$

Using the boundary conditions in system 2.4.1, we could choose : $f_1(i,t) = v_{i,0}(t) = 0$, then we got :

$$\left\| v_{i,k} \right\|_{\infty} \leqslant c^{k-1} \left\| f_2(i,.) \right\|_{\infty} \tag{2.5.18}$$

□

Now :

$$|x_n(t) - x_m(t)| = \left| \sum_{k=m+1}^{n} p^{k} \left(-g(i,t)\left(\frac{d}{dt}\right)^{2} \int_{0}^{t} \frac{v_{i,k-1}(s)}{(t-s)^{\frac{1}{4}}} ds \right) \right|$$

$$\leqslant \left| \underbrace{-g(i,t)\left(\frac{d}{dt}\right)^2 \int_0^t \frac{1}{(t-s)^{\frac{1}{4}}} ds}_{=c} \right| \sum_{k=m+1}^{n} p^k \| v_{i,k-1} \|_\infty = c \sum_{k=m+1}^{n} p^k \| v_{i,k-1} \|_\infty$$

$$\leqslant c \sum_{k=m+1}^{n} p^k c^{k-2} \| f_2(i,.) \|_\infty = \sum_{k=m+1}^{n} p^k c^{k-1} \| f_2(i,.) \|_\infty \leqslant \| f_2(i,.) \|_\infty \sum_{k=m+1}^{n} p^k c^{k-1}$$

Hence, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in a Banach space, it converges.

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