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Existence and uniqueness of solutions to higher order fractional partial differential equations with purely integral conditions

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Abstract: In this paper, we prove the existence and uniqueness of Caputo time fractional pseudo-hyperbolic equations of higher order with purely nonlocal conditions of integral type. We use an a priori estimate method; the so-called energy inequalities method, based on some functional analysis tools, is developed for a Caputo time fractional of $2m$-th and $(2m + 1)$-th order and the density of the range of the operator generated by the considered problem. Using the Laplace transform and homotopy perturbation, we find a semi-analytical solution. Finally, we give some examples for illustration.

Keywords: Energy inequality, integral condition, a priori estimate, existence and uniqueness of solution, fractional problem, homotopy perturbation method, Laplace transform

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1 Introduction

Fractional differential equations (FDEs) can be considered as generalizations of differential equations of integer order to an arbitrary order. These types of problems appear in a large variety of areas such as engineering, physics and applied mathematics. Therefore, they have generated a lot of interest among engineers and scientists in recent years. Some recent developments in FDEs and in partial differential equations (PDEs) were proposed in [4, 7, 11].

Also, existence and uniqueness of solutions to initial and boundary-value problems for FDEs were studied in many articles; see [1, 3, 5]. Some results of the existence and uniqueness in FDEs were obtained by using the well-known Lax–Milgram theorem and fixed point theorems [15, 19].

A large number of problems in modern physics and technology are stated using nonlocal conditions for partial differential equations, which are described using integral conditions. Integral boundary conditions receive a lot of attention because of their applications in population dynamics, blood flow models, chemical engineering and cellular systems; see [6–8].

Mesloub and Aldosari [18] investigated the existence and uniqueness of the solution for a Caputo time fractional $2m$-th order diffusion wave equation with purely nonlocal conditions. They considered the follow-

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ing problem:

$$\begin{cases} Lv = {}^c\partial_t^{\alpha+1} v + (-1)^m \theta(t) \dfrac{\partial^{2m} v}{\partial x^{2m}} = f(x,t), & x \in (0,1),\ t \in (0,T), \\ \ell_1 v = v(x,0) = \varphi(x), \quad \ell_2 v = v_t(x,0) = \psi(x), & x \in (0,1), \\ \int_0^1 x^i v(x,t)\,dx = 0, & i = \overline{0, 2m-1},\ t \in (0,T), \end{cases}$$

where $\theta(t)$, $f(x,t)$, $g(x)$ and $h(x)$ are given functions that satisfy certain conditions, and the operator $\partial_t^{\alpha+1}$ denotes the Caputo left fractional derivative of order $1+\alpha$ with $0<\alpha<1$.

In this paper, we discuss the existence and uniqueness of a solution for a Caputo time fractional pseudo-hyperbolic equation of higher order, with purely nonlocal conditions of integral type, given by

$$\begin{cases} Lu = {}^c\partial_t^{\alpha+1} u + (-1)^m a(t) \dfrac{\partial^{2m+1} u}{\partial x^{2m}\partial t} + (-1)^m b(t) \dfrac{\partial^{2m} u}{\partial x^{2m}} = f(x,t), & x \in (0,1),\ t \in (0,T), \\ \ell_1 u = u(x,0) = \varphi(x), \quad \ell_2 u = u_t(x,0) = \psi(x), & x \in (0,1), \\ \int_0^1 x^i u(x,t)\,dx = 0, & i = \overline{0, 2m-1},\ t \in (0,T), \end{cases}$$

where ${}^c\partial_t^{\alpha+1}$ denotes the Caputo fractional derivative of order $1+\alpha$ with $0<\alpha<1$.

We apply a method from functional analysis, the so-called energy inequality method based mainly on some a priori estimates and on the density of the range of the operator generated by the studied problem.

This paper is organized as follows. In Section 2, we set our fractional initial boundary value problem. In Section 3, we give some preliminaries concerning the used function spaces, some useful tools and write down the given problem in its operator form. In Section 4, we establish an a priori estimate for the solution and deduce some consequences about the uniqueness of the solution and its dependence on the free term and the given data. Section 5 provides proofs of the main result concerning the solvability of the posed problem. In Section 6, we use the homotopy perturbation method with the Laplace transform (LT-HPM). Finally, we give some examples for illustration.

1.1 Setting of the problem

In the rectangle $Q = (0,1)\times(0,T)$, with $T<+\infty$, we consider the time fractional initial boundary problem of higher order with purely integral conditions

$$\begin{cases} Lu = {}^c\partial_t^{\alpha+1} u + (-1)^m a(t) \dfrac{\partial^{2m+1} u}{\partial x^{2m}\partial t} + (-1)^m b(t) \dfrac{\partial^{2m} u}{\partial x^{2m}} = f(x,t), & x \in (0,1),\ t \in (0,T), \\ \ell_1 u = u(x,0) = \varphi(x), \quad \ell_2 u = u_t(x,0) = \psi(x), & x \in (0,1), \\ \int_0^1 x^i u(x,t)\,dx = 0, & i = \overline{0, 2m-1},\ t \in (0,T), \end{cases} \tag{1.1}$$

where $a(t)$, $b(t)$, $f(x,t)$, $\varphi(x)$ and $\psi(x)$ are given functions that satisfy certain conditions which will be specified later, and the operator ${}^c\partial_t^{\alpha+1}$ denotes the Caputo left fractional derivative of order $(1+\alpha)$ with $0<\alpha<1$ defined by (see [20])

$${}^c\partial_t^{\alpha+1} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_{\tau\tau}(x,\tau)}{(t-\tau)^\alpha}\,d\tau, \quad t>0,$$

where $\Gamma(\cdot)$ is the Gamma function.

In this paper, we prove the existence and uniqueness of a strong solution of problem (1.1). We consider the solution of problem (1.1) as a solution of the operator equation $\Xi u = \omega = (f, \varphi, \psi)$, where $\Xi = (L, \ell_1, \ell_2)$

and $\Xi : E \to F$ is an unbounded operator with domain of definition

$$D(\Xi) = \begin{cases} u \in L^2(Q), {}^c\partial_t^{\alpha+1} u, \dfrac{\partial^{2m+1} u}{\partial x^{2m} \partial t}, & \dfrac{\partial^{2m} u}{\partial x^{2m}} \in L^2(Q), \\ \int_0^1 x^i u(x,t)\, dx = 0, & i = \overline{0, 2m-1},\ t \in (0,T), \end{cases}$$

such that u satisfies the initial conditions in problem (1.1), where E is a Banach space of functions u equipped with the finite norm

$$\|u\|_E^2 = \sup_{0 \le t \le T} \Big(D^{\alpha-1} \|\mathfrak{I}_x^m u_t\|_{L^2(0,1)}^2 + \int_0^t \|u_\tau\|_{L^2(0,1)}^2\, d\tau + \int_0^1 u^2\, dx \Big),$$

F is a Hilbert space constituted of the element $\omega = (f, \varphi, \psi)$ equipped by the norm

$$\|\omega\|_F^2 = \|\varphi\|_{L^2(0,1)}^2 + \|\psi\|_{L^2(0,1)}^2 + \int_0^1 \int_0^t f^2\, dx\, d\tau,$$

and L denotes the time fractional differential operator defined as follows:

$$L = {}^c\partial_t^{\alpha+1} + (-1)^m a(t) \frac{\partial^{2m+1}}{\partial x^{2m} \partial t} + (-1)^m b(t) \frac{\partial^{2m}}{\partial x^{2m}}.$$

Then we establish an energy inequality

$$\|u\|_E \le C \|\Xi u\|_F \quad \text{for all } u \in D(\Xi), \tag{1.2}$$

and we show that the operator Ξ has a closure $\overline{\Xi}$.

Definition 1.1. A solution of the operator equation $\overline{\Xi} u = \omega$ is called a strong solution of problem (1.1).

Inequality (1.2) can be extended to $u \in D(\overline{\Xi})$, that is,

$$\|u\|_E \le C \|\overline{\Xi} u\|_F \quad \text{for all } u \in D(\overline{\Xi}).$$

From this inequality, we obtain the uniqueness of a strong solution, if it exists, and the equality of the sets $R(\overline{\Xi})$ and $\overline{R(\Xi)}$. Thus, to prove the existence of the strong solution of problem (1.1) for any $\omega \in F$, it remains to prove that the set $R(\Xi)$ is dense in F.

2 Preliminaries

We begin this section by briefly introducing the basic definitions of fractional calculus. For any $n - 1 < \alpha < n$, $n \in \mathbb{N}$, the commonly used fractional definitions are as follows.

Definition 2.1 ([13]). The partial Riemann–Liouville fractional integral operator of order α with respect to t of a function $u(x,t)$ is defined by

$$I^\alpha u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(x,s)}{(t-s)^{n-\alpha}}\, ds.$$

Definition 2.2 ([13]). The partial Riemann–Liouville fractional derivative of order α of a function $u(x,t)$ with respect to t is of the form

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(x,s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where the function $u(x,t)$ has absolutely continuous derivatives up to order $(n-1)$.

Definition 2.3 ([13]). The Caputo partial fractional derivative of order α with respect to t of a function $u(x,t)$ is defined by

$$\frac{{}^{c}\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{1}{(t-s)^{\alpha-n+1}}\frac{\partial^{n}u(x,s)}{\partial s^{n}}\,ds,$$

where the function $u(x,t)$ has absolutely continuous derivatives up to order $(n-1)$.

Now, we introduce some important lemmas and inequalities needed throughout the sequel.

Lemma 2.4 ([2]). *For any absolutely continuous function $J(s)$ on the interval $[0,T]$, the following inequality holds:*

$$J(s)\partial_{s}^{\alpha}J(s)\geq\frac{1}{2}\partial_{s}^{\alpha}J^{2}(s),\quad 0<\alpha<1.$$

Lemma 2.5 ([14]). *Let $\varphi(t)$ be nonnegative and absolutely continuous on $[0,T]$, and for almost all $t\in[0,T]$ satisfy the inequality*

$$\frac{d\varphi}{dt}\leq C(t)\varphi(t)+B(t),$$

where the functions $C(t)$ and $B(t)$ are summable and nonnegative on $[0,T]$. Then

$$\varphi(t)\leq e^{\int_{0}^{t}C(\tau)\,d\tau}\Big(\varphi(0)+\int_{0}^{t}B(\xi)e^{\int_{0}^{\xi}c(\tau)\,d\tau}\,d\xi\Big)\leq e^{\int_{0}^{t}C(\tau)\,d\tau}\Big(\varphi(0)+\int_{0}^{t}B(\tau)\,d\tau\Big).$$

Lemma 2.5 can be generalized as follows.

Lemma 2.6 ([2]). *Let a nonnegative absolutely continuous function $Z(t)$ satisfy the inequality*

$$\partial_{t}^{\alpha}Z(t)\leq c_{1}Z(t)+c_{2}(t),\quad 0<\alpha<1,$$

for almost all $t\in[0,T]$, where c_1 is a positive constant and $c_2(t)$ is an integrable nonnegative function on $[0,T]$. Then

$$Z(t)\leq Z(0)E_{\alpha}(c_{1}t^{\alpha})+\Gamma(\alpha)E_{\alpha,\alpha}(c_{1}t^{\alpha})D_{t}^{-\alpha}c_{2}(t),$$

where

$$E_{\alpha}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\alpha n+1)}\quad\text{and}\quad E_{\alpha,\alpha'}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\alpha n+\alpha')}$$

are Mittag-Leffler functions.

Lemma 2.7 ([17]). *Let $Z_i(\tau)$, $i=1,2,3$, be nonnegative functions on the interval $[0,T]$, let $Z_1(\tau)$, $Z_2(\tau)$ be integrable functions, and let $Z_3(\tau)$ be nondecreasing. Then*

$$\int_{0}^{t}Z_{1}(\tau)\,d\tau+Z_{2}(t)\leq Z_{3}(t)+C\int_{0}^{t}Z_{2}(\tau)\,d\tau$$

implies

$$\int_{0}^{t}Z_{1}(\tau)d(\tau)+Z_{2}(t)\leq e^{Ct}Z_{3}(t).$$

Lemma 2.8 (Poincaré-type inequality). *For $m\in\mathbb{N}$, we have*

$$\|\mathbb{J}_{x}^{2m}u\|_{L^{2}(0,A)}^{2}\leq\Big(\frac{A}{2}\Big)^{2m}\|u\|_{L^{2}(0,A)}^{2},$$

where

$$\mathbb{J}_{x}^{2m}u=\int_{0}^{x}\int_{0}^{\xi_{1}}\cdots\int_{0}^{\xi_{2m-1}}u(\eta,t)d\eta\,d\xi_{2m-1}\cdots d\xi_{1}=\int_{0}^{x}\frac{(x-\xi)^{2m-1}}{(2m-1)!}u(\xi,t)\,d\xi.$$

Also, we will use Young's inequality with ϵ: For any $\epsilon>0$, we have the inequality

$$ab\leq\frac{1}{p}|\epsilon a|^{p}+\frac{p-1}{p}\Big|\frac{b}{\epsilon}\Big|^{\frac{p}{p-1}},\quad a,b\in\mathbb{R},\ p>1,\tag{2.1}$$

where a and b are nonnegative numbers.

A special case of (2.1) is the Cauchy inequality with ϵ (see [10]):

$$ab \le \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2, \quad \epsilon > 0. \tag{2.2}$$

3 Uniqueness of solution

In this section, we prove the uniqueness of solution of problem (1.1), and we study the effectiveness of the a priori estimate for the solution, from which we deduce the uniqueness of solution of the problem.

Theorem 3.1. *Assume that the functions $a(t)$, $b(t)$ satisfy the conditions*

$$c_2 \le a(t) \le c_1, \quad c_4 \le b(t) \le c_3, \tag{3.1}$$

$$c_6 \le b'(t) \le c_5, \quad c_8 \le a'(t) \le c_7, \quad c_{10} \le a''(t) \le c_9 \quad \text{for all } t \in [0, T], \tag{3.2}$$

where c_i, $i = 1, \dots, 10$, are positive constants. Then for any $u \in D(\Xi)$, there exists a positive constant κ such that the following a priori estimate is satisfied:

$$\|u\|_E \le \kappa \|\Xi u\|_F, \tag{3.3}$$

where $\kappa = \kappa(\eta, \delta, \rho)$ is given by

$$\kappa = \rho\Big(1 + \frac{T^\alpha}{\Gamma(1+\alpha)}\Big),$$

with η, δ and ρ respectively given by (3.13), (3.15) *and* (3.17).

Proof. For $u \in D(\Xi)$, we consider the scalar product in $L^2(0,1)$ of the differential equation in problem (1.1) and the integrodifferential operator $Mu = (-1)^m \mathfrak{I}_x^{2m} u_t$. We have

$$(-1)^m ({}^c\partial_t^{\alpha+1} u, \mathfrak{I}_x^{2m} u_t)_{L^2(0,1)} + \Big(a(t)\frac{\partial^{2m+1} u}{\partial x^{2m}\partial t}, \mathfrak{I}_x^{2m} u_t\Big)_{L^2(0,1)} + \Big(b(t)\frac{\partial^{2m} u}{\partial x^{2m}}, \mathfrak{I}_x^{2m} u_t\Big)_{L^2(0,1)} = (-1)^m (f, \mathfrak{I}_x^{2m} u_t)_{L^2(0,1)}. \tag{3.4}$$

We separately consider the inner products on the left-hand side of equation (3.4) and we integrate by parts and take into account boundary and initial conditions in problem (1.1) to obtain

$$(-1)^m ({}^c\partial_t^{\alpha+1} u, \mathfrak{I}_x^{2m} u_t)_{L^2(0,1)} = (-1)^m ({}^c\partial_t^{\alpha} u_t, \mathfrak{I}_x^{2m} u_t)_{L^2(0,1)} = ({}^c\partial_t^{\alpha} \mathfrak{I}_x^{m} u_t, \mathfrak{I}_x^{m} u_t)_{L^2(0,1)}, \tag{3.5}$$

$$\Big(a(t)\frac{\partial^{2m+1} u}{\partial x^{2m}\partial t}, \mathfrak{I}_x^{2m} u_t\Big)_{L^2(0,1)} = \int_0^1 a(t)\frac{\partial^{2m+1} u}{\partial x^{2m}\partial t}\mathfrak{I}_x^{2m} u_t\,dx = \int_0^1 a(t)\frac{\partial^{2m} u_t}{\partial x^{2m}}\mathfrak{I}_x^{2m} u_t\,dx = a(t)\|u_t\|_{L^2(0,1)}^2, \tag{3.6}$$

$$\Big(b(t)\frac{\partial^{2m} u}{\partial x^{2m}}, \mathfrak{I}_x^{2m} u_t\Big)_{L^2(0,1)} = b(t)\int_0^1 \frac{\partial^{2m} u}{\partial x^{2m}}\mathfrak{I}_x^{2m} u_t\,dx = b(t)\int_0^1 u u_t\,dx. \tag{3.7}$$

Substituting (3.5)–(3.7) into (3.4), we get

$$\int_0^1 ({}^c\partial_t^{\alpha} \mathfrak{I}_x^{m} u_t)(\mathfrak{I}_x^{m} u_t)\,dx + a(t)\|u_t\|_{L^2(0,1)}^2 + b(t)\int_0^1 u u_t\,dx = (-1)^m \int_0^1 f(x,t)\mathfrak{I}_x^{2m} u_t\,dx. \tag{3.8}$$

By Lemmas 2.4 and 2.8 and inequality (2.2), identity (3.8) reduces to

$$\frac{1}{2}\int_0^1 ({}^c\partial_t^{\alpha} \mathfrak{I}_x^{m} u_t)^2\,dx + a(t)\|u_t\|_{L^2(0,1)}^2 + b(t)\int_0^t u u_t\,dx \le \frac{1}{2\epsilon}\int_0^1 f^2\,dx + \frac{\epsilon}{2^{m+1}}\int_0^1 (\mathfrak{I}_x^{m} u_t)^2\,dx.$$

Replacing t by τ, integrating with respect to τ from zero to t and using the given conditions, we obtain

$$\frac{1}{2}\int_0^t\int_0^1 ({}^c\partial_\tau^\alpha \mathfrak{I}_x^m u_\tau)^2\,dx\,d\tau + \int_0^t a(\tau)\|u_\tau\|_{L^2(0,1)}^2\,d\tau + \int_0^t\int_0^1 b(\tau)uu_\tau\,dx\,d\tau \leq \frac{1}{2\epsilon}\int_0^t\int_0^1 f^2\,dx\,d\tau + \frac{\epsilon}{2^{m+1}}\int_0^1\int_0^t (\mathfrak{I}_x^m u_\tau)^2\,dx\,d\tau. \tag{3.9}$$

The third term on the left-hand side of (3.9) can be evaluated as

$$\int_0^1\int_0^t b(\tau)uu_\tau\,dx\,dt = \frac{1}{2}\int_0^1 b(t)u^2\,dx - \frac{1}{2}b(0)\int_0^1 \varphi^2(x)\,dx - \frac{1}{2}\int_0^1\int_0^t b'u^2\,dx\,d\tau.$$

Hence, inequality (3.9) becomes

$$\frac{1}{2}\int_0^t\int_0^1 ({}^c\partial_\tau^\alpha \mathfrak{I}_x^m u_\tau)^2\,dx\,d\tau + \int_0^t a(\tau)\|u_\tau\|_{L^2(0,1)}^2\,d\tau + \frac{1}{2}\int_0^1 b(t)u^2\,dx \leq \frac{1}{2}b(0)\int_0^1 \varphi^2\,dx + \frac{1}{2}\int_0^t\int_0^1 b'u^2\,dx\,d\tau + \frac{1}{2\epsilon}\int_0^t\int_0^1 f^2\,dx\,d\tau + \frac{\epsilon}{2^{m+1}}\int_0^t\int_0^1 (\mathfrak{I}_x^m u_\tau)^2\,dx\,d\tau. \tag{3.10}$$

Now since

$$\int_0^t\int_0^1 ({}^c\partial_\tau^\alpha \mathfrak{I}_x^m u_\tau)^2\,dx\,d\tau = D^{\alpha-1}\|\mathfrak{I}_x^m u_t\|_{L^2(0,1)}^2 - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\|\mathfrak{I}_x^m \psi\|_{L^2(0,1)}^2, \tag{3.11}$$

evoking conditions (3.1) and (3.2) and using (3.11), we infer from (3.10) that

$$D^{\alpha-1}\|\mathfrak{I}_x^m u_t\|_{L^2(0,1)}^2 + \int_0^t \|u_\tau\|_{L^2(0,1)}^2\,d\tau + \int_0^1 u^2\,dx \leq \eta\Big(\int_0^1 \varphi^2\,dx + \int_0^1\int_0^t u^2\,dx\,d\tau + \int_0^t\int_0^1 f^2\,dx\,d\tau + \|\psi\|_{L^2(0,1)}^2 + \int_0^t\int_0^1 (\mathfrak{I}_x^m u_\tau)^2\,dx\,d\tau\Big), \tag{3.12}$$

where

$$\eta = \frac{\max(\frac{1}{2\epsilon}, \frac{\epsilon}{2^{m+1}}, \frac{c_3}{2}, \frac{1}{2}, \frac{T^{1-\alpha}2^{-m}}{(1-\alpha)\Gamma(1-\alpha)})}{\min(\frac{1}{2}, c_2, c_4)}. \tag{3.13}$$

If, in Lemma 2.5, we set

$$\varphi(t) = \int_0^1\int_0^t u^2\,dx\,d\tau, \quad \varphi'(t) = \int_0^1 u^2\,dx = \|u\|_{L^2(0,1)}^2, \quad \varphi(0) = 0.$$

then it yields

$$\int_0^1\int_0^t u^2\,dx\,d\tau \leq e^{\eta T}T\eta\Big(\int_0^1 \varphi^2(x)\,dx + \int_0^1\int_0^t f^2\,dx\,d\tau + \|\psi\|_{L^2(0,1)}^2 + \int_0^1\int_0^t (\mathfrak{I}_x^m u_\tau)^2\,dx\,d\tau\Big).$$

Consequently, (3.12) transforms to

$$D^{\alpha-1}\|\mathfrak{I}_x^m u_t\|_{L^2(0,1)}^2 + \int_0^t \|u_\tau\|_{L^2(0,1)}^2\,d\tau + \int_0^1 u^2\,dx \leq \delta\Big(\|\varphi\|_{L^2(0,1)}^2 + \|f\|_{L^2(Q)}^2 + \|\psi\|_{L^2(0,1)}^2 + \int_0^1\int_0^t (\mathfrak{I}_x^m u_\tau)^2\,dx\,d\tau\Big), \tag{3.14}$$

where

$$\delta = \max(\eta, \eta^2 T e^{\eta T}). \tag{3.15}$$

Now, setting

$$Z(t) = \int_Q (\mathfrak{I}_x^m u_\tau)^2 \, dx \, d\tau = \int_0^t \|\mathfrak{I}_x^m u_\tau\|^2_{L^2(0,1)} \, d\tau, \; \partial_t^\alpha Z(t) = D^{\alpha-1} \|\mathfrak{I}_x^m u_t\|^2_{L^2(0,1)}$$

in Lemma 2.6, we obtain

$$\begin{aligned}\int_{Q^\tau} (\mathfrak{I}_x^m u_\tau)^2 \, dx \, d\tau &\le \delta\Gamma(\alpha) E_{\alpha,\alpha}(\delta t^\alpha) \Big(\frac{T}{\alpha\Gamma(\alpha)} \|\varphi\|^2_{L^2(0,1)} + \frac{T}{\alpha\Gamma(\alpha)} \|\psi\|^2_{L^2(0,1)} + D_t^{-\alpha-1} \|f\|^2_{L^2(0,1)} \Big) \\ &\le \delta\Gamma(\alpha) E_{\alpha,\alpha}(\delta t^\alpha) \max\Big\{1, \frac{T}{\alpha\Gamma(\alpha)}\Big\} (D_t^{-\alpha-1} \|f\|^2_{L^2(0,1)} + \|\varphi\|^2_{L^2(0,1)} + \|\psi\|^2_{L^2(0,1)}).\end{aligned} \tag{3.16}$$

Combining (3.14) and (3.16) gives

$$\begin{aligned}&D^{\alpha-1} \|\mathfrak{I}_x^m u_t\|^2_{L^2(0,1)} + \int_0^t \|u_\tau\|^2_{L^2(0,1)} \, d\tau + \|u\|^2_{L^2(0.1)} \\ &\quad \le \rho\Big(\|\varphi\|^2_{L^2(0,1)} + \|\psi\|^2_{L^2(0,1)} + \int_0^1 \int_0^t f^2 \, dx \, d\tau + D_t^{-\alpha-1} \|f\|^2_{L^2(0,1)} \Big),\end{aligned}$$

where

$$\rho = \max\Big(\delta, \delta^2 \Gamma(\alpha) E_{\alpha,\alpha}(\delta t^\alpha) \max\Big\{1, \frac{T}{\alpha\Gamma(\alpha)}\Big\}\Big). \tag{3.17}$$

It is obvious that

$$D_t^{-\alpha-1} \|f\|^2_{L^2(0,1)} \le \frac{t^\alpha}{\Gamma(1+\alpha)} \int_0^t \|f\|^2 \, d\tau \le \frac{T^\alpha}{\Gamma(1+\alpha)} \int_0^T \|f\|^2 \, d\tau. \tag{3.18}$$

Then it follows from (3.17) and (3.18) that

$$D^{\alpha-1} \|\mathfrak{I}_x^m u_t\|^2_{L^2(0,1)} + \int_0^t \|u_\tau\|^2_{L^2(0,1)} \, d\tau + \|u\|^2_{L^2(0,1)} \le \kappa\Big(\|\varphi\|^2_{L^2(0,1)} + \|\psi\|^2_{L^2(0,1)} + \int_0^1 \int_0^t f^2 \, dx \, d\tau \Big). \tag{3.19}$$

Since the right-hand side of (3.19) is independent of τ, we take the supremum with respect to τ from 0 to T in the left-hand side, and thus obtaining (3.3), where

$$\kappa = \sigma\Big(1 + \frac{T^\alpha}{\Gamma(1+\alpha)}\Big).$$

This completes the proof. □

It follows from (3.3) that there is a bounded inverse Ξ^{-1} on the range $R(\Xi)$ of Ξ. However, since we have no information concerning $R(\Xi)$ except that $R(\Xi) \subset F$, we must extend Ξ (construct its closure $\overline{\Xi}$) so that (3.3) holds for the extension and its range is the whole space.

We first show that $\Xi : E \to F$ with domain $D(\Xi)$ has a closure, that is, the closure of the graph $G(\Xi) \subset E \times F$ of Ξ is a graph $G(\overline{\Xi}) = \overline{G(\Xi)}$ of a new linear operator $\overline{\Xi}$, which we call the closure of Ξ.

Lemma 3.2. *The operator Ξ from E to F admits a closure $\overline{\Xi}$.*

The previous theorem is valid for strong solutions. Then we have the inequalities

$$\|u\|_E \le C \|\overline{\Xi} u\|_F \quad \text{for all } u \in D(\overline{\Xi}).$$

Hence, we obtain the following corollaries.

Corollary 3.3. *A strong solution of problem* (1.1) *is unique if it exists, and it depends continuously on ω.*

Corollary 3.4. *The range $R(\overline{\Xi})$ of the operator $\overline{\Xi}$ is closed in F, and $R(\overline{\Xi}) = \overline{R(\Xi)}$.*

4 Existence of solution of the posed problem

To show the solvability of problem (1.1), it is sufficient to show that $R(\Xi)$ is dense in F for all $u \in E$ and for all arbitrary $\omega = (f, \varphi, \psi) \in F$. The proof is based on the following theorem.

Theorem 4.1. *Suppose that the conditions of Theorem 3.1 are satisfied. Then problem* (1.1) *admits a unique solution* $u = \overline{\Xi}^{-1}\omega = \overline{\Xi^{-1}}\omega$.

Proof. First, we prove that $R(\Xi)$ is dense in F, for the special case where $D(\Xi) \equiv E$ is reduced to $D_0(\Xi)$, where

$$D_0(\Xi) = \{u : u \in D(\Xi),\ \ell_1 u = \ell_2 u = 0\}.$$ □

Proposition 4.2. *Let the conditions of Theorem 4.1 be satisfied. If for some* $\varpi \in L^2(Q)$ *we consider*

$$(Lu, \varpi)_{L^2(Q)} = 0, \tag{4.1}$$

then ϖ *vanishes almost everywhere in* Q.

Proof. Identity (4.1) is equivalent to

$$\Big({}^c\partial_t^{\alpha+1} u + (-1)^m a(t)\frac{\partial^{2m+1} u}{\partial x^{2m}\partial t} + (-1)^m b(t)\frac{\partial^{2m} u}{\partial x^{2m}}, \varpi\Big)_{L^2(Q)} = 0. \tag{4.2}$$

Since (4.1) holds for all functions $u \in D_0(\Xi)$, it can be expressed in the form

$$u(x,t) = \mathbb{J}_t^2 z = \int_0^t \int_0^s z(x,\tau)\, d\tau\, ds,$$

where $z(x,t)$ verifies the boundary and initial conditions in (1.1) and such that

$$z, z_x, \mathbb{J}_t z, \mathbb{J}_t \mathbb{J}_x^{2m} z, \mathbb{J}_t^2 z, \partial_t^{\alpha+1} z \in L^2(Q).$$

Therefore, equation (4.2) becomes

$$\int_0^T \Big({}^c\partial_t^{\alpha+1}\mathbb{J}_t^2 z + (-1)^m a(t)\frac{\partial^{2m+1}\mathbb{J}_t^2 z}{\partial x^{2m}\partial t} + (-1)^m b(t)\frac{\partial^{2m}\mathbb{J}_t^2 z}{\partial x^{2m}}, \varpi\Big)_{L^2(0,1)} dt = 0. \tag{4.3}$$

Now, we express ϖ in terms of z as

$$\varpi(x,t) = \mathbb{J}_t z + (-1)^m \mathbb{J}_x^{2m}\mathbb{J}_t z.$$

By substituting ϖ in equation (4.3) and applying integration by parts, each term can be estimated as

$$({}^c\partial_t^{\alpha+1}\mathbb{J}_t^2 z, \mathbb{J}_t z)_{L^2(0,1)} = \int_0^1 {}^c\partial_t^{\alpha}\mathbb{J}_t z \mathbb{J}_t z dx \ge \frac{1}{2}{}^c\partial_t^{\alpha}\|\mathbb{J}_t z\|_{L^2(0,1)}^2, \tag{4.4}$$

$$({}^c\partial_t^{\alpha+1}\mathbb{J}_t^2 z, (-1)^m\mathbb{J}_x^{2m}\mathbb{J}_t z)_{L^2(0,1)} = \int_0^1 {}^c\partial^{\alpha}\mathbb{J}_x^m(\mathbb{J}_t z)\mathbb{J}_x^m(\mathbb{J}_t z)\, dx \ge \frac{1}{2}{}^c\partial_t^{\alpha}\|\mathbb{J}_x^m\mathbb{J}_t z\|_{L^2(0,1)}^2, \tag{4.5}$$

$$\Big((-1)^m a(t)\frac{\partial^{2m+1}(\mathbb{J}_t^2 z)}{\partial x^{2m}\partial t}, \mathbb{J}_t z\Big)_{L^2(0,1)} = \int_0^1 a(t)\Big(\frac{\partial^m}{\partial x^m}(\mathbb{J}_t z)\Big)^2 dx, \tag{4.6}$$

$$\Big((-1)^m a(t)\frac{\partial^{2m+1}(\mathbb{J}_t^2 z)}{\partial x^{2m}\partial t}, (-1)^m\mathbb{J}_x^{2m}\mathbb{J}_t z\Big)_{L^2(0,1)} = \int_0^1 a(t)(\mathbb{J}_t z)^2\, dx, \tag{4.7}$$

$$\Big((-1)^m b(t)\frac{\partial^{2m}(\mathbb{J}_t^2 z)}{\partial x^{2m}}, \mathbb{J}_t z\Big)_{L^2(0,1)} = \frac{1}{2}\frac{d}{dt}\int_0^1 b(t)\Big(\frac{\partial^m}{\partial x^m}(\mathbb{J}_t^2 z)\Big)^2 dx - \frac{1}{2}\int_0^1 b'(t)\Big(\frac{\partial^m}{\partial x^m}(\mathbb{J}_t^2 z)\Big)^2 dx, \tag{4.8}$$

$$\Big((-1)^m b(t)\frac{\partial^{2m}(\mathbb{J}_t^2 z)}{\partial x^{2m}}, (-1)^m\mathbb{J}_x^{2m}\mathbb{J}_t z\Big)_{L^2(0,1)} = \frac{1}{2}\frac{d}{dt}\int_0^1 b(t)(\mathbb{J}_t^2 z)^2\, dx - \frac{1}{2}\int_0^1 b'(t)(\mathbb{J}_t^2 z)^2\, dx. \tag{4.9}$$

Substituting (4.4)–(4.9) in (4.3), replacing t by τ, integrating with respect to τ from zero to t and using conditions (3.1) and (3.2), we obtain

$$\begin{aligned} D_t^{\alpha-1}\|\mathfrak{I}_t z\|^2_{L^2(0,1)} + D_t^{\alpha-1}\|\mathfrak{I}_x^m \mathfrak{I}_t z\|^2_{L^2(0,1)} + \int_0^t\int_0^1\Big(\frac{\partial^m}{\partial x^m}\mathfrak{I}_\tau z\Big)^2\,dx\,d\tau \\ + \int_0^t\int_0^1(\mathfrak{I}_\tau z)^2\,dx\,d\tau + \int_0^1\Big(\frac{\partial^m}{\partial x^m}(\mathfrak{I}_\tau^2 z)\Big)^2\,dx + \int_0^1(\mathfrak{I}_\tau^2 z)^2\,dx \\ \le \eta_1\Big(\int_0^1\int_0^t\Big(\frac{\partial^m(\mathfrak{I}_\tau^2 z)}{\partial x^m}\Big)^2\,dx\,d\tau + \int_0^1\int_0^t(\mathfrak{I}_\tau^2 z)^2\,dx\,d\tau\Big), \end{aligned}$$

where

$$\eta_1 = \frac{c_5}{\min(1, 2c_2, c_4)}. \tag{4.10}$$

If, in Lemma 2.5, we set

$$\varphi_1(t) = \int_0^t\int_0^1\Big(\frac{\partial^m}{\partial x^m}(\mathfrak{I}_\tau^2 z)\Big)^2\,dx\,d\tau, \quad \varphi_1'(t) = \int_0^1\Big(\frac{\partial^m}{\partial x^m}(\mathfrak{I}_\tau^2 z)\Big)^2\,dx, \quad \varphi_1(0) = 0,$$

then

$$\varphi_1(t) = \int_0^t\int_0^1\Big(\frac{\partial^m}{\partial x^m}(\mathfrak{I}_\tau^2 z)\Big)^2\,dx\,d\tau \le \eta_1 T e^{T\eta_1}\int_0^t\int_0^1(\mathfrak{I}_\tau^2 z)^2\,dx\,d\tau.$$

From (4.10), we have that

$$\begin{aligned} D_t^{\alpha-1}\|\mathfrak{I}_t z\|^2_{L^2(0,1)} + D_t^{\alpha-1}\|\mathfrak{I}_x^m \mathfrak{I}_t z\|^2_{L^2(0,1)} + \int_0^t\int_0^1\Big(\frac{\partial^m}{\partial x^m}\mathfrak{I}_\tau z\Big)^2\,dx\,d\tau \\ + \int_0^t\int_0^1(\mathfrak{I}_\tau z)^2\,dx\,d\tau + \int_0^1\Big(\frac{\partial^m}{\partial x^m}(\mathfrak{I}_\tau^2 z)\Big)^2\,dx + \int_0^1(\mathfrak{I}_\tau^2 z)^2\,dx \\ \le \eta_2\int_0^1\int_0^t(\mathfrak{I}_\tau^2 z)^2\,dx\,d\tau, \end{aligned}$$

where

$$\eta_2 = \max(\eta_1^2 T e^{T\eta_1}, \eta_1). \tag{4.11}$$

If, in Lemma 2.5, we set

$$\varphi_2(t) = \int_0^1\int_0^t(\mathfrak{I}_\tau^2 z)^2\,dx\,d\tau, \quad \varphi_2'(t) = \int_0^1(\mathfrak{I}_\tau^2 z)^2\,dx, \quad \varphi_2(0) = 0,$$

then

$$\varphi_2(t) = \int_0^1\int_0^t(\mathfrak{I}_\tau^2 z)^2\,dx\,d\tau \le 0.$$

Consequently, (4.11) transforms to

$$\begin{aligned} D_t^{\alpha-1}\|\mathfrak{I}_t z\|^2_{L^2(0,1)} + D_t^{\alpha-1}\|\mathfrak{I}_x^m \mathfrak{I}_t z\|^2_{L^2(0,1)} + \int_0^t\int_0^1\Big(\frac{\partial^m}{\partial x^m}\mathfrak{I}_\tau z\Big)^2\,dx\,d\tau \\ + \int_0^t\int_0^1(\mathfrak{I}_\tau z)^2\,dx\,d\tau + \int_0^1\Big(\frac{\partial^m}{\partial x^m}(\mathfrak{I}_\tau^2 z)\Big)^2\,dx + \int_0^1(\mathfrak{I}_\tau^2 z)^2\,dx \le 0 \end{aligned}$$

for all $t \in [0, T]$, and hence $\varpi = 0$ a.e in Q. This proves Proposition 4.2.

We return to the proof of Theorem 4.1. We have already noted that it is sufficient to prove that the set $R(\Xi)$ is dense in F.

Suppose that, for some element $(F_1, \varphi_1, \varphi_2) \in R(\Xi)^\perp$, we have

$$\int_0^T (Lu, F_1)_{L^2(0,1)}\, ds + (\ell_1 u, \varphi_1)_{L^2(0,1)} + (\ell_2 u, \varphi_2)_{L^2(0,1)} = 0. \tag{4.12}$$

Then we must prove that $F_1 = 0$, $\varphi_1 = 0$ and $\varphi_2 = 0$. Putting $u \in D_0(\Xi)$ in (4.12), we obtain

$$\int_0^T (Lu, F_1)_{L^2(0,1)}\, ds = 0 \quad \text{for all } u \in D_0(\Xi).$$

Hence, Proposition 4.2 implies that $F_1 = 0$ a.e in Q. Thus, (4.12) takes the form

$$(\ell_1 u, \varphi_1)_{L^2(0,1)} + (\ell_2 u, \varphi_2)_{L^2(0,1)} = 0 \quad \text{for all } u \in D(\Xi). \tag{4.13}$$

Since the ranges of the trace operators ℓ_1 and ℓ_2 are dense in $L^2(0, 1)$, it follows then from (4.13) that $\varphi_1 = 0$ and $\varphi_2 = 0$.

Hence, $F_1 = 0$, $\varphi_1 = 0$ and $\varphi_2 = 0$ imply that $(\overline{R(\Xi)} = F)$. Thus, the proof of Theorem 4.1 is complete. □

5 Homotopy perturbation method with Laplace transform (LT-HPM)

5.1 Basic idea of He's homotopy perturbation method

The homotopy perturbation method is a combination of the classical perturbation technique and homotopy technique, which has eliminated the limitations of the traditional perturbation methods. This technique can have advantages compared to the traditional perturbation techniques. To illustrate the basic idea of the homotopy perturbation method for solving nonlinear differential equations, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{5.1}$$

subject to the boundary condition

$$B\Big(u, \frac{\partial u}{\partial \eta}\Big) = 0, \quad r \in \Gamma,$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω.

The operator A can, generally speaking, be divided into two parts: a linear part L and a nonlinear part N. Equation (5.1) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0.$$

By the homotopy technique, we construct a homotopy $v(r, p) : \Omega \times [0, 1] \to \mathbb{R}$ which satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0, 1],\ r \in \Omega, \tag{5.2}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{5.3}$$

where $p \in [0, 1]$ is an embedding parameter and u_0 is an initial approximation of (5.1) which satisfies the boundary conditions. It follows from (5.2) and (5.3) that we have

$$H(v, 0) = L(v) - L(u_0) = 0, \quad H(v, 1) = A(v) - f(r) = 0.$$

Thus, the changing process of p from zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called homotopy. According to the HPM, we can first use the embedding parameter p as a “small parameter”, and assume that the solution of (5.2) and (5.3) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots .$$

Set $p = 1$ results in the approximate solution of (5.1):

$$u = \lim_{p\to 1} = v_0 + v_1 + v_2 + \cdots . \tag{5.4}$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated limitations of the traditional perturbation techniques. The series (5.4) is convergent for most cases; however, the convergent rate depends upon the nonlinear operator $A(V)$ (the following assertions are suggested by [9, 12]):

(i) The second derivative of $N(v)$ with respect to v must be minor because the parameter may be comparatively large, i.e. $p \to 1$.
(ii) The norm of $L^{-1}\partial N \setminus \partial v$ must be smaller than one so that the series converges.

5.2 Laplace transform HPM

For solving the partial differential equation (1.1) for $1 < \alpha < 2$, with respect to t, applying the Laplace transformation to both sides of equation (1.1), we obtain

$$L\{u(x,t)\} = -s^{-\alpha-1}L\Big\{(-1)^m a(t)\frac{\partial^{2m+1}u}{\partial x^{2m}\partial t} + (-1)^m b(t)\frac{\partial^{2m}u}{\partial x^{2m}} - f(x,t)\Big\} + s^{-1}\varphi(x) + s^{-2}\psi(x). \tag{5.5}$$

By taking the inverse Laplace transformation L^{-1} of equation (5.5), we find

$$u(x,t) = -L^{-1}\Big\{s^{-\alpha-1}L\Big\{(-1)^m a(t)\frac{\partial^{2m+1}u}{\partial x^{2m}\partial t} + (-1)^m b(t)\frac{\partial^{2m}u}{\partial x^{2m}} - f(x,t)\Big\}\Big\} + \varphi(x) + t\psi(x). \tag{5.6}$$

According to the HPM technique, to determine the approximate solution of equation (5.6), we construct a homotopy proposed by Madani, Fathizadeh, Khan and Yildrim [16] in the following form:

$$u(x,t) - \varphi(x) - t\psi(x) = -P\Big[L^{-1}\Big\{s^{-\alpha-1}L\Big\{(-1)^m a(t)\frac{\partial^{2m+1}u}{\partial x^{2m}\partial t} + (-1)^m b(t)\frac{\partial^{2m}u}{\partial x^{2m}} - f(x,t)\Big\}\Big\}\Big]. \tag{5.7}$$

Now, let us present the solution of equation (5.7) in the following form:

$$u(x,t) = \sum_{j=0}^{\infty} P^j u_j(x,t), \tag{5.8}$$

where $u_j(x,t)$, $j = 0, 1, \ldots$, are functions which should be determined. By substituting (5.8) into (5.7), we get

$$\begin{aligned}\sum_{j=0}^{\infty} P^j u_j(x,t) - \varphi(x) - t\psi(x) = -P\Big[L^{-1}\Big\{s^{-\alpha-1}L\Big\{(-1)^m a(t)&\frac{\partial^{2m+1}(\sum_{j=0}^{\infty} P^j u_j(x,t))}{\partial x^{2m}\partial t}\\ &+ (-1)^m b(t)\frac{\partial^{2m}(\sum_{j=0}^{\infty} P^j u_j(x,t))}{\partial x^{2m}} - f(x,t)\Big\}\Big\}\Big].\end{aligned} \tag{5.9}$$

Equating the coefficients of P with the same powers in (5.9) leads to

$$\begin{aligned}
P^0 &: u_0(x,t) = \varphi(x) + t\psi(x),\\
P^1 &: u_1(x,t) = -L^{-1}\Big\{s^{-\alpha-1}L\Big\{(-1)^m a(t)\frac{\partial^{2m+1}u_0}{\partial x^{2m}\partial t} + (-1)^m b(t)\frac{\partial^{2m}u_0}{\partial x^{2m}} - f(x,t)\Big\}\Big\},\\
P^2 &: u_2(x,t) = -L^{-1}\Big\{s^{-\alpha-1}L\Big\{(-1)^m a(t)\frac{\partial^{2m+1}u_1}{\partial x^{2m}\partial t} + (-1)^m b(t)\frac{\partial^{2m}u_1}{\partial x^{2m}}\Big\}\Big\},\\
&\vdots\\
P^{n+1} &: u_{n+1}(x,t) = -L^{-1}\Big\{s^{-\alpha-1}L\Big\{(-1)^m a(t)\frac{\partial^{2m+1}u_n}{\partial x^{2m}\partial t} + (-1)^m b(t)\frac{\partial^{2m}u_n}{\partial x^{2m}}\Big\}\Big\}, \quad n \geq 1.
\end{aligned}$$

When $P \to 1$, equation (5.8) becomes the approximate solution of equation (1.1), i.e.

$$u(x,t) \cong H_n(x,t) = \sum_{j=0}^{n} u_j(x,t).$$

Example 5.1. Consider

$$\begin{aligned} f(x,t) &= \Big(2t^{\frac{3}{2}} - \frac{3\sqrt{\pi}}{16}t^4\Big)e^x, \quad 0 < x < 1,\ 0 < t \le T,\ \alpha = \frac{3}{2},\ m = 0, \\ \varphi(x) &= 0, \quad \psi(x) = 0, \qquad 0 < x < 1, \\ a(t) &= -t, \quad b(t) = 1, \end{aligned}$$

For an exact solution of the problem $u(x,t) = \frac{\sqrt{\pi}}{16}t^4e^x$, we obtain

$$\begin{aligned} u_0(x,t) &= 0, \\ u_1(x,t) &= \Big(\frac{\sqrt{\pi}}{16}t^4 - \frac{64}{15015}t^{\frac{13}{2}}\Big)e^x, \\ u_2(x,t) &= \Big(\frac{64}{15015}t^{\frac{13}{2}} - \frac{11\sqrt{\pi}}{161280}t^9\Big)e^x, \\ u_3(x,t) &= \Big(\frac{11\sqrt{\pi}}{161280}t^9 - \frac{8192}{3194284275}t^{\frac{23}{2}}\Big)e^x, \\ u_4(x,t) &= \Big(\frac{8192}{3194284275}t^{\frac{23}{2}} - \frac{\sqrt{\pi}}{41932800}t^{14}\Big)e^x, \\ u_5(x,t) &= \Big(\frac{\sqrt{\pi}}{41932800}t^{14} - \frac{131072}{234308649526875}t^{\frac{33}{2}}\Big)e^x, \\ u_6(x,t) &= \Big(\frac{131072}{234308649526875}t^{\frac{33}{2}} - \frac{31\sqrt{\pi}}{9001746432000}t^{19}\Big)e^x, \\ u_7(x,t) &= \Big(\frac{31\sqrt{\pi}}{9001746432000}t^{19} - \frac{31}{93768192000}t^{\frac{5}{2}}\Big)e^x. \end{aligned}$$

For $n = 7$, we have

$$u(x,t) \cong H_7(x,t) = \sum_{i=0}^{7} u_j(x,t) = \frac{\sqrt{\pi}}{16}t^4e^x - \frac{31}{93768192000}t^{\frac{5}{2}}e^x.$$

For $t \in [0.2, 2]$, we calculate u_{exa}, u_{hpm} and Relative Error $= \frac{u_{\text{exa}} - u_{\text{hpm}}}{u_{\text{exa}}}$ in Table 1.

t	u_{exa}	u_{hpm}	Relative Error
0.2	$0.0001772453e^x$	$(0.0001\sqrt{\pi} - 5.9140 \times 10^{-12})e^x$	3.3366×10^{-8}
0.4	$0.0016\sqrt{\pi}e^x$	$(0.00016\sqrt{\pi} - 3.3455 \times 10^{-11})e^x$	1.179664×10^{-8}
0.6	$0.0081\sqrt{\pi}e^x$	$0.0081\sqrt{\pi} - 9.219 \times 10^{-11}$	0.6413529×10^{-8}
0.8	$0.0256\sqrt{\pi}e^x$	$(0.0256\sqrt{\pi} - 1.8925 \times 10^{-10})e^x$	0.4170829×10^{-8}
1	$0.1107783657e^x$	$0.1107783654e^x$	0.270811×10^{-8}
1.2	$0.2297100191e^x$	$(0.1296\sqrt{\pi} - 5.2151 \times 10^{-8})e^x$	$0.227029714 \times 10^{-8}$
1.4	$0.42556616e^x$	$(0.2401\sqrt{\pi} - 7.667 \times 10^{-10})e^x$	$0.180160001 \times 10^{-8}$
1.6	$0.7259970973e^x$	$(0.4096\sqrt{\pi} - 1.705 \times 10^{-9})e^x$	0.14745×10^{-8}
1.8	$1.1629069716e^x$	$(0.6561\sqrt{\pi} - 1.4371 \times 10^{-9})e^x$	0.1235782×10^{-8}
2	$1.7724538504e^x$	$0.19 \times 10^{-8}e^x$	$0.10719602 \times 10^{-8}$

Table 1: Relative error calculation.

6 Conclusions

In this work, we are interested in the existence and the uniqueness of the strong solution of a fractional problem with integral conditions. Then the homotopy perturbation method with Laplace transform (LT-HPM) is used to get approximate solution.

In the field of development of numerical methods, there are many interesting points of view to continue the work in this article.

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