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Theme

Asymptotic Stability of an Epidemic Reaction-Diffusion model

Presented by:

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Ł. Djebara.

RÉSUMÉ

Dans cette thèse nous étudions deux types de problèmes. Pour la première, nous établissons la question de l'existence globale et le comportement à l'infini des solutions et en suite la stabilité asymptotique globale des solutions d'un modèle de réaction-diffusion modélisant un phénomène de propagation d'une épidémie sous des conditions sur la non-linéarité. Le résultat de Global existence est basé sur la méthode de fonctionnelle de Lyapunov. En ce qui concerne le deuxième problème, on calcule le nombre de reproduction de base de phénomène épidémique de réaction-diffusion et en analysant les valeurs propres et une fonction de Lyapunov convenablement construite, nous établissons à la fois localement et globalement dans les cas ODE et PDE étudiés en donnant des exemples d'analyse numérique des deux problèmes.

Mots clés: Réaction-diffusion, existence globale de solutions, stabilité asymptotique, SI modèle épidémique, stabilité asymptotique locale et globale, Lyapunov fonctionnel, équilibre endémique, équilibre sans maladie, taux de reproduction de base.

Abstract

The purpose of this thesis is to study two types of problems, the question of global existence and global asymptotic stability of solutions of epidemic reaction-diffusion models. For the first, we establish a result on the global existence in time of the solutions of a class of disease epidemics systems, our techniques of proof are based on Lyapunov functional method. The asymptotic behavior of these solutions via a Lyapunov functional in particular case and under suitable conditions on the non-linearity we contribute to the study of the behaviour of the solutions. As for the second problem through, we calculate the basic reproduction number of reaction-diffusion epidemic phenomena and by analyzing the eigenvalues and an appropriately constructed Lyapunov functional we establish both locally and globally in the ODE and PDE cases, with giving numerical analysis examples of the two problems.

Key Words: Reaction-diffusion, global existence of solutions, asymptotic stability, SI epidemic model, local and global asymptotic stability, Lyapunov functional, endemic equilibrium, steady state, disease free equilibrium, basic reproduction number.

ملخص

في هذه الأطروحة تم دراسة نوعين من المسائل، حيث أثبت **الوجود الكلي** للحل بالنسبة للزمن لحلول **نظام تفاعل-رد الفعل وانتشار الوباء**؛ وذلك بالاعتماد على **دالية ليابونوف**، ثم في حالة خاصة للنظام وتحت شروط للدالة غير الخطية تم **دراسة سلوك الحلول في اللانهاية**. بالنسبة للمسالة الثانية من خلال تحديد حالة العتبة المعطاة برقم التكاثر وتحليلا للقيم الذاتية ودالية ليابونوف التي تم إنشاؤها بشكل مناسب، تم **دراسة الاستقرار المحلي والكلي** لحلول نظام تفاعل-رد الفعل وانتشار الوباء في حالتي **ODE** و **PDE** مع إعطاء **أمثلة تحليلية وعددية** للمسألتين.

الكلمات المفتاحية : نظام تفاعل- رد الفعل و انتشار، الوجود الكلي للحلول، الاستقرار المقارب، نموذج وباء SI، الاستقرار المقارب المحلي والكلي، دالية **ليابونوف**، التوازن المستوطن ، حالة مستقرة، التوازن دون مرض، معدل التكاثر الأساسي.

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إلى كل من علمني حرفا في هذه الدنيا الفانية.

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Contents

Nomenclature

The following are some nomenclature that are used in this work

- $u^+ = \max(u, 0)$
- $u^- = \max(-u, 0)$
- u^T transpose of a vector u
- Ω open bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$
- $\overrightarrow{\eta}$ outward pointing unit normal on the surface element $d\sigma$
- $u_i = u_i(t, x)$
- $u_t = \dfrac{\partial u}{\partial t}$ partial derivative of u with respect to time t
- $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$ second partial derivative of u with respect to x
- $\dfrac{\partial u}{\partial \eta}(x) = \nabla u(x) . \overrightarrow{\eta}(x)$ outward normal derivative to $\partial\Omega$
- $\nabla u = \left(\frac{\partial u}{\partial x_1}(x), ..., \frac{\partial u}{\partial x_n}(x)\right)$ gradient operator
- $\Delta u = \nabla\nabla u = \nabla^2 u = \sum\limits_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ laplace operator of u
- $\limsup\limits_{t\to\infty} \phi(t) = \lim\limits_{t\to\infty} \sup\{\phi(\tau) \,/\, \tau \geq t\}$
- $u^{(n)}$ n^{th} derivatives of u
- ReZ real part of a complex number$Z$
- TrA trace of a matrix A
- $\det A$ determinant of a matrix A
- $\rho(A) = \max\{|\lambda| : \lambda \in spec(A)\}$ spectral radius of operator A
- R_0 basic reproduction number

- $R(A) = \{Au, u \in \mathcal{D}(A)\} = A(\mathcal{D}(A))$ range of operator A
- $\mathcal{D}(A)$ domain of operator A
- $\mathbb{L}^p(\Omega)$ space of p-integrables functions on Ω
- $\mathbb{C}(\Omega)$ space of continous functions on Ω
- $\mathbb{C}^k(\Omega), k \in \mathbb{N}$ space of the functions continuously differentiable to order k on Ω
- $\mathbb{H}_0^1(\Omega) = \{f \in \mathbb{H}^1(\Omega) \diagup f = 0 \text{ in } \partial\Omega\}$
- $\mathbb{H}^m(\Omega) = \{f \in \mathbb{L}^2(\Omega) \diagup D^\alpha f \in \mathbb{L}^2(\Omega) \text{ for all } \alpha \in \mathbb{N}^n, |\alpha| \leq m\}$, $m \in \mathbb{N}^*$ Sobolev spaces
- $\mathbb{L}^p(0,T;\mathbb{V}) = \left\{f : [0,T] \to \mathbb{V} \text{ measurable, } \int_0^T \|f(t)\|_{\mathbb{V}}^p \partial t < \infty\right\}$
- $\mathbb{L}^p(0,T;\mathbb{L}^p(\Omega)) = \mathbb{L}^p(Q_T) = \mathbb{L}^p([0,T]\times\Omega)$
- $\mathbb{L}^\infty(0,T;\mathbb{V}) = \left\{f : [0,T] \to \mathbb{V} \text{ measurable, } \sup_{t\in[0,T]} ess \|f(t)\|_{\mathbb{V}}\right\}$
- $\|u(t)\|_p^p = \int_\Omega |u(x)|^p dx$ norm in space $\mathbb{L}^p(\Omega)$, $1 \leq p < \infty$
- $\|u\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |u(x)|$ norm in space $\mathbb{C}\left(\overline{\Omega}\right)$
- $\|f\|_{H^1(\Omega)}^2 = \|f\|_{L^2(\Omega)}^2 + \int_\Omega |\nabla f|^2 dx$ norm in space $H^1(\Omega)$
- $\|u\|_\infty = ess \sup_{x\in\Omega} |u(x)| = \inf\{M \diagup |u(x)| \leq M \text{ p.p}\}$ norm in space $\mathbb{L}^\infty(\Omega)$
- $\|f\|_{\mathbb{L}^\infty(0,T;\mathbb{V})} = ess \sup\{\|f(t)\|_{\mathbb{V}}, t \in [0,T]\} < +\infty.$
- $< u, v >$ inner product or the scalar product of u and v
- $\langle f, g\rangle_{\mathbb{L}^2(\Omega)} = \int_\Omega fg dx$ inner product in $\mathbb{L}^2(\Omega)$
- ODE ordinary differential equations
- PDE partial differential equations
- SIR susceptible-infectious-recovered epidemic model

General Introduction

The study of infectious diseases is one of the oldest and richest studies in mathematical biology. Infectious diseases have attracted mathematicians for a century, and with good reason. Of course, mathematics has the possibility to make a positive contribution to the world. When studying infectious diseases, the basic elements of mathematics are quickly understood and well captured in mathematical representations. Dynamic systems are widely used in biological modeling, particularly in epidemiology, and mathematicians are interested in studying the equilibrium stability of these models. The main goal of epidemiological modeling is to understand and control the spread of the infectious disease. Because epidemics that cause infections are rapidly increasing in occurrence or expanding in geographical scope.We give examples of this diseases HIV and AIDS, severe acute respiratory syndrome (SARS), Ebola hemorrhagic fever,... With the global emergence of the COVID-19 virus, mathematical models are essential to inform decision-making as governments and health agencies turn to senior designers for advice, and news services seek clarity they can provide to identify and address emerging infection threats. It's a rare opportunity that is important for those who spend most of their time on abstract exercises.

In the study of contact-borne diseases, specific populations have been established, and this knowledge is used to control health problems as central to understanding disease dynamics.

The assumption that the total population size remains constant is reasonable if a disease spreads rapidly (that is, in less than one year) among the population. It is also plausible when modeling a disease over many years that the births are roughly in balance with the natural mortality rate. But if there are many disease-related deaths or births are out of balance with deaths, then it must be assumed that population size depends on time.

Biologists and mathematicians have long been interested in epidemics and infectious diseases that are a branch of population dynamics, and one of the first mathematical models is the Kermack-McKendrick model describing a pandemic, where the population is divided into three disjoint classes of individuals: the susceptible class, the infective class and the removed class.

In this thesis, we study two main branches of mathematics: global existence and the stability of the solution to a general model of an epidemiological problem of a diffusion reaction system. The work and issues addressed in this thesis has been the subject of two published articles:

1. Global existence and asymptotic behavior of solutions for some coupled systems via a Lyapunov functional [23].

2. Global and local asymptotic stability of an epidemic reaction-diffusion model with a nonlinear incidence [24].

This work is organized into four chapters. Firstly, **the chapter**1 is divided into six sections: The first section is devoted to functional spaces, preliminary results and basic theories that will be useful in studying the two problems and to facilitate readers an understanding of our work. In the second section, we present the results of existence and uniqueness of the solution and we show its positivity in the third section. In the fourth section we will give the difference between global existence and blow up of solution. In section five we have presented the basic means of stability theory and the Lyapunov method that will be use in the chapter 4. In the last section we have created a section for numerical analysis in which we will explain the numerical method that we will rely in the analysis and examples to affirm the validity of the findings.

The second chapter 2 is devoted to explaining and studying the special forms of epidemiological models, basic concepts of epidemiology dynamics and an example of reaction–Diffusion Systems.

The third chapter 3 composed of two special sections: The first is devoted to prove the uniform boundedness and so global existence of solutions for a class of disease epidemics systems formed of 2 parabolic partial differential equations called reaction–diffusion systems with homogeneous Neumann boundary conditions, involved in modeling the spread of the disease AIDS in a population, the proof of this result is based on the technique of Lyapunov function, where we prove global existence of solutions under suitable conditions. In the second section we are interested in the study of the asymptotic behavior of solutions in particular case. The basic idea of this result is a Lyapunov function which is non increasing function. These results are valid for any positive initial data in $C\left(\overline{\Omega}\right)$.

Finally, the aim of **the fourth chapter** 4 is to study the local and global stability of a Reaction-diffusion of HIV/AIDS model with a nonlinear incidence rate, where we derive sufficient conditions for asymptotic stability of the proposed model by means of the linearization method in ODE and PDE cases and a threshold R_0 is calculated. In the ODE case we discusses the global stability of the disease-free equilibrium point and the endemic equilibrium point, where we will prove that the disease-free equilibrium is globally stable if $R_0 < 1$, where the infected of the population disappears and the endemic equilibrium is globally asymptotically stable if the basic reproduction number $R_0 > 1$, the infected persists. Also by applying the direct Lyapunov method, we prove the global stability of the disease free and endemic equilibria respectively, where we give the condition of the global stability in PDE case and we present some numerical examples to illustrate the effectiveness of the theoretical results.

Chapter 1

Basic elements to Study a Reaction–Diffusion System

This chapter consists of three parts: In the first part, we recall some basic notions that are fundamental for the study of a reaction-diffusion system.

In the second part which is the most important and essential of this chapter, we will introduce the main known results concerning the local and global existence and uniqueness of solutions to reaction diffusion systems and preservation of it's positivity also we introduce several definitions, some key results on stability of linear and nonlinear nonnegative dynamical systems.

In the third part we are interested in the study of numerical analysis.

1.1 Notions and Preliminary Results

In this section, we collect some results and notions which will be used throughout these work.

Definition 1 *[12]. Let H be a vector space on $\mathbb{R}$, an unbounded linear operator $A : \mathcal{D}(A) \subset H \to H$ said to be monotone (accretive or positive), if it satisfies*

$$\langle Au, u\rangle \geq 0 \quad \textit{for all } u \in \mathcal{D}(A).$$

Definition 2 *(**Quadratic Form with a Matrix**). [26] Let $Q = (a_{ij})$ be an $n \times n$ symmetric matrix, and let x be an n-vector. Then*

$$Q(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j = x^t Ax$$

is a Quadratic forms where matrix Q its associated quadratic form is called positive definite if $Q(x) = x^t Qx > 0$ for all nonzero x.

Theorem 1 *Suppose Q is the symmetric matrix of $Q(x) = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} a_{ij}x_i x_j$. Then $Q(x)$ is positive definite if and only if all leading principal minors of Q are positive.*

Theorem 2 ***(Green's formula)*** *[4]. In dimension n. For any $u \in \mathbb{H}^2(\Omega)$, and $v \in \mathbb{H}^1(\Omega)$ we have*

$$\int_\Omega (\Delta u) v dx = \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v ds - \int_\Omega \nabla u \nabla v dx,$$

where s is the surface mesure of $\partial\Omega$. Note that in the case when $u \in \mathbb{H}^2(\Omega)$, and $v \in \mathbb{H}_0^1(\Omega)$ Green's identity reads

$$\int_\Omega (\Delta u) v dx = -\int_\Omega \nabla u \nabla v dx.$$

Remark 1 *If we write this formula in dimension one, Ω becomes a segment (a, b) in $\mathbb{R}^1$ and we find the integration by parts formula: So let $u, v \in \mathbb{C}^1[a, b]$ we have a special case of the Green's formula.*

$$\int_a^b u'(x) v(x) dx = -\int_a^b u(x) v'(x) dx + u(b) v(b) - u(a) v(a).$$

Now we recall some basic inequalities which we frequently use. We will start with this basic inequality in the study of elliptic and parabolic problems.

Lemma 1 ***(Gronwall Lemma)*** *[32, page 8] .*

Let $\varphi : [0, T] \longrightarrow \mathbb{R}^+$ satisfying

$$0 \le \varphi(t) \le a + b \int_0^t \varphi(s) ds, \quad \text{for all} \quad 0 \le t < T,$$

Gronwall showed that, if a and b are nonnegative constants. Then

$$0 \le \varphi(t) \le a e^{bt} \quad for \quad 0 \le t < T.$$

Proposition 1 ***(Cauchy-Schwarz Inequality)****. [63] Let $(X, \langle ., . \rangle)$ be an inner product space for all $(u, v) \in X \times X$, the Cauchy-Schwarz inequality is:*

$$|< u, v >| \le < u, u >^{\frac{1}{2}} < v, v >^{\frac{1}{2}}.$$

Equivalently,

$$|< u, v >| \le \|u\| \, \|v\|,$$

where we define $\|u\|^2 = \sqrt{< u, u >}$. The equality hold if and only if the two vectors are linearly dependent.

1.1.1 Fundamental Theorems

To prove the existence of solutions to equations, we will mention the theories mentioned below and one of the most useful theories is the **Intermediate Value Theorem**. We will mention the exact situation that we will use.

Theorem 3 ***(Generalizations of Intermediate Value Theorem)****. Replacement of $\mathbb{R}$ by the real straight completed $\overline{\mathbb{R}}$.*

Let $-\infty \le a < b \le +\infty$ and $h : (a, b) \to \mathbb{R}$ continuous and strictement monotone on (a, b) and having in a and b limits L_a and L_b (finies ou infinies):

$$\lim_{v \to a} h(v) = L_a \text{ and } \lim_{v \to b} h(v) = L_b.$$

Then, for all real k strictly included between L_a and L_b, then there exists a unique real $c \in (a, b)$ with $h(c) = k$.

Theorem 4 ***(The Mean Value Theorem).*** *Suppose that the function f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Then there is a point ζ in the open interval (a,b) at which*

$$f(b)-f(a)=f'(\zeta)(b-a).$$

Theorem 5 *(**Lax–Milgram Theorem**). [20, 63]. Let H be a real Hilbert space. Let $\langle .,. \rangle$ be its inner product, and $\|.\|_H$ the associated norm and $\acute{H}$ be the topological dual of H. Let $a : H \times H \to \mathbb{R}$ be a bi-linear form on H.*
Assume that a is continuous in $H \times H$, i.e, if there exists $c > 0$ such that, $\forall (u,v) \in H \times H$, $|a(u,v)| \leq c \|u\| \|v\|$.
a coercive in H, i.e. if there exists $\alpha > 0$ such that, $\forall u \in H, a(u,u) \geq \alpha \|u\|^2$, then for all $L \in \acute{H}$ be a continuous linear form on H, there exists a unique $u \in H$ solution to problem

$$a(u,v) = L(v), \quad \text{for all } v \in H.$$

1.1.2 ODE Existence and Uniqueness (Theorem of Cauchy-Lipschitz-Picard)

To state the very classical result (Cauchy-Lipschitz Theorem), it is necessary to introduce the following definition.

Definition 3 *(**Local Solution and Global Solution**). Let I an open interval of $\mathbb{R}$, Ω is an open in $\mathbb{R}^n$ and a function*

$$\begin{array}{rcl} F & : & I \times \Omega \to \mathbb{R}^n \\ (t,u) & \longmapsto & F(t,u) = (f_1(t,u), ..., f_n(t,u))^T, \end{array}$$

so $(t_0, u_0) \in I \times \Omega$, let us introduce the Cauchy problem:

$$\left\{ \begin{array}{l} \frac{du}{dt} = F(t,u), \ t \in I \\ u(t_0) = u_0. \end{array} \right. \tag{1.1}$$

A local solution of (1.1) is a solution define in an strictly sub-interval $J \subset I$, a global solution is a solution define in I.

Using the previous definition, one can give the following local and global existence result.

Theorem 6 *Let I an open interval of $\mathbb{R}$, let Ω is an open in $\mathbb{R}^n$, suppose the function F satisfying the following:*

- *F continuous functions on $I \times \Omega$.*
- *F locally lipschitzian with respect to u.*

  Then,

1. *there exists a unique maximal solution of (1.1), define on an open interval $J \subset I$: $u \in C^1(J, \mathbb{R}^n)$.*

2. *If* $\sup J < \sup I$. *(i.e. the maximal solution* u *is now global), then* $\lim\limits_{t\to\sup J} \|u(t)\| = +\infty$.

3. *More generally if the maximum solution* $u : J \to \mathbb{R}^n$ *is bounded in all bounded domain in* J*, then is global.*

Lemma 2 *[66] Let* $F(t,u)$ *be continuously differentiable with respect to* u *in* Ω*. If there exists* $\epsilon > 0$ *such that* $\frac{\partial F}{\partial u}(t,u)$ *is bounded in* $U = \{|t - t_0| < \epsilon\} \times \{|u - u_0| < \epsilon\}$ *, then* F *is Lipschitzian on* U*.*

1.1.3 Reaction–Diffusion Systems

Definition 4 ***(Reaction–diffusion systems).*** *Mathematically, by combining the two reaction* f *and diffusion processes* D*, we obtain an equation of "reaction-diffusion" type evolution equations, which take the form of semi-linear parabolic partial differential equations. They can be represented by the general form*

$$\frac{\partial u}{\partial t} = D\Delta u + f(u), \quad t \geq 0 \text{ and } x \in \Omega \subset \mathbb{R}^n, \tag{1.2}$$

where $u = u(t,x) \in \mathbb{R}^m$ *represents the unknown vector function defined on* $\mathbb{R}^+ \times \Omega$ *with values in* $\mathbb{R}^m$*.* $f : \mathbb{R}^m \to \mathbb{R}^m$ *describes all local reactions and it is a nonlinear function that represents the reaction phenomenon in question (human interaction or chemical reaction, for example).* D *denotes a diagonal diffusion coefficient matrix kinetics. Note that we suppose this system to be isotropic and uniform, so* D *is represented by a scalar matrix, independent on coordinates.* $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u)$, $t \geq 0$ *and* $x \in \mathbb{R}$ *is the simplest reaction–diffusion equation is in one spatial dimension in plane geometry (One-component reaction–diffusion equations) and the general two-component system represented in the form (see figure 1.1)*

$$\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = \begin{pmatrix} D_u\Delta & 0 \\ 0 & D_v\Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F(u,v) \\ G(u,v) \end{pmatrix}.$$

The system (1.2) *can be reduced to a system of form*

$$\frac{dU}{dt}(t) + AU(t) = F(U(t)), \quad t \in [0,\tau)$$

where $U : [0,\tau) \to E$*, is an unknown function of time with values in a Banach space* E*, it could be for example the space* $\mathbb{L}^\infty(\Omega)$*. The diffusion operator* A *acting from* E *to* E *and* $F : E \to E$ *is a nonlinear function. In the case where* E *is of finite dimension, we find the EDO.*

Reaction diffusion systems are used to model a variety of phenomena in the natural sciences such as biology, chemistry , population dynamics, thermics, but also physics, crystallization, medicine, etc.

For example, in chemistry, the individuals represent chemicals. In biochemistry, they can represent molecules. In population dynamics, they are humans. In the environment, they can represent the animals or the plants of a forest...

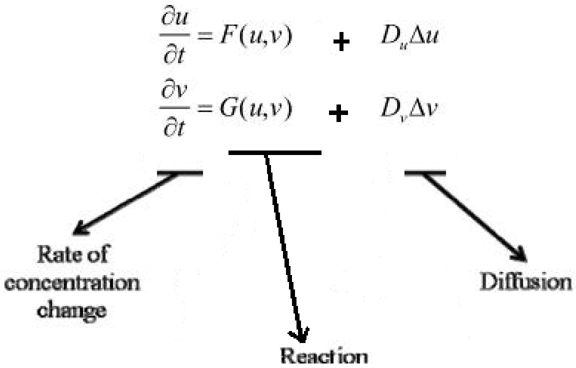


Figure 1.1: reaction-diffusion

1.1.4 Some Results on The Local Existence of the Solutions of the Evolution Equations

The In-homogeneous Initial Value Problem

Cauchy-Lipschitz-Picard's Theorem 6 is of great service in the study of ordinary differential equations, but it is practically useless for solving partial differential equations. The study of local existence and uniqueness of the solutions to the problem of partial differential equations is based on the existence theory for abstract semi-linear differential equations.

Let $A : \mathcal{D}(A) \subset X \to X$ be a linear operator on a Banach space $(X, \|.\|)$. We consider the following nonhomogeneous Cauchy problem

$$\begin{cases} \frac{dU}{dt}(t) + AU(t) = F(U(t)), & t \in [0, \tau), \\ U(0) = U_0 \in \overline{D(A)}, \end{cases} \tag{1.3}$$

where $\tau \in (0, +\infty]$.

In physical applications one encounters many "semi-linear" equations in the form of (1.3). Now, we state the following result which ensures the existence and uniqueness of the local solution for partial differential equations of evolution.

Theorem 7 *[21] Let X is a Banach space and let the problem* (1.3)*: where $A : \mathcal{D}(A) \subset X \to X$ be a linear operator, m-accretive operator in $\overline{\mathcal{D}(A)} = X$, and F is locally Lipschitzian in X, then for any $U_0 \in X$, there exists $T(\|U_0\|_X) > 0$ such that* (1.3) *has a unique classical solution $U \in \mathbb{C}^1((0,T), X) \cap \mathbb{C}([0,T), \mathcal{D}(A))$. In addition we have the alternative*

$$\begin{cases} \textit{either } \ T_{\max} = +\infty \quad \textit{we have the global existence;} \\ \qquad\qquad\qquad\qquad \textit{or} \\ T_{\max} < +\infty \quad \textit{and } \lim_{t \to T_{\max}} \|U(t)\|_X = +\infty, \ \textit{we have} \\ \ \textit{the blow up.} \end{cases}$$

where $T_{\max}$ denotes the greatest of these T's.

If U is a solution of (1.3)*, then U satisfies the variation of constants formula:*

$$U(t) = T_A(t) U_0 + \int_0^t T_A(t-s) F(U(s)) ds, \forall t \in (0, T), \tag{1.4}$$

with $(T_A(t))_{t\geq 0}$ *is a* C_0*- semigroup of operators generated by* $-A$*. The solution* $U(t)$ *of the integral equation* (1.4) *is called a* ***mild solution*** *of equation* (1.3)*.*

1.2 Preservation of Positivity

Let us introduce the general system

$$\begin{cases} \frac{\partial u_i}{\partial t} - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) \ \text{ on } \mathbb{R}^+ \times \Omega, \\ \frac{\partial u_i}{\partial \eta} = 0, \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \\ u_i(0,x) = u_{i0}(x) \geq 0, \ i = 1, ..., m, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain in $\mathbb{R}^n$ with smooth $\partial\Omega$ and $d_i > 0$ for all $i = 1, ..., m$. We start with a very classical result.

Theorem 8 *(Proposition 52.6 page 510 in [61]). Assumed that*

$$\begin{cases} \frac{\partial u_i}{\partial t} - d_i \Delta u_i - f_i(u_1, ..., u_m) \leq \frac{\partial v_i}{\partial t} - \Delta v_i - f_i(v_1, ..., v_m), \\ u_{i0}(x) \leq v_{i0}(x), \ x \in \Omega, \\ \frac{\partial u_i}{\partial \eta} = \frac{\partial v_i}{\partial \eta} = 0. \end{cases}$$

Then $u_i(t,x) \leq v_i(t,x)$*.*

We next give the results for preservation of positivity for classical solutions.

Definition 5 ***(Quasi-Positive or Essentially Nonnegative*** *) [60]. The nonlinearity* $F = (f_i)_{i=1,...,m}$*,* $f_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ *is quasi-positive if*

$$\forall u = (u_1, ..., u_m) \in [0, +\infty)^m, \ i = 1, ..., m, \quad f_i(u_1, u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_m) \geq 0. \tag{1.6}$$

Lemma 3 *[60] If in* (1.5)*, moreover, the nonlinearity* $(f_i)_{i=1,...,m}$ *is quasi-positive (see* (1.6)*) and* $u_{i0}(x) \geq 0$*, for all* $i = 1, ..., m$ *and* $x \in \Omega$*, then the solutions of* (1.5) *satisfies*

$$u_i(x,t) \geq 0, \quad \textit{for all} \ \ x \in \Omega \quad \textit{and} \quad t \in [0,T).$$

To obtain a positivity of u, we use the maximum principle and we can also applying Lemma3.

1.3 Global Existence and blow-up of Classical Solutions

From the general theory of semi–groups, we know that a unique local classical solution exists in some maximal interval $[0, T_{\max})$, where $T_{\max} \in (0, +\infty\ ]$ (e.g. [32], [58, page 109]). Furthermore, if $T_{\max} < +\infty$, then

$$\lim_{t \nearrow T_{\max}} \left(\sum_{1 \leq i \leq m} \|u_i(t)\|_\infty \right) = +\infty,$$

where $T_{\max}(u_{i0})$ denotes the eventual blow-up time. Therefore, if there exists a positive constant C such that

$$\sum_{1\leq i\leq m} \|u_i(t)\|_\infty \leq C \quad \text{for all} \quad t \in [0, T_{\max}), \quad \text{then} \quad T_{\max} = +\infty. \tag{1.7}$$

Consequently, to show the global existence of classical solutions of the systems (1.5), it suffices to show that all the functions $u_i(t)$ are uniformly bounded for any $t \in [0, T)$, $T < T_{\max}$. Thus, a priori L^∞-bounds imply global existence. In other hand to demonstrate the global existence of the solutions of the reaction systems, we use the functional methods based on a priori estimates on Lyapunov functional.

Definition 6 ***(Lyapunov Functional).*** *We call*

$$L : \mathbb{R}^+ \to \mathbb{R}^+,$$

a functional Lyapunov associated with a system (1.5) *if* L *satisfying:*

$$\frac{d}{dt} L(u(t,.)) \leq 0,$$

for all $t > 0$ *and any solution* $u(t,.) = (u_1(t,.), u_2(t,.), ..., u_m(t,.))$ *of system* (1.5).

1.3.1 Global Existence by the Regularizing Effect

Let consider

$$\begin{cases} \frac{\partial u}{\partial t} - d\Delta u = f(t,u) \quad \text{on } \mathbb{R}^+ \times \Omega, \\ \frac{\partial u}{\partial \eta} = 0, \quad \text{on} \quad \mathbb{R}^+ \times \partial\Omega, \\ u(0,x) = u_0 \in L^\infty(\Omega), \end{cases} \tag{1.8}$$

(i_1) the f are $\mathbb{C}^1$ function such that $f(t,0) \geq 0$,

(i_2) $d > 0$.

Proposition 2 *[32, 30] . For system* (1.8)*, if there exists a constant* $c > 0$ *and* $p > \frac{n}{2}$ *such that* $\int_\Omega |f(t,u)|^p \, dx \leq c$, *then there exists* $M > 0$ *such that* $\|u\|_\infty \leq M$*, for all* $t > 0$.

proof. By using Theorem15.11 page 87 of [61]. □

Remark 2 *To show the global existence, it suffices to show, using the regularizing effect that the reaction terms are in* $\mathbb{L}^\infty(T_{\max}; \mathbb{L}^p(\Omega))$ *for some* $p > \frac{n}{2}$.

In this chapter we present several definitions, and some key results on stability of linear and nonlinear nonnegative dynamical systems needed for developing the main results of this monograph.

1.4 Fundamental ODE Stability Theory

In this section we present the tools of Lyapunov stability theory. That will be used in the fourth chapter to analyze and study the stability of an epidemic model. For details we refer to [40, 77], and [66, page230].

1.4.1 Elements of the Lyapunov Stability Theory

One of various types and most important type of stability problems may be discussed for the solutions of differential equations or difference equations describing dynamical systems is concerning the stability of solutions near to a point of equilibrium. This may be discussed by the theory of Aleksandr Lyapunov, a Russian mathematician and engineer who laid the foundation of the theory, which now carries his name. In simple terms, if the solutions that start out near an equilibrium point u^* stay near forever u^*, then is Lyapunov stable. More strongly, if u^* is Lyapunov stable and all solutions that start out near u^* converge to u^*, then is asymptotically stable.

We first consider the general dynamical system which satisfies

$$\begin{cases} \frac{du(t)}{dt} = F\left(t, u\left(t\right)\right) \text{ on } I \subset \mathbb{R}, \\ u\left(0\right) = u_0, u \in \mathbb{R}^m, \end{cases} \tag{1.9}$$

is called **non-autonomous** (ordinary) differential equations or time varying. We will assume that $F\left(t, u\right)$ satisfies the standard conditions for the existence and uniqueness of solutions. Such conditions are, for instance, that $F\left(t, u\right)$ is Lipschitz continuous with respect to u, uniformly in t, and piecewise continuous in t. A special case of (1.9) arises when the function of F does not depend explicitly on t; that is, we considering the general nonlinear **autonomous** dynamical system

$$\begin{cases} \frac{du(t)}{dt} = F\left(u\left(t\right)\right) \text{ on } t \in I \subset \mathbb{R}, \\ u\left(0\right) = u_0, \ u\left(t\right) \in \Omega \subseteq \mathbb{R}^n, \end{cases} \tag{1.10}$$

where

- $u = \left(u_1, ..., u_m\right)^T$ is the unknown function,
- $F = \left(f_1, ..., f_m\right)^T : \Omega \to \mathbb{R}^m$ is a given function on an open subset $\Omega \subset \mathbb{R}^n$, with the assumption F is a locally Lipschitz map. Suppose $u^* \in \Omega$ is an equilibrium point of (1.10). Our goal is to study the stability of u^*.

The system is said to be autonomous or time-invariant, where I is the maximal interval of existence for the solution $u(\cdot)$.

Equilibrium points (states) are essential in studying the dynamics of any physical system in biology, economics, physics, etc. They are used in the study of stability theory, which is a topic of great importance to scientists. It is desirable that all states (solutions) of a given system tend to be Equilibrium (point). We are now giving the following definitions.

Definition 7 ***(Equilibrium Points).*** *A point $u^* \in \mathbb{R}^m$ in the state space is said to be an equilibrium point of* (1.10) *if $\frac{du}{dt} = 0$ ie $F(u^*) = 0$. Means that, whenever the state of the system starts at u^*, it will remain at u^* for all future time. An equilibrium point of a dynamical system represents a stationary condition for the dynamics. A dynamical system can have zero, one or more equilibrium points. If there are multiple equilibrium points, we will need to examine the stability of each.*

Definition 8 ***(Invariant Set)*** *[27]. A subset $\mathcal{D} \subseteq \Omega$ is an invariant set relative to* (1.10) *if $\mathcal{D}$ contains the orbits of all its points.*

Definition 9 ***(A Positively Invariant Set).*** *A positively invariant set is a set with the following properties: Given a dynamical system* (1.10) *and trajectory $u(t, u_0)$ where u_0 is the initial point. Let $\mathcal{D} = \{u \in \mathbb{R}^n; \phi(u) = 0\}$ where ϕ is a real valued function.The set D is said to be positively invariant if $u_0 \in \mathcal{D}$ implies that $u(t, u_0) \in \mathcal{D}$ for all times $t \geq 0$. In other words, a solution that starts in $\mathcal{D}$ remains in $\mathcal{D}$ for all times $t \geq 0$.*

Definition 10 ***(The Region of Attraction of the Equilibrium).*** *Assume that $u = u^*$ is an equilibrium point of* (1.10) *and let ϕ be the solution of the system. The set*

$$\mathcal{D} = \left\{ \zeta \in \Omega \diagup \limsup_{t \to \infty} \phi(t, \zeta) = u^* \right\}$$

is called the region of attraction of the equilibrium u^*.

1.4.2 Stability For Nonlinear autonomous Systems

Stability is one of the essential aspects in the study of linear and nonlinear dynamical systems. An important question is to know the behavior of the trajectories initially close to the equilibrium, for this one defines the notion of stability of a point of equilibrium that is, whether a particle near such a point remains near or wanders off into another part. It is not necessary to calculate detailed differential solutions; it is sufficient to know the fixed points (equilibrium points) and their stability. To study non-linear systems this simplified result is of great importance by studying non-linear systems in the vicinity of these points.

Definition 11 *[40, 66],(**Locally Stable**). An equilibrium point is locally stable if all solutions which start near u^* remain near u^* for all time. Means that, An equilibrium point u^* for* (1.10) *is called stable if given any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ (meaning that the initial conditions are in a neighborhood of u^*) so that*

$$\|u(0) - u^*\| < \delta \text{ implies } \|u(t) - u^*\| < \varepsilon,$$

for all $t > 0$. Other wise it is unstable. Note that local stability of an equilibrium point means that if you put the system somewhere nearby the point then it will move itself to the equilibrium point in some time. If u^* is not stable it is said to be unstable .

- ***(Global stability).*** *Global stability means that the system will come to the equilibrium point from any possible starting point (i.e., there is no "nearby" condition).*

- ***(Locally Asymptotically Stable- Globally asymptotically stable)*** *[15]. The point u^* is said to be locally asymptotically stable equilibrium if, in addition, for $\|u(0)-u^*\|<\delta$ one has $\lim\limits_{t\to\infty}\|u(t)-u^*\|=0$, or an equilibrium point is locally asymptotically stable if it is locally stable and also all solutions starting near u^* tend towards u^* as $t\to\infty$. This corresponds to the case where all nearby trajectories converge to the equilibrium point for large time. If the system is asymptotically stable whatever the initial state vector $u(0)$, then the equilibrium point is globally asymptotically stable.*

Theorem 9 *[73]. Let $F:\mathbb{R}^n\to\mathbb{R}^n$ be a C^1 function, suppose that $F(u^*)=0$. Then $u=u^*$ is an equilibrium solution to the system of ordinary differential equations* (1.11)*. The equilibrium is locally asymptotically stable if the Jacobian matrix $J=F'(u^*)$ is stable; i.e. all the eigenvalues of J have negative real parts.*

1.4.3 Second-Order autonomous systems

The second autonomous system occupy an important place in the study of nonlinear systems, is represented by two scalar differential equations in the form

$$\begin{cases} u_t = f(u,v), \\ v_t = g(u,v). \end{cases} \tag{1.11}$$

Here autonomous" means that the functions f,g do not depend explicitly on time t. If $t\mapsto U(t)=(u(t),v(t))$ be the solution of (1.11) that starts at a certain initial state $U(0)=U_0=(u_0,v_0)$.

Definition 12 ***(Trajectory or Orbit).*** *The locus in the $u-v$ plane of the solution $U(t)$ for all $t\geq 0$ is a curve that passes through the point U_0. This curve is called a trajectoury or orbit of* (1.11) *from U_0 or the set of points*

$$\mathcal{O}=\{(u(t),v(t));\ t\in I\}\subset\mathbb{R}^2,$$

is called an orbit.

Definition 13 ***(The Phase Plane).*** *The phase plane diagram in the $u-v$ plane for the system* (1.11) *is obtained by drawing orbits and equilibrium points, and marking the direction of motion along the orbits.*

Definition 14 ***(The Phase Portrait).*** *Is the family of all trajectories or solution curves.*

If we compare the case of linear and nonlinear systems, one realizes that the dynamic behaviors of nonlinear systems are: Much more complicated. For example, talking about equilibrium stability or system stability is the same in a linear state. In the case of a non-linear system, the study of the stability of the equilibrium point is not limited to determining the nature of the equilibrium point or its non-asymptotic stability, but also in determining the field of attraction, that is, the set of initial conditions whose solutions converge with equilibrium. Consequently, we are talking about local or global stability or instability, and local stability, which signifies the convergence of solutions with close initial conditions.

Phase Plane Diagrams for Linear Systems

The study of stability of linear systems about the equilibrium point $(0,0)$ is important because any equilibrium's point can be shifted to the origin and the local stability of a nonlinear system near an equilibrium point can be deduced by linearising the system about that point and studying the stability of the resultant linear system.

Let consider the linear homogeneous system (F is linear) that is to say that the system (1.10) is of the form.

$$U' = AU, \tag{1.12}$$

where $U = \begin{pmatrix} u \\ v \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and the coefficients a_{ij} are real numbers. If a_{11} $a_{22} - a_{21}a_{12} \neq 0$ the unique equilibrium point of (1.12) is $(0,0)$ and the solution of (1.12) for a given initial state U_0 is given by $U(t) = Pe^{J_r}P^{-1}U_0$, where $J_r = P^{-1}AP$ is the real Jordan form of A he may take one of three forms depending on the eigenvalues of A,

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda & k \\ 0 & \lambda \end{pmatrix} \text{ and } \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

where k is either 0 or 1. Using the following linear change of coordinates

$$\begin{pmatrix} u \\ v \end{pmatrix} = P^{-1} \begin{pmatrix} w \\ z \end{pmatrix}, \tag{1.13}$$

then $\begin{pmatrix} w \\ z \end{pmatrix}' = PAP^{-1} \begin{pmatrix} w \\ z \end{pmatrix}$. Various cases arise.

Case 1. Both eigenvalues are real : $\lambda_1 \neq \lambda_2 \neq 0$ Let v_1, v_2 be corresponding eigenvectors the change of coordinates (1.13) transforme the system into two decoupled first order differential equations and the general solution is thus

$$c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2,$$

the phase portrait depends on the signes of λ_1 and λ_2.

Case 1a. $\lambda_1 < \lambda_2 < 0$ The term $c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$ tend to zero as $t \to +\infty$ and all trajectories flow into the origin. In this situation, the equilibrium point is called a **stable node**.

Case 1b. $0 < \lambda_1 < \lambda_2$ The term $c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$ tend to $\pm\infty$ and trajectories flow away from the origin. In this situation, the equilibrium point is called a **unstable node**.

Case 1c. $\lambda_1 < 0 < \lambda_2$ The terme $w(t) = w_0 e^{\lambda_1 t} \to 0$ and $z(t) = z_0 e^{\lambda_2 t} \to \pm\infty$. In this case the equilibrium point is called a **saddle**.

Case 2. Complex eigenvalues : $\lambda_{1,2} = \alpha \pm i\beta,\ \ \beta \neq 0$ Next, assume that the matrix A has complex eigenvalues and the change of coordinates (1.13) transform the system into

$$y' = (\alpha + i\beta)\, y, \text{ where } y = w + iv. \tag{1.14}$$

We write y in the form $y(t) = r(t)\, e^{i\theta t}$ and substituting this in (1.14), where we get two uncoupled first-order differential equations and his solution for a given initial state (r_0, θ_0) is given by

$$\begin{cases} r(t) = r_0 e^{\alpha t}, \\ \theta(t) = \beta t + \theta_0. \end{cases}$$

Case 2a. $\alpha = 0$ When $\alpha = 0$, the trajectory is a cerle of radius r_0. In this situation, the equilibrium point is called a center.

Case 2b. $\alpha < 0$ When $\alpha < 0$, we get $r(t) \to 0$ as $t \to +\infty$. In this situation, the equilibrium point is called a asymptotically stable focus.

Case 2c. $\alpha > 0$ When $\alpha > 0$, we get $r(t) \to \pm\infty$ as $t \to +\infty$. It diverges a way from the origin. In this situation, the equilibrium point is called a unstable focus.

Case 3. $\lambda_1 = \lambda_2 = \lambda \neq 0$ The change of coordinates (1.13) transform the system and his solution for a given initial state is given by

$$\begin{cases} w(t) = w_0 e^{\lambda t} \\ z(t) = z_0 e^{\lambda t}. \end{cases}$$

The equation of the trajectory given by $z = \frac{z_0}{w_0} w$. The phase portrait depends on the signs of λ.

Case 3a. $\lambda > 0$ When $\lambda > 0$, we get $u, v \to \pm\infty$ as $t \to +\infty$. The equilibrium point is called a unstable node.

Case 3b. $\lambda < 0$ When $\lambda < 0$, we get $u, v \to 0$ as $t \to +\infty$. The equilibrium point is called a asymptotically stable node.

Case 4. One or both eigenvalues are zero Using the change of coordinates (1.13), we get a system his solution is

$$\begin{cases} w(t) = w_0, \\ z(t) = z_0 e^{\lambda_2 t}, \end{cases}$$

all trajectories converge to the equilibrium when $\lambda_2 < 0$ and diverge away from it when $\lambda_2 > 0$. For details we refer to ([40, , page 43], [66, page236]).

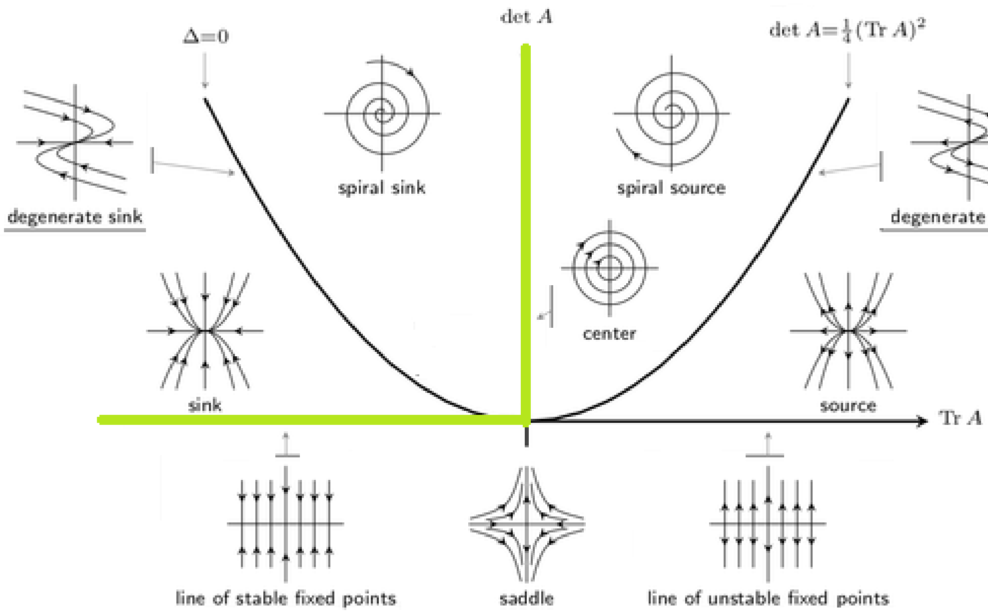


Figure 1.2: Stability-Diagram

Typology Classification of the Solutions of the Linear Systems in the $(tr, \det)$-Plane

Conclusion 1 *For system* (1.12) *the characteristic polynomial is*

$$p(\lambda) = \lambda^2 - (TrA)\,\lambda + \det A = 0,$$

where

$$\begin{cases} TrA = a_{11} + a_{22} = \lambda_1 + \lambda_2, \\ \det A = a_{11}a_{22} - a_{21}a_{12} = \lambda_1\lambda_2, \\ \triangle = (TrA)^2 - 4\det A. \end{cases} \tag{1.15}$$

Since we are assuming that A is a real matrix, this polynomial has real coefficients and either its roots are both real or they are complex conjugates. The typology of the solutions of the planar linear systems which we established depends on the nature of the eigenvalues of the matrix of the system (1.12). *This can also be summarized in the plan shown in Figure 1.2.*

The nature of the eigenvalues depends on the sign of the discriminant $\triangle$. In the plan $(Tr, \det)$, the equation $\triangle = 0$ is that of a parabola passing by the origin: $TrA^2 = 4\det A$. This parabola divides the plan into two great areas: above the parabola ($\triangle < 0$), one finds the portraits of phase of the hearths and the centers, below ($\triangle > 0$), one finds them nodes and the points saddles.

- The case $\triangle = 0$. In this case $TrA^2 = 4\det A$ and from (1.15), we get $(\lambda_1 - \lambda_2)^2 = 0$, then $\lambda_1 = \lambda_2 = \lambda$, i.e. $TrA = 2\lambda$ and $\det A = \lambda^2$. Consequently,

1. if $TrA > 0$, we have a star or an unstable degenerated node.

2. If $TrA < 0$, we have a star or a stable degenerated node.

- The case $\triangle > 0$. In this case we have two distinct real eigenvalues then is in the area under the parabola who still can is shared in three zones:

1. $\det A < 0$. Then λ_1 and λ_2 are of opposed sign, the origin is a point saddles.

2. $\det A > 0$ and $TrA > 0$, then $\lambda_1 > \lambda_2 > 0$, the equilibrium point $(0,0)$ is called an unstable node.

3. $\det A > 0$ and $TrA < 0$, then $\lambda_1 < \lambda_2 < 0$, the equilibrium point $(0,0)$ is called an asymptotically stable node.

- The case $\triangle < 0$. In this case the matrix A has two complex eigenvalues $\lambda_{1,2} = \alpha \pm i\beta$, then $\det A = \alpha^2 + \beta^2 > 0$ and $TrA = 2\alpha$, which division there still in three distinct zones:

1. $TrA > 0$. Then, the real part of the eigenvalues is positive, the equilibrium point $(0,0)$ is called an unstable focus.

2. $TrA < 0$. Then, the real part of the eigenvalues is negative, the equilibrium point $(0,0)$ is called an asymptotically stable focus.

3. $TrA = 0$. Then the eigenvalues of A are imaginary number and the origin is a stable center.

Remark 3 ***1.*** *The asymptotic behavior of solutions to the linear two–component system based on the nature of the eigenvalues of A. The first stability case, which is the asymptotically stable node, can be guaranteed if the trace of A is negative and its determinant is positive, i.e.*

$$\begin{cases} TrA < 0, \\ \det A > 0. \end{cases} \tag{1.16}$$

2. *The origin is stable, but not asymptotically stable, in the following cases.*

2.1 $\triangle > 0$, $TrA < 0$, $\det(A) = 0$,

2.2 $\triangle < 0$, $TrA = 0$.

Theorem 10 *[28, Theorem 4.1 page 15], [77, Theorem 1.2.5 page 11] The equilibrium of the homogeneous equation $U' = AU$ is asymptotically stable if all the characteristic roots of the matrix A have negative real parts ($Re\lambda_i < 0$). If at least one of the characteristic roots has a positive real part then the equilibrium is unstable. If all the characteristic roots have non-positive real parts and some of them actually have a zero real part then the equilibrium is stable, but not asymptotically stable.*

proof. See [77, Theorem 1.2.5 page 11]. □

Lyapunov stability, in his original 1892 work, proposed two methods for demonstrating stability.

1.4.4 Lyapunov's first method (The linearization techniques)

In Lyapunov's first method, the linearization techniques is applied to study the stability. The study of stability in the case of nonlinear systems poses a very difficult problem, however, Lyapunov and others have noticed by studying the trajectories of integral curves in the neighborhood of equilibrium that, in the majority of cases, the equilibrium points of nonlinear systems can be brought back to the same types of equilibrium points of linear systems. So the study of a linear system is easy since it is solved in a purely algebraic criterion and the determination of the nonlinear stability of the equilibrium point is reduced to the linearization of the system (1.10) at this equilibrium point then, to answer the questions of stability, we must consider the linearization of the system (1.10) around the point of equilibrium gives the following linear system:

$$\begin{aligned} \frac{du(t)}{dt} &= \left.\frac{\partial F(u(t))}{\partial u}\right|_{u=u^*} u(t), \\ &= Au(t), \end{aligned} \tag{1.17}$$

may be found by forming the Jacobian matrix, where $a_{ij} = \left[\frac{\partial f_i(u)}{\partial u_j}\right]_{u=u^*}$ with $i, j = 1, 2, ..., n$ and $A = (a_{ij})$. In the case of the second order autonomous systems, we get

$$A = \left.\frac{\partial F(u(t))}{\partial u}\right|_{u=u^*} = \left.\begin{pmatrix} (f_1)_u(u) & (f_1)_v(u) \\ (f_2)_u(u) & (f_2)_v(u) \end{pmatrix}\right|_{u=u^*} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where A is the Jacobian matrix of $F(u(t)) = (f_1, f_2)^T$ evaluated at the equilibrium u^*.

For the stability of the nonlinear autonomous system we have the following theorem:

Theorem 11 ***(Linearization theorems)***

1. *If all the eigenvalues of* $A = \left.\frac{\partial F(u(t))}{\partial u}\right|_{u=u^*}$ *are strictly negative real parts, the nonlinear system* (1.10) *is asymptotically stable.*
2. *If all the eigenvalues of* $A = \left.\frac{\partial F(u(t))}{\partial u}\right|_{u=u^*}$ *have negative real parts, the nonlinear system* (1.10) *is stable.*
3. *If at least one eigenvalue of A has a real part strictly greater than 0, the nonlinear system is unstable.*

1.4.5 Lyapunov's second method for stability (The direct Method of Lyapunov)

Lyapunov functions or direct method (also called the Lyapunov's second method for stability) are used to study the global stability of an equilibrium points of dynamic systems. The problem is that the method rests on knowledge about a certain function having certain properties, and there is no general technique for constructing Lyapunov functions.

Theorem 12 *[40] Let $u = u^*$ be an equilibrium points for* (1.10) *and $\Omega \subset \mathbb{R}^n$ be a domain containing u^*. Let $\mathcal{V} : \Omega \to \mathbb{R}$ be a continuously differentiable function such that*

- $\mathcal{V}(u^*) = 0$ *and* $\mathcal{V}(u) > 0$ *in* $\Omega - \{u^*\}$,

- $\mathcal{V}'(u) = \nabla\mathcal{V}.F$, *given by*

$$\begin{aligned}\mathcal{V}'(u(t)) &= \frac{d}{dt}\mathcal{V}(u)(t) = \sum_{i=1}^{n}\frac{\partial\mathcal{V}}{\partial u_i}u_i' = \sum_{i=1}^{n}\frac{\partial\mathcal{V}}{\partial u_i}f_i = \langle\nabla\mathcal{V}, F(u)\rangle \\ &= \left[\frac{\partial\mathcal{V}}{\partial u_1}, ..., \frac{\partial\mathcal{V}}{\partial u_n}\right][f_1, ..., f_n]^T = \frac{\partial\mathcal{V}}{\partial u}F(u) \leq 0, \ in \in \Omega.\end{aligned}$$

Then, $\mathcal{V}(u)$ is called a Lyapunov function and when we use direct Lyapunov method, we can get from the system if it is stable, asymptotic or unstable, but in some cases this method fails to prove stability, in this case we apply Lasalle invariance principle to get asymptotic stability.

Theorem 13 *[40](**LaSalle's theorem**) Let $u = u^*$ be an equilibrium points for* (1.10) *and $\Omega \subset \mathbb{R}^n$ be a domain containing u^*. Let $\mathcal{V} : \Omega \to \mathbb{R}$ be a continuously differentiable function such that*
$\mathcal{V}'(u) \leq 0$ in Ω. Let $M = \{u \in \Omega : \mathcal{V}'(u) = 0\} = \{u^*\}$. Then, u^* is asymptotially stable.

Remark 4 *If $\mathbb{R}^n = \Omega$, in the last theorem, u^* is globally asymptotially stable.*

1.4.6 Stability For Nonlinear Non-Autonomous Systems

The concepts of stability and asymptotic stability of equilibrium points for non-autonomous systems are essentially the same as those presented in definition 11 for an autonomous system. We note that, the solution of the non autonomous system may depend on both t and t_0 and while the autonomous system solution depends only on $t - t_0$. The definition of global uniform asymptotic stability in an autonomous system is equivalent to global asymptotic stability in Non-Autonomous Systems.

1.5 Fundamental Stability and unstable Theory of Reaction Diffusion Systems

We consider the system of reaction diffusion equations (1.5), where $f = (f_1, ..., f_n)$, $D =$ diagonal matrix $(d_1, ..., d_n)$, $\Delta u = (\Delta u_1, ..., \Delta u_n)$ and $u_0 = (u_{01}, ..., u_{0n})$, now we can write (1.5) in the following form

$$\frac{\partial u}{\partial t} = D\Delta u + f(u), \ t \geq 0 \ \text{ and } \ x \in \Omega \subset \mathbb{R}^n, \tag{1.18}$$

with Neumann boundary data. The solutions of

$$D\Delta u + f(u) = 0,$$

with the Neumann boundary condition are called equilibrium solutions.

1.5.1 Eigenvalues of the Laplacian with Neumann boundary conditions

The Laplacian is an unbounded operator. To apply the Local Stability in the PDE case from the previous chapter we use the Neumann Laplacians (see [50]).

$$\begin{cases} -\Delta u_j = \lambda_j u_j \text{ in } \Omega, \\ \dfrac{\partial u_j}{\partial \eta} = 0 \text{ on } \partial\Omega. \end{cases}$$

We will denote the eigenvalues of the Laplacian with Neumann boundary conditions by

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \nearrow^{+\infty} .$$

The first eigenvalue for the Laplace operator with zero Neumann boundary conditions data case is zero : $\lambda_1 = 0$, corresponding to constant functions, then eigenfunction $u_1 \equiv$ const $\neq 0$ (This constant function belongs to $\mathbb{H}^1(\Omega)$).

1.5.2 The technique of eigenfunction expansions

The eigenfunction expansion method is a total analytic method or a method of transformation of a PDE (in space and time) into a system of ODEs in time.

Let denote $\left\{\lambda_i, (\Phi_{ij})_{j=1}^{m_i}\right\}_{i=0}^{\infty}$ be the sequence of eigenvalues and the normalized eigenfunctions corresponding to λ_i for the elliptic operator $(-\Delta)$ subject to the homogeneous Neumann boundary condition on Ω, where $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$, and each λ_i has the algebraic multiplicity $m_i \geq 1$. Also let recall that $\Phi_0 = $ const and $\lambda_i \to \infty$ at $i \to \infty$. That is, $d_1, ..., d_n$ and λ_i satisfy $-\Delta\Phi_{ij} = \lambda_i \Phi_{ij}$ in Ω, with $\frac{\partial \Phi_{ij}}{\partial \nu} = 0$ in $\partial\Omega$, and $\int_\Omega \Phi_{ij}^2(x)dx = 1$.

Define the linearization of the system (1.18) around the point of equilibrium gives the following linear system:

$$\frac{\partial u}{\partial t} = D\Delta u + Au, \tag{1.19}$$

with the same initial and boundary conditions as given with (1.18). The solution to the linear system (1.19) can be written in the form $u(t,x) = \sum\limits_{i=0}^{\infty} \Phi_{ij}(x) e^{A_i t} u_{0i}$.

Let denote the linearizing operator by $\mathcal{L} = D\Delta + A$. Similar to the ODE case, the asymptotic stability of the steady state solution (u^*, v^*) can be determined by examining the eigenvalues of the operator $\mathcal{L}$. That is the solution is asymptotically stable if all the eigenvalues of $\mathcal{L}$ have negative real parts. For example in the case where $N = 2$, to achieve that, suppose $(\phi(x), \psi(x))^T$ is an eigenfunction of $\mathcal{L}$ corresponding to an eigenvalue ξ. By definition, the characteristic equation can be given by

$$\mathcal{L}(\phi(x), \psi(x))^T = \xi(\phi(x), \psi(x))^T,$$

leading to

$$(\mathcal{L} - \xi I) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let $(\phi(x), \psi(x))^T$ forms a complete orthonormal basis in $L^2(\Omega)$, for this we can write $(\phi(x), \psi(x))^T = \sum_{0\leq i\leq\infty, 1\leq j\leq m_i} (a_{ij}, b_{ij})^T w_{ij}$ or

$$\phi = \sum_{0\leq i\leq\infty, 1\leq j\leq m_i} a_{ij}\phi_{ij}, \quad \psi = \sum_{0\leq i\leq\infty, 1\leq j\leq m_i} b_{ij}\psi_{ij},$$

This can be rearranged to the form

$$\sum_{0\leq i\leq\infty, 1\leq j\leq m_i} (A - \lambda_i D - \xi I) \begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix} \Phi_{ij} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Theorem 14 *[15].*

(*i*) *The equilibrium point of the nonlinear problem*(1.18) *is (locally) asymptotically stable, if the equilibrium point of the linearized problem* (1.19) *is asymptotically stable.*

(*ii*) *The equilibrium point of* (1.18) *system is globally asymptotically stable, if for each nonnegative integer i the eigenvalues of $(A - \lambda_i D)$ have negative real parts.*

(*iii*) *The equilibrium point is unstable, if for some n there exists an eigenvalue of $A_i = (A - \lambda_i D)$ with either positive real part or zero real part with a non simple elementary divisor.*

proof. See [15]. □

1.6 Numerical analysis

This part aims to find approximate numerical solutions for the system of Reaction-diffusion with elementary-bound conditions using the finite difference method, and the importance of this method comes in that these approximate solutions enable the researcher to understand and clarify the behavior of the physical or chemical phenomenon studied, as well as predict future results for it.

We need approximation because we either cannot solve the procedure analytically or because the analytical method is difficult and using numerical methods are techniques for approximating mathematical procedures to compute the numerical solutions to ordinary differential equations and non-linear partial differential equations by implicit methods. So we must understand what is the finite difference method and how to use it to solve (1.18) in one-dimensional problem that we will use in the numerical examples.

The finite difference method (FDM) seems to be the simplest approach for the numerical solution of PDEs. In numerical analysis, FDM are discretizations used for solving differential equations by approximating them with difference equations that finite differences approximate the derivatives. FDMs convert a linear ordinary differential equations (ODE) or non-linear partial differential equations (PDE) into a system of equations that can be solved by matrix algebra techniques. The reduction of the differential equation to a system of algebraic equations makes the problem of finding the solution to a given ODE/PDE ideally suited to modern computers.

1.6.1 Finite Difference Method for Solving ODEs Equations

Finite-Difference Methods for Linear Problems

The Finite-Difference Method is a way of obtaining a numerical solution for linear and nonlinear problems, but they generally require more computation to obtain a specified accuracy. In this subsection we show how to approximate the solution to boundary-value problems (differential equations with conditions imposed at different points) where, we solve the linear second-order ordinary differential equations with two boundary values of the form (see [53])

$$\begin{cases} u_{xx} = p(x)\,u_x + q(x)\,u + r(x), \quad x \in [a,b] \quad \text{(Governing equation)}, \\ \qquad u(a) = \alpha, \quad u(b) = \beta \quad \text{(Boundary values)}. \end{cases} \tag{1.20}$$

It can be noted that, for first-order differential equations, only one condition is specified, so there is no distinction between initial-value and boundary-value problems.

To find u as a function of x. First, we select an integer $n > 0$ and follow the steps below:

Step1: Divide solution domain $[a,b]$ **into a series of discrete nodes** separated by $\triangle x = h =$ distance between nodes, i.e, into $n+1$ equal subintervals whose endpoints are the mesh points $x_i = a + i\triangle x$ for $0 \le i \le n+1$, where $\triangle x = \frac{b-a}{n+1}$.

So at the interior mesh points x_i, for $0 \le i \le n$, the differential equation to be approximated is

$$u_{xx}(x_i) = p(x_i)\,u_x(x_i) + q(x_i)\,u(x_i) + r(x_i). \tag{1.21}$$

We need to approximate derivatives in the Governing equation in x_i, by using the following step.

Step2: (Write the ODE for each node using finite difference formulas for the derivatives)

Assuming that $u \in C^4([x_{i-1}, x_{i+1}])$, developing u in a third Taylor polynomial about x_i evaluated at x_{i+1} and x_{i-1}, we get

$$\begin{aligned} u(x_{i+1}) &= u(x_i + h) = u(x_i) + hu_x(x_i) + \tfrac{h^2}{2}u_{xx}(x_i) \\ &+ \tfrac{h^3}{6}u_{xx}(x_i) + \tfrac{h^4}{24}u^{(4)}(\zeta_i) \text{ for some } \zeta_i \text{ in } (x_i, x_{i+1}), \end{aligned} \tag{1.22}$$

$$\begin{aligned} u(x_{i-1}) &= u(x_i - h) = u(x_i) - hu_x(x_i) + \tfrac{h^2}{2}u_{xx}(x_i) \\ &- \tfrac{h^3}{6}u_{xx}(x_i) + \tfrac{h^4}{24}u^{(4)}(\eta_i) \text{ for some } \eta_i \text{ in } (x_{i-1}, x_i). \end{aligned} \tag{1.23}$$

Adding (1.22), (1.23), solving the resulting for $u_{xx}(x_i)$ and by applying the Intermediate Value Theorem **3** can be used to simplify the error term and we obtain the following centered-difference formula

$$u_{xx}(x_i) = \frac{1}{h^2}\left[u(x_{i+1}) - 2u(x_i) + u(x_{i-1})\right] - \frac{h^2}{12}u^{(4)}(\theta_i), \tag{1.24}$$

for some θ_i in (x_{i-1}, x_{i+1}). In a similar manner, we can get

$$u_x(x_i) = \frac{1}{2h}\left[u_{i+1} - u_{i-1}\right] - \frac{h^2}{6}u^{(3)}(\rho_i), \tag{1.25}$$

for some ρ_i in (x_{i-1}, x_{i+1}).

Substituting $u_x(x_i)$, $u_{xx}(x_i)$ with its values in (1.21), we get

$$\begin{array}{c}\frac{1}{h^2}\left[u\left(x_{i+1}\right)-2u\left(x_i\right)+u\left(x_{i-1}\right)\right]=p\left(x_i\right)\left[\frac{1}{2h}\left[u_{i+1}-u_{i-1}\right]\right]\\ +q\left(x_i\right)u\left(x_i\right)+r\left(x_i\right)-\frac{h^2}{12}\left[2p\left(x_i\right)u^{(3)}\left(\rho_i\right)-u^{(4)}\left(\theta_i\right)\right].\end{array} \tag{1.26}$$

Step3: (**Apply the boundary conditions and write system of algebraic equations**)

Approximations v_i to $u(x_i)$ for each $i=0,1,...,n+1$.

A Finite-Difference method with truncation error of order $O(h^2)$, results by using (1.26) together with the boundary conditions $u(a)=\alpha,\ \ u(b)=\beta$ to define the system of linear equations.

At node $i=0$ and $i=n+1$, we have (Boundary Condition)

$$v_0=u\left(x_0\right)=u\left(a\right),\ v_{n+1}=u\left(x_{n+1}\right)=u\left(b\right)$$

and

$$\begin{array}{c}\frac{1}{h^2}\left[-v_{i+1}+2v_i-v_{i-1}\right]+p\left(x_i\right)\left[\frac{1}{2h}\left[v_{i+1}-v_{i-1}\right]\right]\\ +q\left(x_i\right)v_i=-r\left(x_i\right),\end{array} \tag{1.27}$$

for each $i=1,2,...,n$. We can rewrite the last equation in the following form

$$\begin{array}{c}-\left(1+\frac{h}{2}p\left(x_i\right)\right)v_{i-1}+\left(2+h^2q\left(x_i\right)\right)v_i\\ +\left(1-\frac{h}{2}p\left(x_i\right)\right)v_{i+1}=-h^2r\left(x_i\right),\end{array} \tag{1.28}$$

for each $i=1,2,...,n$. The resulting system of equations is expressed in the tridiagonal $n\times n$ matrix form

$$AV=E, \tag{1.29}$$

where

$$A=\begin{bmatrix} 2+h^2q\left(x_1\right) & -1+\frac{h}{2}p\left(x_2\right) & 0 & \cdots & 0\\ -1-\frac{h}{2}p\left(x_2\right) & 2+h^2q\left(x_2\right) & 1+\frac{h}{2}p\left(x_3\right) & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 2+h^2q\left(x_{n-1}\right)\\ 0 & \ldots & 0 & -1-\frac{h}{2}p\left(x_n\right) & 2+h^2q\left(x_n\right) \end{bmatrix},$$

$$V=\begin{pmatrix} v_1, & v_2, & ... & v_n\end{pmatrix}^T \quad \text{and} \quad E=\begin{bmatrix} -h^2r\left(x_1\right)+\left(1+\frac{h}{2}p\left(x_1\right)\right)v_0\\ -h^2r\left(x_2\right)\\ \vdots\\ -h^2r\left(x_{n-1}\right)\\ -h^2r\left(x_n\right)+\left(1-\frac{h}{2}p\left(x_n\right)\right)v_{n+1}\end{bmatrix}.$$

Step4: Solve the resulting system of linear algebraic equations by using the tridiagonal matrix algorithm, also known as the Thomas algorithm (named after Lewellyn Thomas), is a simplified form of Gaussian elimination that can be used to solve tridiagonal systems of equations to obtain an approximate solution $\left(\begin{array}{cccc} u_0, & u_1, & ...., u_{n-1}, ..., & u_n \end{array}\right)^T$.

The following theorem gives conditions under which the tridiagonal linear system (1.29) has a unique solution.

Theorem 15 *Suppose that $p, q,$ and r are continuous on $[a,b]$. If $q(x) \geq 0$ on $[a,b]$. Then, the tridiagonal linear system (1.29) has a unique solution provided that $h < \frac{2}{L}$, where $L = \max\limits_{a \leq x \leq b} |p(x)|$.*

Remark 5 *It should be noted that, the hypotheses of this last Theorem guarantee a unique solution to the boundary-value problem (1.20), but they do not guarantee that $u \in C^4[a,b]$. We need to establish that $u^{(4)}$ is continuous on $[a,b]$ to ensure that the truncation error has order $O(h^2)$.*

Finite-Difference Methods for non Linear Problems

For the general nonlinear boundary-value problem

$$\left\{ \begin{array}{c} u_{xx} = f(x, u, u_x), \quad x \in [a,b], \\ u(a) = \alpha, \quad u(b) = \beta. \end{array} \right. \tag{1.30}$$

We assume that f satisfies the following conditions:

1. f and the partial derivatives f_u and f_{u_x} are all continuous on

$$D = \{(x, u, u_x) \, / \, x \in [a,b], \text{ with } -\infty < u < \infty \text{ and } -\infty < u_x < \infty\}.$$

2. The constants $k = \max\limits_{(x,u,u_x) \in D} |f(x, u, u_x)|$ and $L = \max\limits_{(x,u,u_x) \in D} |f_{u_x}(x, u, u_x)|$ exist.

The difference method is similar to the method applied to linear problems in the last Subsection. Requires the replacement of u_{xx} and u_x by difference quotients, which results in a nonlinear system (Here, the system of equations (1.29) will not be linear). This system is solved using Newton's method, so an iterative process is required to solve it.

1.6.2 Finite Difference Method for Solving Parabolic PDEs

We consider the numerical solution to a problem involving the heat, or a parabolic partial differential equation of the form

$$\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < l, \quad t > 0, \tag{1.31}$$

subject to the initial data

$$u(x, 0) = f(x), \; 0 \leq x \leq l, \tag{1.32}$$

and to one of the typical sets of the boundary conditions of this problem

$$u(0, t) = u(l, t) = 0, \; t > 0 \tag{1.33}$$

or

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(l,t) = 0. \tag{1.34}$$

The approach we use to approximate the solution to this problem involves finite differences. Method of finite differences in a single spatial dimension depends on the conversion of the following physical solution region

$$\mathcal{D} = \{(x,t);\ 0 \leq x \leq l,\ 0 < t\},$$

where x is the space variable, the spatial domain is $[0,l]$, t is the time variable and $u(x,t)$ denote the exact solution to the initial value problem. To get the finite difference scheme, we follow the following steps.

Step1: $\mathcal{D}$ is discretized into cells described by network of nodes set (x_i,t_j) by dividing them into equal distances. The grid points for this situation are (x_i,t_j),

$$\begin{cases} x_i = i\triangle x = ih, & i = 0,1,...,n, \\ t_j = j\triangle t = jk, & j = 0,1, \end{cases}.$$

$$n = \frac{l-0}{\triangle x} = \frac{l}{h}, \tag{1.35}$$

where n is integer (Number of steps), $\triangle x$ distance between nodes and $\triangle t$ time step.

Step2: The lines $x = x_i$ and $t = t_j$ are grid lines, and their intersections are the mesh points of the grid. For each mesh point in the interior of the grid, (x_i,t_j), for $i = 1,2,...,n-1$ and $j = 1,2,...$, we can use the Taylor series in the variable x about x_i to generate the centered-difference formula

$$u_{xx}(x_i,t_j) = \frac{1}{h^2}[u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)] - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\theta_i,t_j),$$

for some θ_i in (x_{i-1},x_{i+1}). In a similar manner we can get

$$u_t(x_i,t_j) = \frac{1}{k}[u(x_i,t_{j+1}) - u(x_i,t_j)] - \frac{k}{2}u_{tt}(x_i,\rho_j),$$

for some ρ_j in (t_j,t_{j+1}). Using these formulas in the parabolic partial differential equation (1.31) allows us to express the equation at the points (x_i,t_j) as

$$\begin{array}{c} \frac{1}{k}[u(x_i,t_{j+1}) - u(x_i,t_j)] - \alpha^2\frac{1}{h^2}[u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)] \\ = \frac{k}{2}u_{tt}(x_i,\rho_j) - \alpha^2\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\theta_i,t_j) = \tau, \end{array} \tag{1.36}$$

for $j = 1,2,...$ and $i = 1,2,...,n-1$.

Remark 6 *The local truncation error for this difference equation is*

$$\tau = \frac{k}{2}u_{tt}(x_i,\rho_j) - \alpha^2\frac{h^2}{12}\frac{\partial^4 u}{\partial u^4}(\theta_i,t_j).$$

Step3: Approximations v_i^j to $u(x_i,t_j)$. (A Finite-Difference method with truncation error of order $O(h^2+k^2)$ results).

We denote the approximate solution v_i^j at an arbitrary point (x_i,t_j), i.e. $v_i^j \approx u(x_i,t_j) = u(i\triangle x, j\triangle t)$.

Replace $u(x_i, t_j)$ with v_i^j in (1.36) (in a single spatial dimension), gives

$$\frac{1}{k}\left[v_i^{j+1} - v_i^j\right] - \alpha^2 \frac{1}{h^2}\left[v_{i+1}^j - 2v_i^j + v_{i-1}^j\right] = 0,$$

we obtain the explicit scheme for (1.31)

$$v_i^{j+1} - v_i^j - \frac{k}{h^2}\alpha^2\left[v_{i+1}^j - 2v_i^j + v_{i-1}^j\right] = 0, \tag{1.37}$$

for all $j = 0, 1, 2, ...$ and $i = 1, 2, ..., n-1$.

Solving equation (1.37) for v_i^{j+1}, gives

$$\begin{array}{l} v_i^{j+1} = (1-2\Upsilon)\, v_i^j + \Upsilon\left[v_{i+1}^j + v_{i-1}^j\right], \quad \text{for each} \\ \quad i = 1, ..., n-1 \text{ and } \quad j = 0, 1, ..., \end{array} \tag{1.38}$$

where we choose $\Upsilon = \left(\frac{k}{h^2}\right)\alpha^2$.

So we have from the initial data (1.32)

$$u(x_i, t_0) = u(x_i, 0) = v_i^0 = f(x_i) \text{ for all } i = 0, 1, ..., n. \tag{1.39}$$

For all the interior nodes, putting $j = 0$ and $i = 1, ..., n-1$, in equation (1.38), we have

$$\begin{cases} v_0^1 = u(x_0, t_1) = u(0, t_1) = 0; \\ v_1^1 = (1-2\Upsilon)\, v_1^0 + \Upsilon\left[v_2^0 + v_0^0\right]; \\ v_2^1 = (1-2\Upsilon)\, v_2^0 + \Upsilon\left[v_3^0 + v_1^0\right]; \\ \quad\vdots \\ v_{n-1}^1 = (1-2\Upsilon)\, v_{n-1}^0 + \Upsilon\left[v_n^0 + v_{n-2}^0\right]; \\ v_n^1 = u(x_n, t_1) = u(l, t_1) = 0. \end{cases} \tag{1.40}$$

Now we can use the v_i^1 values to generate all the v_i^2 values and so on.

The typical equation (1.38) can be solved explicitly because it can be written for each internal location node for time node $j+1$ in terms of u at time node j. In other words, if we know values of u at node $j = 0$, and knowing the boundary values of u, which is the values of u at the external nodes, we can find values of u at the next time step. We continue the process by first finding the values of u at all nodes $j = 1$, and using these to find the values of u at the next time node $j = 2$. This process continues till we reach the time at which we are interested in finding the values of u.

In other case equation (1.38) can be written for all nodes (except the external nodes), at a particular time level. This results in simultaneous linear equations which can be solved to find u at a particular time. The explicit nature of the difference method implies that the $(n-1) \times (n-1)$ matrix associated with this system can be written in the following tridiagonal matrix form

$$AV^{j+1} = V^j, \tag{1.41}$$

such that

$$A=\begin{bmatrix} 1-2\Upsilon & \Upsilon & 0 & \cdots & 0 \\ \Upsilon & 1-2\Upsilon & \Upsilon & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \Upsilon \\ 0 & \ldots & 0 & \Upsilon & 1-2\Upsilon \end{bmatrix},$$

$V^j=\left(v_1^j,v_2^j,...,v_{n-1}^j\right)^t$ for each $j=1,2,...,$ the difference approximation of the exact solution $u\left(x,t\right)$ and $V^0=\left(f\left(x_1\right),\ f\left(x_2\right),...,\ f\left(x_{n-1}\right)\right)^t$.

Remark 7 *Special algorithms such as Thomas algorithm, can be used to solve simultaneous linear equation with tri-diagonal coefficient matrices.*

Chapter 2

Modeling in epidemic dynamics and basic concepts

To study infectious diseases in mathematical biology, the basic elements of this science will first be quickly understood by providing examples of simple epidemics in terms of the basic SIR or SIS models for constant population size. In a simple epidemic, where the population consists only of susceptibles and infectives, and are well captured in mathematical representations. Therefore, we introduce basic mathematical disease models in the first part to facilitate understanding of the models we will study in the third and fourth chapters.

2.1 The theory of epidemic dynamics

The theory of epidemic dynamics is so rich that it is impossible for us to cover all of its aspects. We only introduce basic mathematical disease models, the ideas for modeling, and fundamental concepts and techniques.

2.1.1 Special forms of epidemiological models

An epidemic model is a simplified means of describing the transmission of communicable disease through individuals. Two fundamental dynamic models of epidemics dynamic models for infectious diseases are mostly based on compartment structures that were initially proposed by Kermack and McKendrick in 1927 and 1932 models of infectious diseases were presented systematically in [38, 39], who obtained a known threshold result for the model they proposed. Since then great literature about such models has grown. Their work was later developed by many other biomedical scientists. To formulate a dynamic model for the transmission of an epidemic disease, the population of a particular area is often divided into several different groups.

Kermack–Mckendrick SIR model

Kermack–Mckendrick SIR model studied by Kermack and Mckendrick in [38], the population is divided into three groups: a susceptible group, labeled S, in which all the individuals are susceptible if they contact with a disease; an infected group, labeled I, in which all the individuals are infected by the disease; and a removed group, labeled R, in

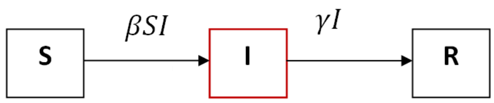


Figure 2.1: The SIR model without vital dynamics.

which all the individuals recovered from the infection. Denote the numbers of individuals in the groups S, I, and R, at time t, as $S(t)$, $I(t)$, and $R(t)$, respectively. The following three assumptions were made by them:

(i) The disease spreads in a closed environment; that is, there is no immigration, and neither birth nor death in the population, so that the total population remains a constant N for all t, that is,

$$S(t) + I(t) + R(t) \equiv N.$$

(ii) The number of susceptible who are infected by an infected individual per unit of time, at time t, is proportional to the total number exposed to injury by the relative parameter called the transmission factor β, so that the total number of newly invective's, at time t, is $\beta S(t)I(t)$.

(iii) The number removed (recovered) individuals from the infected group per unit time is $\gamma I(t)$ at time t, where γ is the recovery rate coefficient, and the recovered individuals gain permanent immunity and the model equations are given in the system

$$\begin{cases} S_t = -\beta S(t) I(t), \\ I_t = \beta S(t) I(t) - \gamma I(t), \\ R_t = \gamma I(t) \ , \ t \in \mathbb{R}^+. \end{cases} \tag{2.1}$$

and see Figure 2.1 for the following corresponding flowchart of the SIR model.

Because the formula for the variable R is separate from the first two equations for system (2.1), we only need to consider the following system

$$\begin{cases} S_t = -\beta S(t) I(t), \ t \in \mathbb{R}^+. \\ I_t = \beta S(t) I(t) - \gamma I(t) \end{cases} \tag{2.2}$$

From (2.2), we have

$$\frac{\partial S}{\partial I} = -1 + \frac{\gamma}{\beta S},$$

if the initial number of susceptible $S(0) = S_0 > \frac{\gamma}{\beta}$, the number of infectives increases and if $S(0) = S_0 < \frac{\gamma}{\beta}$, the number of infectives decreases. Define $R_0 = \frac{\beta}{\gamma} S_0$. Then the epidemic spreads when $R_0 > 1$ and dies when $R_0 < 1$. We know that γI is the number of people who recovered from the affected group I, per unit of time, at time t. Hence after the period of time $\frac{1}{\gamma}$, all the infectives $I(t)$ recover. Therefore, $\frac{1}{\gamma}$ is actually the mean duration of infection, and R_0 is the number of newly infectives infected by an infected individual during the whole infection period when all of the individuals in the population are initially susceptible. The number of infectives decreases if $R_0 < 1$ and increases if $R_0 > 1$. Therefore, to control the spread of an epidemic, one of the key factors is to estimate the value of R_0 and then reduce it to less than one. So the quantity R_0 determines the transmission threshold.

It follows from the second equation of (2.2) that $I(t) \to 0$, as $t \to \infty$, if $R_0 < 1$. Then since $S(t)$ is monotone decreasing and bounded below, $\lim_{t\to\infty} S(t) = S_\infty$.

Kermack–Mckendrick SIS model

For epidemiological viral diseases, such as influenza, measles, and covid-19 and smallpox, individuals who have recovered generally acquire immunity to the virus (SIR). But for the study of the dynamics of transmission of bacterial diseases, such as encephalitis and gonorrhea, the people who have recovered do not gain immunity and the infection can be repeated. For this study, Kermack and Mckendrick [39] suggest the SIS model equations are given in the system

$$\begin{cases} S_t = -\beta S(t) I(t) + \gamma I(t), \\ \quad I_t = \beta S(t) I(t) + \gamma I(t), \ t \in \mathbb{R}^+. \end{cases}$$

In the SIS model, infected individuals become susceptible to infection again once they recover from the infection, and this model is one of the simplest and is the primary form of infectious disease that is transmitted from person to person, and many other models are nothing but derivatives of this form. This model consists of two parts: S the number of people exposed to the infection and I for the number of infectious people, Where healing gives lasting resistance, like measles and rubella. If there is a migration to the population and the birth and death rates differ among the population, this means that the size of the population will change, or that there are other diseases that have caused the death of the population, then the population will be different and it is assumed that a small number of infections are introduced to the population at risk. With an risk person, the infected person becomes infected immediately. That is, S turns into I and the graph of the SIS model is shown in Figure 2.2.

The basic forms of epidemiological models fall into the following groups:

Models without vital dynamics: If a disease spreads rapidly in a population for a short period, such as influenza or chickenpox, then vital factors, such as natural population death and childbirth, can usually be ignored. We call these models, models without biodynamics.

Models without latent periods: Models without latent periods are models in which individuals are directly injured. These models are represented by SI, SIS, SIR, $SIRS$,

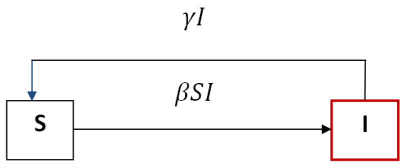


Figure 2.2: The SIS model without vital dynamics.

$SIRI$ models, ... For example, in the SI model, the infectives cannot be recovered from infection and in the SIS model, infected individuals become susceptible to re-infection immediately after they recover from the infection (no immunity after recovery). As for the $SIRS$ model, recovered individuals have temporary immunity after the infection is cured, and their immunity will wear off over time.

In the case of dynamic models there are two cases:

First case: If we assume that birth and death rates in the population are equal during the period of the epidemic and that there are no deaths due to the disease, then the size of the population will remain constant in a closed environment, denoted by N.

The second case: If the birth and death rates among the population are different, or there are deaths, then there is a migration into the population, then the size of the population is variable. For further understanding, we present an example of a SIS model in which newborns or individuals with many diseases, such as hepatitis C and hepatitis B, can be infected, a model in which newborns can also be infected by individuals with many diseases. μ the natural death rate coefficient, b is the birth rate coefficient, Λ is the input rate of the total population, B is the output rate coefficient of the susceptible and the infected and α is the coefficient of death rate caused by the disease. See Figure 2.3.

2.1.2 Basic Concepts of Epidemiology Dynamics

Basic reproduction number, disease-free equilibrium and endemic equilibrium

The possibility of an propagation of an infectious disease and its spread of the population is one of the most important concerns, and for many large epidemiological models, a disease-free equilibrium E_0 to keep the population in the absence of disease.

Definition 15 *(**Basic Reproductive Number**) The basic reproduction number, denoted by R_0, is the most important mathematical quantity in the infectious diseases. Biologically, R_0 can be interpreted as the average number of infections expected for secondary cases resulting, in a population of susceptible individuals, Typically contagious during the time in which he have an infection.*

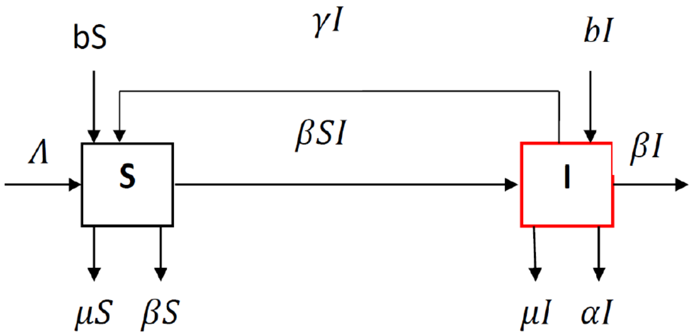


Figure 2.3: The SIS model with input output and disease-induced death.

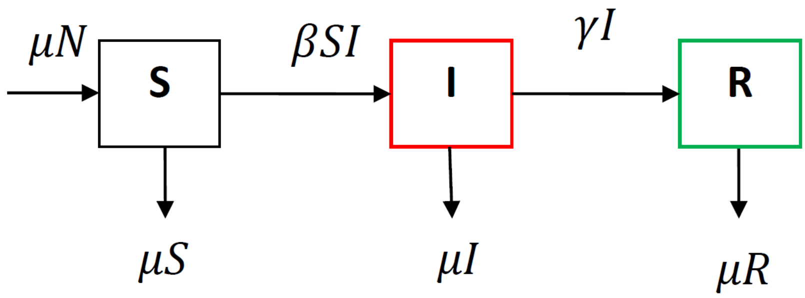


Figure 2.4: The SIR model without vertical transmission

If $R_0 > 1$, then each infected individual produces, more than one new infection, and the disease can invade the population and creating an epidemic situation. If $R_0 < 1$, then on average an infected individual produces less than one new infected individual over the course of its infectious period, and the infection cannot grow. (See [17, 22, 71].

Definition 16 *$R_0 = \rho(A)$, where $A = FV^{-1}$ is the next generation matrix for the epidemiological model and to understand what are F and V the $m \times m$ matrices (see [71, page 32]).*

Remark 8 *R_0 is the very important number for the study of our model in chapter four and important number in the study of Kermack–Mckendrick SIR model* (2.1)*, where a disease dies out if $R_0 < 1$ (the disease will disappear over time) and the disease can spreads if $R_0 > 1$ (the disease continues in the population), it is known this law the famous threshold standard.*

To better understand this number R_0 and to find out a disease-free equilibrium and endemic equilibrium, we propose to study the example below, which was introduced in [79].

Example 16 *We consider now the SIR model after changes to the previous model (2.1) and its corresponding flow chart in figure 2.4*

$$\begin{cases} S_t = \mu N - \beta S(t) I(t) - \mu S(t), \\ I_t = \beta S(t) I(t) - \mu I(t) - \gamma I(t), \\ R_t = \gamma I(t) - \mu R(t), \ t \in \mathbb{R}^+. \end{cases} \tag{2.3}$$

The variable R is not included in the first two equations of (2.3), and we are only interested in the spread of the disease, we only investigate the system consisting of the first two equations

$$\begin{cases} S_t = \mu N - \beta S(t) I(t) - \mu S(t) := F(S, I), \\ \\ I_t = \beta S(t) I(t) - (\mu + \gamma) I(t) \ := G(S, I), \ \text{in } \mathbb{R}^+, \end{cases} \tag{2.4}$$

where $(S, I) \in D = \{(S, I) : S > 0, \ I \geq 0, \ S + I \leq N\}$, so that the equilibrium of the system (2.4) are ***a disease free equilibrium*** *$E_0 = (N, 0)$ and* ***an endemic equilibrium*** *$E^* = \left(\frac{\mu+\gamma}{\beta}, \frac{\mu[\beta N - (\mu+\gamma)]}{\beta(\mu+\gamma)}\right)$.*

We then discuss the dynamics of these equilibrium as follows:

1. *If $R_0 = \frac{\beta N}{(\mu+\gamma)} < 1$. Then, there is only one equilibrium E_0 in region D. The Jacobian matrix of (2.4) at this equilibrium, E_0, is given by*

$$J(E_0) = \begin{pmatrix} -\mu & -\mu N \\ 0 & \beta N - \gamma - \mu \end{pmatrix},$$

   $J(E_0)$ has two eigenvalues $\lambda_1 = -\mu < 0, \ \ \lambda_2 = \beta N - \gamma - \mu < 0 \ \ if \ \ R_0 < 1$. Then the equilibrium E_0 is locally asymptotically stable and we note that region D is positively invariant for system (2.4) such that all orbits of (2.4) started inside D cannot go out of D. This implies that in this population, the epidemic cannot continue and dies out eventually. The point $E_0 = (N, 0)$ is called ***a disease-free equilibrium****.*

2. *Assume $R_0 = \frac{\beta N}{(\mu+\gamma)} > 1$. In addition to E_0, there is a positive equilibrium E^* in region D. The Jacobian matrix of (2.4) at the equilibrium E^* is given by*

$$J(E^*) = \begin{pmatrix} -\mu \frac{\beta N}{(\mu+\gamma)} & -(\mu + \gamma) \\ \frac{\mu[\beta N - (\mu+\gamma)]}{\mu+\gamma} & 0 \end{pmatrix}.$$

   Thus, it follows from $tr\ J(E^*) = -\mu \frac{\beta N}{(\mu+\gamma)} < 0$ and $\det J(E^*) = \mu[\beta N - (\mu + \gamma)] > 0$, the equilibrium E^* is locally asymptotically stable if it exists. They proved that the equilibrium E^* is globally asymptotically stable in region D. This explains that once the disease spreads in the population, the epidemic will continue, so that the disease becomes endemic and the point E^* is called an ***endemic equilibrium****, where the epidemic invades the population and $\lim_{t\to\infty} S(t) = \frac{\mu+\gamma}{\beta}$ and $\lim_{t\to\infty} I(t) = \frac{\mu[\beta N - (\mu+\gamma)]}{\beta(\mu+\gamma)}$.*

3. $R_0 = 1$ *is a threshold that determines whether a disease persists or goes. The epidemic does not persist if* $R_0 < 1$*, and spreads and eventually forms an endemic if* $R_0 > 1$*. This can be interpreted. From the second equation of system* (2.4)*, we can see that* $\frac{1}{(\mu+\gamma)}$ *is the mean infective period,* N *is the number of the individuals in the population, also the number of susceptibles at the disease-free equilibrium* E_0*. Therefore,* $R_0 = \frac{\beta N}{(\mu+\gamma)}$ *is actually the average number of secondary infections produced by one infected individual during the mean course of infection in a completely susceptible population. If* $R_0 < 1$*, then on average, the number of new infections by one infected individual over the mean course of the disease is less than one, which implies that the disease dies out eventually. If* $R_0 > 1$*, then the number of new infections produced by one infected individual is greater than one, which leads to the persistence of the infection.*

2.1.3 Example of Reaction–Diffusion System

Example 17 *(**An epidemic propagation model**). The spread of an epidemic(contagious disease) or a rumor in a population, divides the latter automatically into two separate groups:*

- (S) *the group of susceptible individuals who are not infected, but who may become so.*
- (I) *the group of contagious or infected individuals carrying the virus and therefore capable of transmitting it to individuals in group* (S)*.*
- *Let us denote by* $S(x,t)$*,* $I(x,t)$ *the densities at point* x *of space (position) at time* $t > 0$ *of the groups* (S) *and* (I) *respectively. We suppose that Susceptibles* (S) *become infectious* (I) *after contact with an infectious and at a rate proportional to the number of contacts between individuals in groups* (S) *and class* (S) *and another function of class* (I) *(i.e .* $\psi(S)\varphi(I)$ *).*
- *Individuals are removed from the infected class at a rate proportional to a function of* (I)*.*
- *The total population is constant and is confined to a bounded domain, we are led to the following system of equations:*

$$\begin{cases} S_t = d_1 \Delta S - \psi(S)\varphi(I), \\ \quad I_t = d_2 \Delta I + \psi(S)\varphi(I) - h(I), \quad \text{in } \Omega \times \mathbb{R}^+. \end{cases}$$

If we assume that there is no migration across the border of Ω *we have the conditions with homogeneous Neumann boundary conditions:*

$$\frac{\partial S}{\partial \eta} = \frac{\partial I}{\partial \eta} = 0 \qquad \text{on} \qquad \mathbb{R}^+ \times \partial\Omega,$$

with the nonnegative continuous and bounded initial data

$$u(0,x) = u_0(x), v(0,x) = v_0(x) \quad \text{in } \Omega,$$

Remark 9 *When* $\psi(S) = kS$ *with* $k > 0$ *and* $\varphi(I) = h(I) = I$ *is called a diffusive epidemic model.*

Chapter 3

Global existence and asymptotic behavior of solutions for some coupled systems via a Lyapunov functional

The purpose of this chapter is to study the global existence and asymptotic behaviour of solutions to a coupled reaction-diffusion systems, which describes epidemiological or chemical situations. Our techniques are based on Lyapunov functional methods.

In the first part we will prove the local existence, the positivity of the solutions, global existence and bounded solutions for all positive time of a class of disease epidemics systems formed of 2 parabolic partial differential equations called reaction–diffusion systems. In second part we give a simple proof of global existence and bounded solutions for all positive time for an epidemic system in the case where, $f(u,v) = g(u,v)$ and under suitable conditions on the the non-linearity $\varphi(v)$, we contribute to the study of the behaviour of the solutions for $f(u,v) = g(u,v) = \lambda u\varphi(v)$.

The work of this chapter is the subject of an international publication [23].[1]

3.1 System Model

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is an open bounded region with smooth boundary $\partial\Omega$. In this section we consider the following mathematical model in epidemiology

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\Delta u = \Lambda - f(u,v) - \mu u, \\ \\ \dfrac{\partial v}{\partial t} - b\Delta v = g(u,v) - \sigma v, \end{cases} \tag{3.1}$$

in $\mathbb{R}^+ \times \Omega$, with the non negative continuous and bounded initial data

$$u(0,x) = u_0(x), v(0,x) = v_0(x) \qquad \text{in } \Omega, \tag{3.2}$$

[1] L. Djebara, S. Abdelmalek, S. Bendoukha, *Global existence and asymptotic behavior of solutions for some coupled systems via a Lyapunov functional*, Journal.Acta Mathematica Scientia, Chinese Academy of Sciences, 39B(6): 1538–1550, (2019).

and Neumann boundary conditions of this problem

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \qquad \text{on} \qquad \mathbb{R}^+ \times \partial\Omega. \tag{3.3}$$

3.1.1 Parameters of The model and its Assumptions

- t is the time variable and x is the position.
- $u = u(x,t)$ and $v = v(x,t)$ is used to represent the number of non dimensional population densities of the susceptibles and the number of infectious individuals at location x and time t, where the susceptibles u, which can get the disease and the infected v, who have the disease and can transmit it respectively and hence have to be positive.
- a, b the diffusion coefficients.
- The constant Λ denote the growth of susceptibles corresponding to births or migration (the recruitment rate of the population).
- μ is the natural death rate.
- σ is the rate at which invective's recover from the disease.
- $\frac{1}{\sigma}$ the period of infection.

We also assume that a, b, Λ, μ and σ are positive constants such that

$$a > 0, b > 0, \mu > 0, \sigma > 0 \qquad \text{and} \qquad \Lambda \geq 0 \text{ and that } a \neq b.$$

In addition, we suppose that f and g are nonnegative continuously differentiable function on $\mathbb{R}^2_+$ satisfying

$$\textbf{(A1)} \quad f(0,\eta) = 0 \text{ and } g(\xi, 0) \geq 0 \quad \text{for any } \xi, \eta \geq 0,$$

$$\textbf{(A2)} \quad g(\xi,\eta) \leq \psi(\eta)\ f(\xi,\eta),$$

where ψ is a non–negative and continuously differential function on $\mathbb{R}^+$ such that there exists a constant $\beta \geq 1$ satisfying

$$\lim_{\eta \to +\infty} \eta^{\beta-1} \psi(\eta) = \ell,$$

for $\ell \geq 0$, and

$$\textbf{(A3)} \quad g(\xi,\eta) \leq C\phi(\xi)\, e^{\alpha\eta^{\beta}} \quad C > 0 \text{ and } \alpha > 0,$$

for certain constants $C > 0$ and $\alpha > 0$ where β is the same as in (**A2**) and ϕ is all non negative continuously differentiable function on $\mathbb{R}^+$ such that $\phi(0) = 0$.

The diffusive model (3.1)–(3.3) subject to conditions with (**A1**)-(**A3**) is related to epidemiology. It describes the propagation of an infectious disease with u and v representing the population densitities of susceptible and infective individuals, respectively [13, 16].

3.1.2 Interpretation of model

System (3.1)–(3.3) it is a model derived from the simple basic model uv. The model is for the spread of an infectious disease in a confined population in a region Ω (this corresponds with $\dfrac{\partial u}{\partial \eta} = \dfrac{\partial v}{\partial \eta} = 0$, which means no migration through Ω) and the population divided into two health states: susceptible to the infection of the pathogen (often denoted by u); and the class of infected by the pathogen v. In this case, there is an arrival of new susceptible Λ individuals into the population and assuming (The population of susceptible individuals is increased at any time "t" by the recruitment rate Λ. We consider all the new births are susceptible and join the susceptible class $u(t,x)$), a death rate μ, so we have $\Lambda - \mu u$. During the epidemic, the number of susceptible person is supposed to become infectious over time through contact with other infectious (The population of susceptible individuals is decreased by the contact rate of susceptible with infected individuals by $f(u,v)$), and then he leaves the community of susceptible. Hence the term $-f(u,v)$. The infection is supposed to leave an infectious layer at the rate σ of infection, quarantine and then die, so we have $g(u,v) - \sigma v$.

3.2 Objectives of This Study

In this paper, we aim to establish the global existence of a unique solution to system (3.1)–(3.3) with homogeneous Neumann boundary conditions and exponential growth conditions on the nonlinear terms by the constraint $\|u_0\|_\infty$, where the following assumption is imposed

$$\|u_0\|_\infty < M < \frac{8ab}{\alpha\beta\ell n\,(a-b)^2},\ \ell > 0, \tag{3.4}$$

$$\|u_0\|_\infty < M,\ \ell = 0. \tag{3.5}$$

For example, our results are applicable to

$$\begin{cases} f(\xi,\eta) = \xi\,(1+\eta^m)\,e^{\eta^\beta}, \\ g(\xi,\eta) = \xi e^{\eta^\beta}, m > \beta - 1 \geq 0, \\ \psi(\eta) = (1+\eta^m)^{-1}, \end{cases} \tag{3.6}$$

in this case $\ell = 0$.

The local and global existence results will be discussed in Section 3.5, 3.6, respectively. In section 3.7 we deal with the long time behaviour of the solution as the time goes to $+\infty$ in the case where $f(u,v) = g(u,v) = \lambda u \varphi(v)$, we show that every global solution uniformly converges to the corresponding constant function as $t \to +\infty$, where we demonstrate that the infective class v converges asymptotically to zero in $\mathbb{L}^\infty(\Omega)$, and the susceptible class u converges asymptotically to $\frac{\Lambda}{\mu}$. Our approach for the global existence and the asymptotic stability is based on the use of Lyapunov functionals [1]. Section 3.8 illustrates the analytical conditions through numerical examples.

3.3 Previous results for our first work

The diffusive model (3.1)–(3.3) is related to epidemiology. It describes the propagation of an infectious disease with u and v representing the population densities of susceptible and infective individuals, respectively [13, 16]. Consider two component version of system (3.1)–(3.3) with $\Lambda = \mu = 0$ and nonlinearities

$$f(u,v) = -\, g(u,v) = -uv^{\beta}.$$

This corresponds to a modification of the system considered in [55] and numerous studies have examined its dynamics. For instance, Alikakos [9] established the global existence and $\mathbb{L}^{\infty}$-bounds of solutions for positive initial data when $1 \leq \beta < \frac{n+2}{n}$.

In 1983 Masuda [56] continued the work and showed that solutions of this system exist globally for $\beta \geq 1$. Haraux and Youkana [31] proved the global existence of solutions by employing Lyapunov functional with the nonlinearities

$$f(u,v) = -g(u,v) = -uF(v) = -ue^{v^{\gamma}},$$

for $\gamma < 1$. Kouachi and Youkana [47] generalised the method further and added the term $-c\triangle u$ to the right hand side of the second equation with the reaction terms

$$f(u,v) = -\lambda F(u,v) \quad \text{and} \quad g(u,v) = \mu\, F(u,v),$$

for some $F(u,v) \geq 0$. The authors imposed the condition

$$\lim_{s\to+\infty}\left[\frac{\log(1+F(r,s))}{s}\right] < \alpha^{*},$$

for all $r \geq 0$, with

$$\alpha^{*} = \frac{2ab}{n(a-b)^{2}\|u_0\|_{\infty}} \min\left\{\frac{\lambda}{\xi}, \frac{a-b}{c}\right\}.$$

They assumed positive diffusivities $a > b$, and constants c, λ, $\zeta > 0$ that reflect the weak exponential growth of the reaction term. Another important study that followed is [34], where the authors established the global existence of positive solutions subject to with the boundary conditions (3.3) with

$$\begin{array}{lll} \beta_1, \beta_2 \geq 0 & \text{and} & 0 < \lambda_1, \lambda_2 < 1,\ \lambda_1 = \lambda_2 = 1,\ \text{or}\ \lambda_1, \lambda_2 = 0,\ \text{or} \\ \beta_1 = \beta_2 = 0 & \text{and} & \lambda_1, \lambda_2 = 0. \end{array}$$

The considered that for each $r > 0$ there exist numbers $L_0(r)$ and $\mu_0(r)$ such that for $r \leq v$

$$\left\{\begin{array}{l} \gamma \geq 1, |g(u,v)| \leq L_0(r)(1+v)^{\gamma}, \\ f(u,v) + g(u,v) \leq \mu_0(r). \end{array}\right.$$

The global existence of solutions to this problem was established in [64] for $\Lambda = \mu = \sigma = 0$.

Note that in case $\Lambda \geq 0$, and $\mu > 0$, it is not obvious to obtain the global existence of solutions when the non-linearity has an exponential growth or at most polynomial growth (see the work of L. Melkemi et al. [57]). Finally, Abdelmalek et al. [6] added the term $-d\Delta v$ to the right hand side of the second equation with the reaction terms $f(u,v) = g(u,v)$, and $\mu = \sigma$.

3.4 The case that a = b = 0 and a = b ≠ 0

If we looks at the corresponding system without diffusion we get the ordinary differential equations case ($a = b = 0$), local existence of a unique classical solution holds by using the Cauchy-Lipschitz theorem 6. Note that in the simple case when the diffusion coefficients are moreover equal $a = b \neq 0$, we use the Hill-Yosida Theorem 7 for the local existence of a unique classical solution and under the conditions (**A1**) and (**A2**) the classic local solutions on $[0, T)$ can be extended on a maximal interval $[0, T^*)$. If we assume, $u_0(x), v_0(x) \geq 0$ and thanks to the hypothese (**A1**), we conclude that the solutions is nonnegative. Moreover, in particular we deduce by applying the maximum principle (see proposition6)

$$u(t,x) \leq \|u(t,.)\|_\infty \leq \max\left(\|u_0\|_\infty, \frac{\Lambda}{\mu}\right) := K \quad \text{for all } t \in (0,T), \ x \in \overline{\Omega}.$$

The problem is then reduced to obtaining a uniform estimate of v. Multiplying the first equations of (3.1) by the constant $\theta \geq \boldsymbol{\ell} + \mathbf{1}$, adding it to second equation of (3.1), lead to

$$\frac{\partial(\theta u + v)}{\partial t} - a\Delta(\theta u + v) \leq \theta\Lambda - \theta f(u,v) + g(u,v).$$

From the assumption (**A2**), we get

$$\begin{aligned} & \theta\Lambda - \theta f(\xi,\eta) + g(\xi,\eta) \\ \leq \ & \theta\Lambda + [-(\boldsymbol{\ell}+\mathbf{1}) + \psi(\eta)] \ f(\xi,\eta). \end{aligned}$$

Since $\eta^{\beta-1}\psi(\eta)$ goes to $\boldsymbol{\ell}$ as η goes to infinity, there exists $\tilde{\eta} > 0$ such that for all $\eta \geq \tilde{\eta}$, we have

$$\frac{\psi(\eta)}{\eta^{1-\beta}} < \boldsymbol{\ell} + \mathbf{1}.$$

Then,

$$\begin{aligned} & [-(\boldsymbol{\ell}+\mathbf{1}) + \psi(\eta)] \ f(\xi,\eta) \\ \leq \ & \left[-(\boldsymbol{\ell}+\mathbf{1}) + (\boldsymbol{\ell}+\mathbf{1})\eta^{1-\beta}\right] \ f(\xi,\eta). \end{aligned}$$

Since $\eta^{1-\beta}$ goes to $\mathbf{0}$ as η goes to infinity, there exists $\tilde{\eta}_0 > 0$ such that for all $\eta \geq \tilde{\eta}_0$, we have $\eta^{1-\beta} \leq 1$ and

$$\left[-(\boldsymbol{\ell}+\mathbf{1}) + (\boldsymbol{\ell}+\mathbf{1})\eta^{1-\beta}\right] \ f(\xi,\eta) \leq 0.$$

On the other hand, if η is in the compact interval $[0, \tilde{\eta}_0]$, then the continuous function

$$\eta \mapsto \theta\Lambda + [-(\boldsymbol{\ell}+\mathbf{1}) + \psi(\eta)] \ f(\xi,\eta)$$

is uniformly bounded for $\eta \geq 0$ and $0 \leq \xi < K$. In this way one can deduce that there exists a positive constant $\tilde{\delta}$ such that

$$\theta\Lambda + [-(\boldsymbol{\ell}+\mathbf{1}) + \psi(\eta)] \ f(\xi,\eta) \leq \tilde{\delta}$$

and we have

$$\begin{cases} \dfrac{\partial(\theta u+v)}{\partial t}-a\Delta(\theta u+v)\leq\tilde{\delta} \text{ in } \mathbb{R}^{+}\times\Omega, \\ \theta u(0,x)+v(0,x)=\theta u_0(x)+v_0(x) \quad \text{in } \Omega, \\ \dfrac{\partial(\theta u+v)}{\partial\eta}=0, \end{cases}$$

by taking

$$\begin{cases} B=\dfrac{\partial}{\partial t}-a\Delta \text{ and } \; w=\theta u+v, \\ \qquad h(t)=\|\theta u_0+v_0\|_{\infty}+\tilde{\delta}t. \end{cases}$$

We obtain

$$\begin{cases} B(w)\leq B(h), \\ w(0,x)\leq h(0), \\ \dfrac{\partial w}{\partial\eta}\leq\dfrac{\partial h}{\partial\eta}, \end{cases}$$

thanks to the comparison principle Theorem8, we get

$$0\leq w(t,x)\leq h(t) \quad \text{for all } \; (t,x)\in\,]0,T[\,\times\Omega,$$

and the following estimate is verified

$$\|\theta u+v\|_{L^{\infty}(Q_T)}\leq\|\theta u_0+v_0\|_{\infty}+CT.$$

Together with the positivity of $u(t)$, $v(t)$ and θ, then according to the criterion of global existence (1.7), we deduce the global existence of the solutions of the problem. But the situation turns out to be quite more complicated when the diffusion coefficients are not equal. It is assumed throughout this study that the diffusion coefficients a and b are different.

3.5 Preliminary results

3.5.1 The local existence and uniqueness

Proposition 3 *(We can see [42, page 331]) An operator A is m-accretive in $L^p(\Omega)$ for $1\leq p<+\infty$ and $(e^{-tA})_{t\geq 0}$ is an analytical semi-group in $\mathbb{L}^p(\Omega)$ for $1<p<\infty$. Moreover, the restriction of A to $X=\mathbb{C}(\overline{\Omega})$ is m-accretive.*

proof. See [42, 30]. □

Proposition 4 *For all u_0, v_0 in $X=\mathbb{C}(\overline{\Omega})$, the local existence of a classical solution holds for system (3.1)–(3.3), where there is a maximum time $T_{\max}>0$ and only one solution (u,v)*

$$\begin{aligned} u &\in \mathbb{C}^1(]0,T[;X)\cap\mathbb{C}(]0,T[;\mathcal{D}(A)), \\ v &\in \mathbb{C}^1(]0,T[;X)\cap\mathbb{C}(]0,T[;\mathcal{D}(A)), \forall T\leq T_{\max}, \end{aligned}$$

Proposition 5 *for some* $T_{\max} \in (0, +\infty]$ *, such that*

$$\mathcal{D}(A) = \left\{ (u,v) \in \mathbb{H}^2(\Omega) \times \mathbb{H}^2(\Omega) ,\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \ in \ \mathbb{R}^+ \times \partial\Omega \right\}.$$

proof. First, we convert (3.1)–(3.3) to an abstract first order system in the Banach space $X \times X$ of the form

$$\begin{cases} \frac{dU}{dt} + AU = F(U), \\ U(0) = U_0, \\ \frac{\partial U}{\partial \nu} = 0, \end{cases} \tag{3.7}$$

where the unbounded linear operator $A : \mathcal{D}(A) \subset X \times X \to X \times X$, is defined by

$$A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -a\Delta u & 0 \\ 0 & -b\Delta v \end{pmatrix} \text{ for each } U = \begin{pmatrix} u \\ v \end{pmatrix} \in D(A),$$

where we will consider U as the function $U : \mathbb{R}^+ \to X \times X$ and

$$F(U) = \begin{pmatrix} \Lambda - f(u,v) - \mu u \\ g(u,v) - \sigma v \end{pmatrix}.$$

Using Theorem 7 and Proposition 3, we will state and prove some results regarding the local existence, uniqueness and regularity of solutions U in $X \times X$ with $X = \mathbb{C}\left(\overline{\Omega}\right)$ to (3.6). We can prove A that is a maximal monotone operators in $\mathbb{L}^p(\Omega) \times \mathbb{L}^p(\Omega), 1 \leq P < \infty$ and consequently in $X \times X$ (see [30, page 272] and [42, page 330]).

For a simple study, we consider the case of the Hilbert space $\mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$ and prove that $A : \mathcal{D}(A) \subset \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega) \to \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$ is a maximal monotone operators in $\mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$.

(i) A is monotone in $\mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$ because $\forall U = (u,v) \in \mathcal{D}(A)$,

$$\begin{aligned} \langle AU, U \rangle_{L^2(\Omega) \times L^2(\Omega)} &= \int_\Omega \begin{pmatrix} -a\Delta & 0 \\ 0 & -b\Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} dx \\ &= a \int_\Omega |\nabla u|^2 \, dx + b \int_\Omega |\nabla v|^2 \, dx \geq 0. \end{aligned}$$

(ii) To show that A is maximal monotone in $\mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$ enough to show that $(I + A)$ is surjective, that is to say

$$\forall H = (h_1, h_2) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega), \exists U \in \mathcal{D}(A), U + AU = H,$$

which is equivalent to the following strong formulation of the problem is: Find u, v define on $\mathbb{R}^+ \times \Omega$ such that

$$\begin{cases} u - a\Delta u = h_1 & \text{in } \Omega, \\ v - b\Delta v = h_2 & \text{in } \Omega, \end{cases} \tag{3.8}$$

with $(u,v) \in D(A)$. Multiplying (3.8) by the functions tested φ and $\psi \in \mathcal{H} = \left\{ u \in \mathbb{H}^1(\Omega), \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \mathbb{R}^+ \times \Omega \right\}$, we have

$$\begin{cases} \langle u, \varphi \rangle - a \langle \Delta u, \varphi \rangle = \langle h_1, \varphi \rangle, \\ \langle v, \psi \rangle - b \langle \Delta v, \psi \rangle = \langle h_2, \psi \rangle. \end{cases}$$

By Green's identity and because of the homogeneous condition at the border, we have the variational formulation of the problem (3.7) is to find $(u,v)\in\mathcal{H}\times\mathcal{H}$ such that

$$\begin{cases} \int_\Omega u\varphi dx + a\int_\Omega \nabla u\nabla\varphi dx = \int_\Omega h_1\varphi dx, & \forall\varphi\in\mathcal{H},\\ \int_\Omega v\psi dx + b\int_\Omega \nabla v\nabla\psi dx = \int_\Omega h_2\psi dx, & \forall\psi\in\mathcal{H}, \end{cases} \quad \text{(weak formulation)}. \quad (3.9)$$

A solution (u,v) is naturally solution of the following problem: Applying the Lax-Milgram Theorem5, (3.9) suggests that the appropriate bilinears forms in this case are

$$\begin{aligned} a_1(u,\varphi) &= \int_\Omega (u\varphi dx + a\nabla u\nabla\varphi)\,dx, \\ a_2(v,\psi) &= \int_\Omega (v\psi dx + b\nabla v\nabla\psi)\,dx, \end{aligned} \quad (3.10)$$

and linears forms

$$L(\varphi) = \int_\Omega h_1\varphi dx, \quad L(\psi) = \int_\Omega h_2\psi dx,$$

1. a_1, a_2 are bilinears. Let $\alpha_1,\alpha_2\in\mathbb{R}$ and $u_{1,}u_{2,}\varphi\in\mathcal{H}$,

$$\begin{aligned} a_1(\alpha u_1+\beta u_{2,}\varphi) &= \int_\Omega [(\alpha u_1+\beta u_2)\varphi dx + a\nabla(\alpha u_1+\beta u_2)\nabla\varphi]\,dx \\ &= \alpha(\int_\Omega [u_1\varphi dx + a\nabla u_1\nabla\varphi dx]) + \beta(\int_\Omega [u_2\varphi dx + a\nabla u_2\nabla\varphi dx]) \\ &= \alpha a_1(u_1,\varphi) + \beta a_1(u_2,\varphi). \end{aligned}$$

The same result for a_2

$$\begin{aligned} a_2(u,\alpha\varphi_1+\beta\varphi_2) &= \int_\Omega u(\alpha\varphi_1+\beta\varphi_2)dx + a\int_\Omega \nabla u\nabla(\alpha\varphi_1+\beta\varphi_2)dx \\ &= \alpha(\int_\Omega u\varphi_1 dx + a\int_\Omega \nabla u\nabla\varphi_1 dx) + \beta(\int_\Omega u\varphi_2 dx + a\int_\Omega \nabla u\nabla\varphi_2 dx \\ &= \alpha a_2(u,\varphi_1) + \beta a_2(u,\varphi_2). \end{aligned}$$

2. $L\in(\mathcal{H})'$, L is linear because, for all $\alpha,\beta\in\mathbb{R}$ and $\varphi_1,\varphi_2\in\mathcal{H}$ we have

$$L(\alpha\varphi_1+\beta\varphi_2) = \int_\Omega \alpha h\varphi_1 dx + \int_\Omega \beta h\varphi_2 dx = \alpha L(\varphi_1) + \beta L(\varphi_2),$$

and is continuous (bounded) because, by the Cauchy-Schwarz inequality that,

$$\begin{aligned} |L(\varphi)| &= \left|\int_\Omega h\varphi dx\right| \le ||h||_{L^2}\,||\varphi||_{L^2}, \\ &\le ||h||_{L^2}\,||\varphi||_{\mathcal{H}} \le C_1\,||\varphi||_{\mathcal{H}}. \end{aligned}$$

3. a_1 is bounded on $\mathcal{H}$ because $\forall u, \varphi \in \mathcal{H}$ and applying Cauchy-Schwarz inequality, we get

$$\begin{aligned} |a_1(u,\varphi)| &= \left| \int_\Omega u\varphi dx + a \int_\Omega \nabla u \nabla \varphi dx \right| = |\langle u, \varphi\rangle|_{\mathcal{H}} \\ &\leq 1. \|u\|_{\mathcal{H}} \|\varphi\|_{\mathcal{H}} . \end{aligned}$$

The same result for all Ψ, $v \in \mathcal{H}$, we get

$$|a_2(v, \Psi)| \leq C_3 \, ||v||_{\mathcal{H}} . ||\Psi||_{\mathcal{H}} ,$$

then a_2 is bounded on in $\mathcal{H}$.

4. a_1 coercive : It must be shown that there is a constant $\alpha \in \mathbb{R}^+$such that

$$|a_1(u, u)| \geq \alpha \, ||u||^2_{\mathcal{H}} , \quad \text{for all } u \in \mathcal{H}.$$

By hypothesis on a_1, we have:

$$\begin{aligned} a_1(u,u) &= \int_\Omega [u.udx + a\nabla u \nabla u] \, dx \\ &= \|u\|^2_{\mathcal{H}} = 1 \times \|u\|^2_{\mathcal{H}} , \end{aligned}$$

which demonstrates the coercivity of a_1 in $\mathcal{H}$. The hypotheses of Lax-Milgram Theorem5 are thus verified and we can deduce the variational formulation has a unique solution. Therefore Lax-Milgram says that $\exists! \, (u, v) \in \mathcal{H} \times \mathcal{H}$ such that (3.9), which means precisely that there is a unique weak solution (u, v) to (3.7).

Step3. (F is locally Lipschitz in $X \times X$) In the other hand, it is clear that the map F is locally Lipschitz in $X \times X$, where

$$F(u,v) = \begin{pmatrix} \Lambda - f(u,v) - \mu u \\ g(u,v) - \sigma v \end{pmatrix} = \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \end{pmatrix},$$

that is to say we will prove that f_1, f_2 are locally Lipschitz in $X \times X$.

To show that f_1 is locally Lipschitz in $X \times X$, it's enough to prove that $\frac{\partial f_1}{\partial u}(u,v)$, $\frac{\partial f_1}{\partial v}(u,v)$ are locally bounded in $\mathbb{R}^+$. On a $\frac{\partial f}{\partial u}(u,v)$, $\frac{\partial f}{\partial v}(u,v)$ are locally bounded in $\mathbb{R}^+$ (because $f \in C^1(\Omega)$), we have

$$\begin{cases} \left|\frac{\partial f_1}{\partial u}(u,v)\right| = \left|-\frac{\partial f}{\partial u}(u,v) - \mu\right| \leq K_{1loc}, \\ \left|\frac{\partial f_1}{\partial v}(u,v)\right| = \left|-\frac{\partial f}{\partial v}(u,v)\right| \leq K_{2loc}. \end{cases}$$

Then f_1 is locally Lipschitz in $X \times X$.

Similarly, f_2 is locally Lipschitz in $X \times X$. Then F is locally Lipschitz in $X \times X$. This concludes the proof of Proposition.

3.5.2 The positivity of the solutions

For the global existence we need the fact that the solutions are positive.

Lemma 4 *For the initial data u_0 and v_0 are nonnegative functions, if we assume,* (**A1**), *the solution of system* (3.1)–(3.3) *is nonnegative for all $t > 0$.*

proof. [62, Lemma5.4.1] We are going to give two simple proofs of this lemma .

(1) In the first step, we start by studying the case of variable u, we prove that

$$u(t,x) \geq 0, \forall (t,x) \in (0, T_{\max}) \times \Omega.$$

By applying maximum principle to the first equation in system (3.2), multiplying this last by $(-u^-)$, and integrating over Ω, to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(u^-\right)^2 dx + a\int_\Omega (\Delta u)u^- dx = -\Lambda \int_\Omega u^- dx + \int_\Omega f(u,v)\, u^- dx + \mu \int_\Omega uu^- dx.$$

Using Green's formula (see Lemma 2) and the Neumann boundary conditions, it holds that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(u^-\right)^2 dx + a\int_\Omega |\nabla u|^2 dx \leq \int_\Omega f(u,v)\, u^- dx - \mu \int_\Omega \left(u^-\right)^2 dx.$$

But in the other hand, by **the Mean Value Theorem** (Theorem **4)** and using $(\mathbf{A}_1)$ there exists $c \in [0, v]$ with $f(u,v) - f(0,v) = u\frac{\partial f}{\partial u}(c,v)$, then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \left(u^-\right)^2 dx &\leq \int_\Omega \frac{\partial f}{\partial u}(c,v)\, uu^- dx \\ &\leq k_{loc}\int_\Omega \left(u^-\right)^2 dx,\end{aligned}$$

integrating in time $[0,t]$, give the following

$$\frac{1}{2}\int_\Omega \left(u^-\right)^2 dx - \frac{1}{2}\int_\Omega \left(u_0^-\right)^2 dx \leq k_{loc}\int_0^t \int_\Omega \left(u^-\right)^2 dx d\tau,$$

and the non-negativity of u_0, implies that $\frac{1}{2}\int_\Omega \left(u_0^-\right)^2 dx = 0$, applying now the Gronwall Lemma 1 we have $u^- \equiv 0$, witch implies, $u \geq 0$ on $(0, T_{\max})$. To prove the positive of $v(t,x)$, in an analogue way, we get $v^- = 0$ witch implies $v \geq 0$ in $(0, T_{\max}) \times \Omega$. Lemma4 is completely proved.

(2) The second proof is as follows: The quasi-positivity condition of f_1 and f_2 is satisfied because

$$f_1(0,v) = \Lambda \geq 0 \quad \text{and} \quad f_2(u,0) = g(u,0) \geq 0.$$

From Lemma3 we deduce the preservation of the positivity of the solution.

Remark 10 *The positivity of the solution is always preserved if the condition* (**A1**) *is replaced by the condition*

$$g(\zeta,0) \geq 0 \text{ and } \Lambda \geq f(0,\eta) \quad \text{for all } \zeta, \eta \geq 0.$$

3.6 Global existence and boundedness of solutions

3.6.1 Uniform boundedness of the Solutions

For proving the global existence of solutions for the problem (3.1)–(3.3) it suffices to prove that the solutions (u, v) remains bounded in $(0, T_{\max}) \times \Omega$. Using the regularizing effect it's enough to show that the reaction terms are uniformly bounded in $\mathbb{L}^p(\Omega)$, for some $p > \frac{n}{2}$. We introduce the following proposition.

Proposition 6 *Let $(u(t,.), v(t,.))$ be a solution of (3.1)–(3.3) on $[0, T]$ for some $T < T_{\max}$, it follows from the maximum principle that, there exists $M > 0$ such that*

$$0 \leq u(t,x) \leq \max\left(\|u_0\|_\infty, \frac{\Lambda}{\mu}\right) = K < M, \tag{3.11}$$

for all $(t,x) \in (0, T_{\max}) \times \Omega$.

proof. Multiplying the first equation in system (3.2) by $(u-K)^+$ and integrating over Ω, we get

$$\begin{aligned}\int_\Omega \frac{du}{dt}(u-K)^+ dx - a\int_\Omega \triangle u (u-K)^+ dx &= +\Lambda\int_\Omega (u-K)^+ dx - \int_\Omega f(u,v)(u-K)^+ dx \\ &\quad -\mu\int_\Omega u(u-K)^+ dx, \quad \forall t \in (0, T^*),\end{aligned}$$

using $u = u - K + K = (u-K)^+ - (u-K)^- + K$ and $(u-K)^+(u-K)^- = 0$, we have

$$\int_\Omega \frac{du}{dt}(u-K)^+ dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \left((u-K)^+\right)^2 dx,$$

and

$$\begin{aligned}-\mu\int_\Omega u(u-K)^+ dx &= -\mu\int_\Omega \left[(u-K)^+\right]^2 dx - \mu k\int_\Omega (u-K)^+ dx \leq \mu\int_\Omega \left[(u-K)^+\right]^2 dx \\ &\quad -\mu k\int_\Omega (u-K)^+ dx.\end{aligned}$$

On the other hand using Green's formula, the Neumann boundary conditions and $\max\left(\|u_0\|_\infty, \frac{\Lambda}{\mu}\right) = K$, it holds that

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega ((u-K)^+)^2 dx &= -a\int_\Omega \left(\nabla (u-K)^+\right)^2 dx - \int_\Omega f(u,v)(u-K)^+ dx \\ &\quad +\Lambda\int_\Omega (u-K)^+ dx - \mu\int_\Omega u(u-K)^+ dx \\ &\leq \Lambda\int_\Omega (u-K)^+ dx + \mu\int_\Omega \left[(u-K)^+\right]^2 dx - \mu k\int_\Omega (u-K)^+ dx \\ &\leq \mu\int_\Omega \left[(u-K)^+\right]^2 dx,\end{aligned}$$

integrating the resulting over $[0, T]$, we get

$$\frac{1}{2}\int_{\Omega}((u-K)^{+})^{2}dx-\frac{1}{2}\int_{\Omega}((u_{0}-K)^{+})^{2}dx\leq\mu\int_{0}^{t}\int_{\Omega}\left[(u-K)^{+}\right]^{2}dxd\tau,$$

from $u_0(x) \leq \|u_0\|_\infty \leq K$, we obtain

$$(u_0-K)^{+}=0,$$

which implies

$$0\leq\frac{1}{2}\int_{\Omega}\left((u-K)^{+}\right)^{2}dx\leq\mu\int_{0}^{t}\int_{\Omega}\left[(u-K)^{+}\right]^{2}dxd\tau,$$

so that Gronwall's Lemma 1 permits to conclude that $0\leq\frac{1}{2}\int_{\Omega}\left((u-K)^{+}\right)^{2}dx\leq 0$, it follows readily that

$$(u-K)^{+}=0,\ \text{ and } u(t,x)\leq K,$$

for all $(t,x)\in(0,T_{\max})\times\Omega$. □

Now, to derive a uniform bound of $\|v(t)\|_\infty$ on $(0,+\infty)\times\Omega$, we will make use the result established in Haraux-Kirane [30] and D.Henry [32, pp. 35-62], it's enough to show that $g(u,v)-\sigma v$ is uniformly bounded in $\mathbb{L}^p(\Omega)$, for some $p>\frac{n}{2}$ and established the global existence of unique solutions for (3.1)–(3.3).
First, let use define, for any $t\in[0,T_{\max})$ the functional

$$L(t)=\int_{\Omega}\left[(M-u)^{-\gamma}e^{\alpha p(v+1)^{\beta}}\right](t,x)\,dx, \tag{3.12}$$

where α,β,γ,M are positive constants such that

$$\begin{cases}\|u_0\|_\infty\leq K<M<\frac{2\gamma}{\alpha\beta n\ell}, & \ell>0\\ \|u_0\|_\infty\leq K<M, & \ell=0\end{cases},\quad 0<\gamma\leq\frac{4ab}{(a-b)^2},\ \text{ and } \beta\geq 1. \tag{3.13}$$

$$\text{and}\qquad \frac{n}{2}<p<\frac{\gamma}{\alpha\beta\boldsymbol{\ell}M},\boldsymbol{\ell}>0. \tag{3.14}$$

It is important to observe that when $\boldsymbol{\ell}=0$, the initial data $u_0\in\mathbb{L}^\infty(\Omega)$. Before presenting the global existence results, let us state the following lemmas and proposition, which will be useful in the global existence proof.

Lemma 5 *Let Φ and Ψ be two non-negative continuous functions on $[0,+\infty[$ with $\Phi(\eta)\to+\infty$ as $\eta\to+\infty$. Then there exists a positive constant A such that*

$$(1-\Phi(\eta))\Psi(\eta)\leq A.$$

proof. Because $\Phi(\eta)\to+\infty$ as $\eta\to+\infty$, it follows from the limit definition that, for all $\alpha>0$ there exists $\eta_0>0$ such that for all $\eta>\eta_0$, $\Phi(\eta)>\alpha$. Now, we choose $\alpha=1$ and multiplying by $\Psi(\eta)$, clearly, we get

$$(1-\Phi(\eta))\Psi(\eta)<0,\ \text{ for all } \eta>\eta_0.$$

On the other hand, if η is in the compact interval $[0,\eta_0]$, then the continuous function

$$\eta\mapsto(1-\Phi(\eta))\Psi(\eta)$$

is uniformly bounded for $\eta\in[0,\eta_0]$. Consequently bounded by a positive constant A for all $\eta>0$. This completes the proof of Lemma 5. □

Lemma 6 *Let $(u(t,.),v(t,.))$ be a solution of* (3.1)–(3.3) *on* $[0,T]$ *for some* $T<T_{\max}$*. Under the assumptions* (***A1***)*-*(***A2***)*, there exists a positive constant* δ_1 *and* $p>\frac{n}{2}$ *satisfies the inequality*

$$\int_\Omega \left(\alpha\beta pM(v+1)^{\beta-1}g(u,v)-\gamma f(u,v)\right)e^{\alpha p(v+1)^\beta}dx\leq \delta_1|\Omega|, \tag{3.15}$$

where α,β,γ *and* M *are positive constants satisfying* (3.13)*.*

proof. Using the conditions (3.13), we obtain

$$\frac{n}{2}<\frac{\gamma}{\alpha\beta\ell M}.$$

Then, we can choose p such that

$$\frac{n}{2}<p<\frac{\gamma}{\alpha\beta\ell M}.$$

From the assumption (**A2**), we get

$$\begin{aligned}&\left[\alpha\beta pM(v+1)^{\beta-1}g(u,v)-\gamma f(u,v)\right]e^{\alpha p(v+1)^\beta}\\ \leq\;&\left[\alpha\beta pM(v+1)^{\beta-1}\psi(v)-\gamma\right]e^{\alpha p(v+1)^\beta}f(u,v).\end{aligned}$$

Since $\alpha\beta pM\boldsymbol{\ell}<\gamma$ and $(v+1)^{\beta-1}\psi(v)$ goes to $\boldsymbol{\ell}$ as v goes to infinity, there exists $\tilde{v}>0$ such that for all $v\geq\tilde{v}$, we have

$$\left[\alpha\beta pM(v+1)^{\beta-1}\psi(v)-\gamma\right]e^{\alpha p(v+1)^\beta}f(u,v)\leq 0.$$

On the other hand, if v is in the compact interval $[0,\tilde{v}]$, then the continuous function

$$v\mapsto\left[\alpha\beta pM(v+1)^{\beta-1}\psi(v)-\gamma\right]e^{\alpha p(v+1)^\beta}f(u,v)$$

is uniformly bounded for $v\geq 0$ and $0\leq u<M$. In this way one can deduce that there exists a positive constant δ_1 such that

$$\int_\Omega\left[\alpha\beta pM(v+1)^{\beta-1}\psi(v)-\gamma\right]e^{\alpha p(v+1)^\beta}f(u,v)\,dx\leq\delta_1|\Omega|.$$

So that (3.15) immediately follows and the Lemma 6 is completely proven.

□

Proposition 7 *Suppose that* (***A1***) *and* (***A2***) *hold. For all positive classical solution (u,v) of the problem* (3.1)–(3.3) *on* $[0,T_{\max})$*, let* $L(t)$ *be the function defined by* (3.12) *with* α,β,γ,p *and* M *satisfying the restrictions* (3.13)*. Then there exist positive constants* δ_2 *and* $p>\frac{n}{2}$ *such that for all* $t\in[0,T],T<T_{\max}$ *the functional* L *satisfies the following inequality*

$$\frac{d}{dt}L(t)\leq-\sigma L(t)+B, \tag{3.16}$$

where $B=|\Omega|\left[(M-K)^{-\gamma-1}\delta_1+\sigma\delta_2(M-K)^{-\gamma}\right]$.

proof. Differentiating L with respect to t yields

$$\frac{d}{dt}L(t)=\int_{\Omega}\left[\gamma(M-u)^{-\gamma-1}\frac{\partial u}{\partial t}+p\alpha\beta(v+1)^{\beta-1}(M-u)^{-\gamma}\frac{\partial v}{\partial t}\right]e^{\alpha p(v+1)^{\beta}}dx,$$

on the other hand, it follows from (3.1)

$$\begin{cases}\frac{\partial u}{\partial t}=\Lambda-f(u,v)-\mu u+a\bigtriangleup u(t,x),\\ \frac{\partial v}{\partial t}=g(u,v)-\sigma v+b\bigtriangleup v(t,x),\end{cases}$$

substituting $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$ with its values, we get

$$\begin{aligned}\frac{d}{dt}L(t) &= \int_{\Omega}\left[\Lambda-f(u,v)-\mu u+a\bigtriangleup u(t,x)\right]\gamma(M-u)^{-\gamma-1}\exp^{\alpha p(v+1)^{\beta}}dx\\ &\quad+\int_{\Omega}\left[g(u,v)-\sigma v+b\bigtriangleup v)\right](M-u)^{-\gamma}\alpha\beta p(v+1)^{\beta-1}\exp^{\alpha p(v+1)^{\beta}}dx\end{aligned}$$

and we can write

$$\frac{d}{dt}L(t)=I+J_1+J_2+J_3, \tag{3.17}$$

where

$$I=\int_{\Omega}\left[a\gamma(M-u)^{-\gamma-1}e^{\alpha p(v+1)^{\beta}}\triangle u+bp\alpha\beta(M-u)^{-\gamma}(v+1)^{\beta-1}e^{\alpha p(v+1)^{\beta}}\triangle v\right],$$

$J_1=-\sigma L(t)$,

$$J_2=\sigma\int_{\Omega}\left[\frac{\gamma}{\sigma}\left(\frac{\Lambda-\mu u}{M-u}\right)+1-\alpha\beta p(v+1)^{\beta-1}v\right](M-u)^{-\gamma}e^{\alpha p(v+1)^{\beta}}dx,$$

and

$$J_3=\int_{\Omega}\left[\alpha\beta p(M-u)(v+1)^{\beta-1}g(u,v)-\gamma f(u,v)\right](M-u)^{-\gamma-1}e^{\alpha p(v+1)^{\beta}}dx.$$

By simple use of Green's formula on I, we obtain

$$\begin{aligned}I &= -a\gamma(\gamma+1)\int_{\Omega}(M-u)^{-\gamma-2}\exp^{\alpha p(v+1)^{\beta}}|\nabla u|^{2}dx\\ &\quad-a\gamma\int_{\Omega}(M-u)^{-\gamma-1}\alpha\beta p(v+1)^{\beta-1}\exp^{\alpha p(v+1)^{\beta}}\nabla u\nabla v dx\\ &\quad+a\gamma\int_{\partial\Omega}(M-u)^{-\gamma-1}\exp^{\alpha p(v+1)^{\beta}}\frac{\partial u}{\partial\eta}d\sigma\\ &\quad-b\alpha\beta p\int_{\Omega}(M-u)^{-\gamma-2}e^{\alpha p(v+1)^{\beta}}(M-u)^{2}\left[\beta-1+\alpha\beta p(v+1)^{\beta}(v+1)^{\beta-2}\right]|\nabla v|^{2}\\ &\quad-\int_{\Omega}bp\alpha\beta(v+1)^{\beta-1}\exp^{\alpha p(v+1)^{\beta}}\gamma(M-u)^{-\gamma-1}\nabla u\nabla v dx\\ &\quad+bp\alpha\beta\int_{\partial\Omega}(v+1)^{\beta-1}\gamma(M-u)^{-\gamma}\exp^{\alpha p(v+1)^{\beta}}\frac{\partial v}{\partial\eta}d\sigma.\end{aligned}$$

We may write I as follows

$$\begin{aligned} I &= \int_{\partial\Omega}\left[\alpha\gamma\frac{\partial u}{\partial\eta}+bp\alpha\beta\frac{\partial v}{\partial\eta}(M-u)(v+1)^{\beta-1}\right](M-u)^{-\gamma-1}\exp^{\alpha p(v+1)^{\beta}}d\sigma \\ &\quad -\alpha\beta p(a+b)\int_{\Omega}(M-u)^{-\gamma-1}(v+1)^{\beta-1}\exp^{\alpha p(v+1)^{\beta}}\nabla u\nabla v dx \\ &\quad -a\gamma\left(\gamma+1\right)\int_{\Omega}(M-u)^{-\gamma-2}\exp^{\alpha p(v+1)^{\beta}}\left|\nabla u\right|^{2}dx \\ &\quad -\alpha\beta pb\int_{\Omega}(M-u)^{2}\left[((\beta-1)+\alpha\beta p\left(v+1\right)^{\beta})(v+1)^{\beta-2}\left|\nabla v\right|^{2}\right](M-u)^{-\gamma-2}\exp^{\alpha p(v+1)^{\beta}}dx. \end{aligned}$$

We next infer from (3.3) that

$$\begin{aligned} I &= a\gamma\left(-\gamma-1\right)\int_{\Omega}\left(M-u\right)^{-\gamma-2}e^{\alpha p(v+1)^{\beta}}\left|\nabla u\right|^{2}dx- \\ &\quad \left(a+b\right)\gamma p\alpha\beta\int_{\Omega}\left(M-u\right)^{-\gamma-2}e^{\alpha p(v+1)^{\beta}}\left(M-u\right)\left(v+1\right)^{\beta-1}\nabla u\nabla v dx- \\ &\quad b\alpha\beta p\int_{\Omega}\left(M-u\right)^{-\gamma-2}e^{\alpha p(v+1)^{\beta}}\left(M-u\right)^{2}\left[\beta-1+\alpha\beta p\left(v+1\right)^{\beta}\left(v+1\right)^{\beta-2}\right]\left|\nabla v\right|^{2}, \end{aligned}$$

and using $\beta\geq 1$, leads to

$$\begin{aligned} I &\leq -a\gamma\left(\gamma+1\right)\int_{\Omega}(M-u)^{-\gamma-2}\exp^{\alpha p(v+1)^{\beta}}\left|\nabla u\right|^{2}dx \\ &\quad -\alpha\beta p(a+b)\int_{\Omega}(M-u)^{-\gamma-2}(M-u)(v+1)^{\beta-1}\nabla u\nabla v\exp^{\alpha p(v+1)^{\beta}}dx \\ &\quad -\alpha\beta pb\int_{\Omega}(M-u)^{2}\left[(\alpha\beta p\left(v+1\right)^{\beta})(v+1)^{\beta-2}\left|\nabla v\right|^{2}\right](M-u)^{-\gamma-2}\exp^{\alpha p(v+1)^{\beta}}dx, \end{aligned}$$

which can be written as,

$$I\leq-\int_{\Omega}Q\left(\nabla u,\nabla v\right)\left(M-u\right)^{-\gamma-2}e^{\alpha p(v+1)^{\beta}}dx,$$

this new form involves a quadratic form Q with respect to ∇u and ∇v,

$$\begin{aligned} Q\left(\nabla u,\nabla v\right) &= a\gamma\left(\gamma+1\right)\left|\nabla u\right|^{2}+\left(a+b\right)\gamma p\alpha\beta\left(M-u\right)\left(v+1\right)^{\beta-1}\nabla u\nabla v \\ &\quad +b\left[\alpha\beta p\left(M-u\right)\left(v+1\right)^{\beta-1}\right]^{2}\left|\nabla v\right|^{2}. \end{aligned}$$

Let D, the discriminant of Q given by

$$\begin{aligned} D &= \left[\left(a+b\right)\gamma p\alpha\beta\left(M-u\right)\left(v+1\right)^{\beta-1}\right]^{2} \\ &\quad -4ab\gamma\left(\gamma+1\right)\left[\alpha\beta p\left(M-u\right)\left(v+1\right)^{\beta-1}\right]^{2} \\ &= \gamma\left[\alpha p\beta\left(M-u\right)\left(v+1\right)^{\beta-1}\right]^{2}\left[\gamma\left(a-b\right)^{2}-4ab\right]. \end{aligned}$$

Subject to (3.13), the discriminant D of Q is negative, which implies that $Q(\nabla u,\nabla v)\geq 0$ and we deduce immediately

$$I\leq 0 \text{ on } \left(0,T_{\max}\right). \tag{3.18}$$

Since $J_1 \leq 0$ is evident, we move to estimate J_2 can be estimated by taking advantage of condition (3.12) and Lemma 5, we obtain

$$\begin{aligned} J_2 &\leq \sigma \int_\Omega \left[\frac{\gamma}{\sigma} \frac{\Lambda}{M} + 1 - \alpha\beta p \left(v+1\right)^{\beta-1} v \right] \left(M-u\right)^{-\gamma} e^{\alpha p (v+1)^\beta} dx \\ &\leq \sigma \left(M-K\right)^{-\gamma} \int_\Omega \Phi\left(v\right) e^{\alpha p (v+1)^\beta} dx \end{aligned}$$

which implies that

$$J_2 \leq \sigma \left(M-K\right)^{-\gamma} \delta_2 \left|\Omega\right|, \tag{3.19}$$

where δ_2 is a positive constant and

$$\Phi\left(v\right) = \frac{\gamma}{\sigma} \frac{\Lambda}{M} + 1 - \alpha\beta p \left(v+1\right)^{\beta-1} v.$$

Also, since $0 \leq u \leq K < M$, we observe that

$$J_3 \leq \left(M-K\right)^{-\gamma-1} \int_\Omega \left[\alpha\beta p M \left(v+1\right)^{\beta-1} g\left(u,v\right) - \gamma f\left(u,v\right) \right] e^{\alpha p (v+1)^\beta} dx.$$

From (3.15), it is evident that there exists $\delta_1 > 0$ such that

$$J_3 \leq \delta_1 \left(M-k\right)^{-\gamma-1} \left|\Omega\right|. \tag{3.20}$$

Finally, adding I, J_1, J_2, J_3 and by virtue of (3.17), (3.18), (3.19) and (3.20) it is seen that

$$\frac{d}{dt} L\left(t\right) \leq -\sigma L\left(t\right) + \sigma \left(M-k\right)^{-\gamma} \delta_2 \left|\Omega\right| + \delta_1 \left(M-k\right)^{-\gamma-1} \left|\Omega\right|.$$

This concludes the proof of Proposition 7. □

With these results in mind, our main finding concerning the global existence of solutions can be stated as in the following theorem.

Theorem 18 *Suppose that the assumptions (**A1**)-(**A3**) are satisfied. Then for any non-negative and bounded initial data* (u_0, v_0) *the classical solution* (u, v) *of the parabolic system* (3.1)–(3.3) *is global and belongs to* $\left(\mathbb{L}^\infty\left((0,+\infty);\mathbb{L}^\infty\left(\Omega\right)\right)\right)^2$.

proof. For the proof of this Theorem, we use the result established in Haraux Kirane [30]. First, we know that The functional L satisfies the differential inequality (3.16). A simple integration gives a uniform bound of the functional L over the interval $[0, T]$, where

$$L\left(t\right) \leq L\left(0\right) e^{-\sigma t} + \frac{B}{\sigma}, \text{ for all t} \in [0, T_{\max}). \tag{3.21}$$

Since $M^{-\gamma} \leq \left(M-\zeta\right)^{-\gamma}$, for any $\zeta \in [0, K]$, it follows that

$$\begin{aligned} L\left(t\right) &\geq \int_\Omega \left(M-u\right)^{-\gamma} \exp^{\alpha p (v+1)^\beta} dx, \\ &\geq M^{-\gamma} \int_\Omega \exp^{\alpha p (v+1)^\beta} dx \\ &\geq M^{-\gamma} \int_\Omega \exp^{\alpha p v^\beta} dx, \end{aligned}$$

on the other hand, by using this last and (**A3**) we get

$$\begin{aligned}
\|g(u,v)\|_p^p &= \int_\Omega |g(u,v)|^p\, dx \\
&\leq \left[\int_\Omega C\ \phi(u)\ \ \exp^{\alpha v^\beta}\right]^p dx \\
&\leq C^p \left[\max_{0\leq\zeta\leq K} \phi(\zeta)\right]^p \int_\Omega \exp^{\alpha p v^\beta}\, dx \\
&\leq C^p R^p M^\gamma L(t),
\end{aligned}$$

where

$$R = \max_{0\leq\zeta\leq K} \phi(\xi).$$

From (3.21) we obtain the following inequality

$$\|g(u,v)\|_p^p \leq C^p R^p M^\gamma \left(L(0)\, e^{-\sigma t} + \frac{B}{\sigma}\right).$$

Hence $g(u,v) - \sigma v$ is uniformly bounded in $\mathbb{L}^p(\Omega)$, for some $p > \frac{n}{2}$ (using the results of [30], S. Abdelmalek, M. Kirane and A. Youkana [6]). Finally by the regularizing effect of the heat equation [32], we conclude that

$$v \in \left(\mathbb{L}^\infty\left((0,+\infty);\mathbb{L}^\infty(\Omega)\right)\right).$$

which, together, with the fact that $\|u(t)\|_\infty \leq K$ for $t \in (0, T_{\max})$, implies that the solutions of the system (3.1)–(3.3) are global and uniformly bounded on $(0,+\infty)\times\Omega$. This completes the proof of Theorem 18. ☐

3.7 Asymptotic behaviour of solutions for epidemic system

In this section, we will study the asymptotic behaviour of global solutions obtained in Theorem 18 for the following diffusive epidemic model

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\Delta u = \Lambda - \lambda u\varphi(v) - \mu u, & \text{in } \mathbb{R}^+\times\Omega \\ \dfrac{\partial v}{\partial t} - b\Delta v = \lambda u\varphi(v) - \sigma v, & \text{in } \mathbb{R}^+\times\Omega \end{cases} \tag{3.22}$$

with the non negative continuous and bounded initial data

$$u(0,x) = u_0(x),\, v(0,x) = v_0(x) \quad \text{in } \Omega, \tag{3.23}$$

and Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \mathbb{R}^+\times\partial\Omega, \tag{3.24}$$

where u and v represent the non dimensional population densities of susceptible and infective, respectively. We can consider the system (3.22)–(3.24) as a model describing the spread of an infection disease (such as AIDS for instance) within a population assumed to be divided into the susceptible and infective classes (see [29]). We can consider the constants Λ, μ, σ and λ are such that

$$\mu > 0, \sigma > 0, \lambda > 0 \qquad \text{and} \qquad \Lambda \geq 0, \tag{3.25}$$

and the constants of diffusion a, b are positive, $a \neq b$. The non-linearity $\varphi(v)$ is a non-negative and continuously differentiable function on $[0, +\infty)$ satisfying

$$\varphi(\eta) \leq Ce^{\alpha\eta} \ \text{ and } \varphi(0) = 0, \tag{3.26}$$

for some positive constants C, α and we impose the following assumption

$$(\mathbf{A4}) \sup_{0<\eta\leq K_0} |\varphi'(\eta)| \leq \frac{\mu+\sigma}{\lambda\left(\theta\frac{\Lambda}{\mu}+\frac{\Lambda}{\sigma}\right)} \quad \text{with } \theta \geq \frac{(a+b)^2}{ab}.$$

First, thanks to Theorem 18 (the result established in section 3.6), in the case where $g(\xi, \eta) = f(\xi, \eta) = \lambda\xi\varphi(\eta)$ with $\psi(\eta) = 1$ and $\beta = 1$, under the conditions (3.13), (3.25)–(3.26) it is clear that the system (3.22)–(3.24) possesses a unique global and uniformly bounded solution (u, v) on $[0, +\infty) \times \Omega$.

Before stating the results, Let us expose some simple facts concluded from the result of the previous section. From Proposition 7 and by using classical method of the semi group and the fractional powers of operators (see [57]) we can find the positive constants K and K_0 for any t in $[0, +\infty)$

$$\|u(t,.)\|_\infty \leq K \quad \text{and } \|v(t,.)\|_\infty \leq K_0.$$

Our principal result of this section is stated in the following key proposition.

Proposition 8 *Let* (u, v) *be a solution of* (3.22)–(3.24), *suppose that assumption* (**A4**) *is fulfilled. Then the function*

$$F(t) = \int_\Omega \left[uv + \frac{\theta}{2}\left(u - \tfrac{\Lambda}{\mu}\right)^2 + \frac{1}{2}v^2 + \tfrac{\Lambda}{\sigma}v\right](x,t)\,dx, \tag{3.27}$$

satisfies the inequality

$$\frac{d}{dt}F(t) \leq -\mu\theta \int_\Omega \left(u - \tfrac{\Lambda}{\mu}\right)^2 dx - \sigma \int_\Omega v^2 dx \tag{3.28}$$
$$-\tfrac{b}{2}\int_\Omega |\nabla v|^2 dx - \tfrac{a\theta}{2}\int_\Omega |\nabla u|^2 dx \leq 0.$$

proof. First, we differentiate $F(t)$ with respect to time

$$\frac{d}{dt}F(t) = \int_\Omega \left(\tfrac{\partial u}{\partial t}v + u\tfrac{\partial v}{\partial t}\right)dx + \theta\int_\Omega \left(u - \tfrac{\Lambda}{\mu}\right)\tfrac{\partial u}{\partial t}dx + \int_\Omega v\tfrac{\partial v}{\partial t}dx + \tfrac{\Lambda}{\sigma}\int_\Omega \tfrac{\partial v}{\partial t}dx,$$

substituting $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ with its values and in the light of Green's formula and using Neumann boundary conditions, we get

$$\frac{d}{dt}F(t) = I_1 + I_2 + I_3,$$

where

$$
\begin{aligned}
I_1 &= \int_\Omega (d_1\Delta u+\Lambda-\lambda u\varphi(v)-\mu u)\,v+\int_\Omega u\,(d_2\Delta v+\lambda u\varphi(v)-\sigma v)\,dx\\
&= -(d_1+d_2)\int_\Omega \nabla u\nabla v dx+\Lambda\int_\Omega v dx-\lambda\int_\Omega uv\varphi(v)\,dx\\
&\quad +\lambda\int_\Omega u^2\varphi(v)\,dx-(\sigma+\mu)\int_\Omega uv dx,
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \theta\int_\Omega \left(u-\tfrac{\Lambda}{\mu}\right)(d_1\Delta u+\Lambda-\lambda u\varphi(v)-\mu u)\,dx\\
&= -d_1\theta\int_\Omega |\nabla u|^2 dx+\theta\int_\Omega\left(u-\tfrac{\Lambda}{\mu}\right)(\Lambda-\mu u-\lambda u\varphi(v))\,dx\\
&= -d_1\theta\int_\Omega |\nabla u|^2 dx-\mu\theta\int_\Omega\left(u-\tfrac{\Lambda}{\mu}\right)^2 dx-\lambda\theta\int_\Omega u^2\varphi(v)\,dx+\lambda\theta\frac{\Lambda}{\mu}\int_\Omega u\varphi(v)\,dx,
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \int_\Omega v\tfrac{\partial v}{\partial t}dx+\tfrac{\Lambda}{\sigma}\int_\Omega \tfrac{\partial v}{\partial t}dx\\
&= \int_\Omega v\,(b\Delta v+\lambda u\varphi(v)-\sigma v)\,dx\\
&\quad +\tfrac{\Lambda}{\sigma}\int_\Omega (b\Delta v+\lambda u\varphi(v)-\sigma v)\,dx\\
&= -b\int_\Omega |\nabla v|^2 dx+\lambda\int_\Omega uv\varphi(v)\,dx-\sigma\int_\Omega v^2dx\\
&\quad +\lambda\tfrac{\Lambda}{\sigma}\int_\Omega u\varphi(v)\,dx-\Lambda\int_\Omega v dx.
\end{aligned}
$$

Finally, adding I_1, I_2, I_3 it is seen that

$$
\begin{aligned}
\frac{d}{dt}F(t) &= -a\theta\int_\Omega |\nabla u|^2 dx-(a+b)\int_\Omega \nabla u\nabla v dx-b\int_\Omega |\nabla v|^2 dx\\
&\quad +\lambda(1-\theta)\int_\Omega u^2\varphi(v)\,dx-(\sigma+\mu)\int_\Omega uv dx-\mu\theta\int_\Omega\left(u-\tfrac{\Lambda}{\mu}\right)^2 dx\\
&\quad -\sigma\int_\Omega v^2dx+\lambda\left(\theta\frac{\Lambda}{\mu}+\tfrac{\Lambda}{\sigma}\right)\int_\Omega u\varphi(v)\,dx\\
&= -\frac{a}{2}\theta\int_\Omega |\nabla u|^2 dx-(a+b)\int_\Omega \nabla u\nabla v dx-\frac{b}{2}\int_\Omega |\nabla v|^2 dx\\
&\quad -\frac{a}{2}\theta\int_\Omega |\nabla u|^2 dx-\frac{b}{2}\int_\Omega |\nabla v|^2 dx\\
&\quad -(\sigma+\mu)\int_\Omega uv dx-\sigma\int_\Omega v^2dx+\lambda\left(\theta\frac{\Lambda}{\mu}+\tfrac{\Lambda}{\sigma}\right)\int_\Omega u\varphi(v)\,dx\\
&\quad -\lambda(\theta-1)\int_\Omega u^2\varphi(v)\,dx-\mu\theta\int_\Omega\left(u-\tfrac{\Lambda}{\mu}\right)^2 dx.
\end{aligned}
$$

Finally, we get

$$
\frac{d}{dt}F(t)=I+J+H,
$$

where

$$I=\frac{a}{2}\theta\int_{\Omega}\left|\nabla u\right|^{2}dx-(a+b)\int_{\Omega}\nabla u\nabla vdx-\frac{b}{2}\int_{\Omega}\left|\nabla v\right|^{2}dx,$$

$$J=-\frac{a}{2}\theta\int_{\Omega}\left|\nabla u\right|^{2}dx-\frac{b}{2}\int_{\Omega}\left|\nabla v\right|^{2}dx,$$

and

$$\begin{aligned}H &= \left(\lambda\theta\frac{\Lambda}{\mu}+\lambda\frac{\Lambda}{\sigma}\right)\int_{\Omega}u\varphi\left(v\right)dx-(\mu+\sigma)\int_{\Omega}uvdx\\ &\quad -\lambda\left(\theta-1\right)\int_{\Omega}u^{2}\varphi\left(v\right)dx-\mu\theta\int_{\Omega}\left(u-\frac{\Lambda}{\mu}\right)^{2}dx-\sigma\int_{\Omega}v^{2}dx.\end{aligned}$$

We observe that $I=-\int_{\Omega}Q\left(\nabla u,\nabla v\right)dx$ involves a quadratic form Q with respect to ∇u and ∇v

$$Q\left(\nabla u,\nabla v\right)=\frac{a}{2}\theta\left|\nabla u\right|^{2}+(a+b)\nabla u\nabla v+\frac{b}{2}\left|\nabla v\right|^{2},$$

is non-negative because θ, a and b satisfies $\theta\geq\frac{(a+b)^{2}}{ab}$ is chosen such that the discriminant D of Q given by

$$D=(a+b)^{2}-ab\theta\leq 0$$

is negative ($D\leq 0$), from which it is obviously deduced that

$$I\leq 0.$$

On the other hand, it follows from $\theta\geq\frac{(a+b)^{2}}{ab}>1$ that

$$\begin{aligned}H &\leq \left(\lambda\theta\tfrac{\Lambda}{\mu}+\lambda\tfrac{\Lambda}{\sigma}\right)\int_{\Omega}u\varphi\left(v\right)dx-(\mu+\sigma)\int_{\Omega}uvdx\\ &\quad -\mu\theta\int_{\Omega}\left(u-\tfrac{\Lambda}{\mu}\right)^{2}dx-\sigma\int_{\Omega}v^{2}dx.\end{aligned}$$

Adding I, J and H, leads to

$$\frac{d}{dt}F\left(t\right)\leq\left(\lambda\theta\tfrac{\Lambda}{\mu}+\lambda\tfrac{\Lambda}{\sigma}\right)\int_{\Omega}u\varphi\left(v\right)dx-(\mu+\sigma)\int_{\Omega}uvdx$$

$$-\frac{b}{2}\int_{\Omega}\left|\nabla v\right|^{2}dx-\frac{a}{2}\theta\int_{\Omega}\left|\nabla u\right|^{2}dx-\mu\theta\int_{\Omega}\left(u-\tfrac{\Lambda}{\mu}\right)^{2}dx-\sigma\int_{\Omega}v^{2}dx,$$

using $\varphi\left(0\right)=0$, the Mean Value Theorem 4, there exists $0<\xi<v$, such that

$$\varphi\left(v\right)-\varphi\left(0\right)=\varphi'\left(\zeta\right)v,\quad\text{for all}\quad v\in[0,K_{2}]$$

from this we can write

$$\frac{d}{dt}F\left(t\right)\leq\int_{\Omega}\left[\left(\lambda\theta\tfrac{\Lambda}{\mu}+\lambda\tfrac{\Lambda}{\sigma}\right)\varphi'\left(\zeta\right)-(\mu+\sigma)\right]uvdx$$

$$-\frac{b}{2}\int_{\Omega}\left|\nabla v\right|^{2}dx-\frac{a}{2}\theta\int_{\Omega}\left|\nabla u\right|^{2}dx-\mu\theta\int_{\Omega}\left(u-\tfrac{\Lambda}{\mu}\right)^{2}dx-\sigma\int_{\Omega}v^{2}dx,$$

and consequently we deduce that

$$\begin{aligned}\frac{d}{dt}F(t) &\leq \int_\Omega \left[\left(\lambda\theta\tfrac{\Lambda}{\mu}+\lambda\tfrac{\Lambda}{\sigma}\right)\sup_{0<\zeta\leq K_0}\left|\varphi'(\zeta)\right|-(\mu+\sigma)\right]uvdx\\ &\quad -\mu\theta\int_\Omega\left(u-\tfrac{\Lambda}{\mu}\right)^2dx-\sigma\int_\Omega v^2dx\\ &\quad -\frac{a}{2}\theta\int_\Omega|\nabla u|^2dx-\frac{b}{2}\int_\Omega|\nabla v|^2dx.\end{aligned}$$

Inequality (3.28) follows directly from (**A4**). □

Before we present the main result concerning the asymptotic behavior of the solutions, the following lemma is important. The complete proof of the lemma can be found in [75].

Lemma 7 *(Barbalat's Lemma) Let $c>0$ be constants, $\psi(t)\geq 0$, and $\lim\limits_{t\to\infty}\int_0^t h(t)\,dt$ exists and is finite assume that $\varphi\in\mathbb{C}^1([0,\infty))$, φ is bounded from below and satisfies*

$$\varphi'(t)\leq -c\psi(t)+h(t)\quad in\ [0,\infty). \tag{3.29}$$

If $\psi\in\mathbb{C}^1([0,\infty))$ and $\psi'(t)\leq k$ in $[0,\infty)$ for some constant $k>0$. Then $\lim\limits_{t\to\infty}\psi(t)=0$.

proof. On the contrary, assume that our conclusion $\lim\limits_{t\to\infty}\psi(t)=0$ is not true. Then there exist $\zeta>0$ and $t_i\to\infty$ as $i\to\infty$ such that $\psi(t_i)>\zeta$ for all $i\in\mathbb{N}$. Choose $0<\sigma<\min\left(\frac{\zeta}{2k},t_i-t_{i-1}\right)$ and because $\psi'(t)\leq k$, we have by using the Mean Value Theorem

$$\psi(t_i)-\psi(t)\leq k(t_i-t),t\in[t_i-\sigma,t_i],$$

then

$$\psi(t)\geq\psi(t_i)-k(t_i-t)>\zeta-k\sigma>\frac{\zeta}{2}.$$

Integrating (3.29) over $[0,t]$ for all $t\geq t_n$ and using the condition $\psi(t)\geq 0$ we get

$$\begin{aligned}\varphi(t) &\leq \varphi(0)-c\int_0^t\psi(s)\,ds+\int_0^t h(s)\,ds\\ &\leq \varphi(0)-c\sum_{i=1}^n\int_{t_i-\sigma}^{t_i}\psi(s)\,ds+\int_0^t h(s)\,ds\\ &\leq \varphi(0)-\frac{1}{2}n\zeta\sigma c+\int_0^t h(s)\,ds,\end{aligned}$$

because $\lim\limits_{t\to\infty}\int_0^t h(t)\,dt$ exists and is finite, we have $\lim\limits_{t\to\infty}\varphi(t)=-\infty$, a contradiction. The proof of the lemma can be get in [75, Lemma 1.1, page 2]. □

Theorem 19 *Let (u,v) the global and bounded solution to the system* (3.22)–(3.24) *in* $[0,+\infty)$ *and suppose that the assumptions* (**A4**) *holds. Then as $t\to+\infty$*

$$\left\|u(t,.)-\tfrac{\Lambda}{\mu}\right\|_\infty\to 0 \tag{3.30}$$

and

$$\|v(t,.)\|_\infty\to 0. \tag{3.31}$$

proof. Integrating from 0 to t the inequality (3.28), we have

$$\begin{array}{c} F(t)+\frac{b}{2}\int_0^t\int_\Omega(\nabla v)^2\,dxd\tau+\frac{a}{2}\theta\int_0^t\int_\Omega(\nabla u)^2\,dxd\tau \\ +\mu\theta\int_0^t\int_\Omega\left(u-\frac{\Lambda}{\mu}\right)^2dxd\tau+\sigma\int_0^t\int_\Omega v^2dxd\tau\le F(0), \end{array}$$

from this and (3.27), if we applying Lemma 7 with

$$t\to\psi(t)=\int_\Omega(u-\frac{\Lambda}{\mu})^2\partial x,\quad C=\mu\theta,\quad t\to\varphi(t)=F(t)$$

and

$$h(t)=-\left[\frac{b}{2}\int_\Omega|\nabla v|^2\,dx+\frac{a}{2}\theta\int_\Omega|\nabla u|^2\,dx+\sigma\int_\Omega v^2dx\right]$$

where $C>0$, $\psi(t)\ge 0$, $F\in\mathbb{C}^1([0,\infty))$ and

$$\begin{array}{rcl} F(t) & \le & F(0) \\ & = & \int_\Omega\left[uv(0)+\frac{\theta}{2}\left(u(0,x)-\frac{\Lambda}{\mu}\right)^2+\frac{1}{2}(v(0,x))^2+\frac{\Lambda}{\sigma}v(0,x)\right]dx, \end{array}$$

we deduces from (3.27) that

$$\left(u-\frac{\Lambda}{\mu}\right)\in\mathbb{L}^2(\Omega),\int_0^{+\infty}\int_\Omega\left[\nabla(u-\frac{\Lambda}{\mu})\right]^2\partial x\partial\tau<+\infty,$$

$$\int_0^{+\infty}\int_\Omega(u-\frac{\Lambda}{\mu})^2\partial x\partial\tau<+\infty$$

and

$$v\in\mathbb{L}^2(\Omega),\int_0^{+\infty}\int_\Omega(\nabla v)^2\,\partial x\partial\tau<+\infty,\int_0^{+\infty}\int_\Omega v^2\partial x\partial\tau<+\infty.$$

By virtue of Lemma 7, we get

$$\lim_{t\to+\infty}\|u(t,.)-\tfrac{\Lambda}{\mu}\|_2=\lim_{t\to+\infty}\|v(t,.)\|_2\to 0. \tag{3.32}$$

On the other hand from Haraux Kirane [30] because the solution (u,v) is uniformly bounded on $[0,+\infty)\times\Omega$, we can show that this solutions are uniformly bounded in $\mathbb{C}^1\left(\overline{\Omega}\right)\times\mathbb{C}^1\left(\overline{\Omega}\right)$. As a consequence of Arzela-Ascoli's theorem we establishes that

$\{v(t,.),\ t\ge 0\}$ and $\left\{u(t,.)-\frac{\Lambda}{\mu},\ t\ge 0\right\}$ the orbits of the dynamical system (3.22)–(3.24) are relatively compact (precompact subset) in $\mathbb{C}\left(\overline{\Omega}\right)\times\mathbb{C}\left(\overline{\Omega}\right)$, we deduce that there exists a subsequence $\{t_n\}_{n\ge 0}, t_n\to\infty$, such that

$$\lim_{n\to+\infty}v(t_n,x)=\widetilde{v}(x)\qquad\text{in }\mathbb{C}\left(\overline{\Omega}\right) \tag{3.33}$$

and

$$\lim_{n\to+\infty}\left(u(t_n,x)-\tfrac{\Lambda}{\mu}\right)=\widetilde{u}(x)\qquad\text{in }\mathbb{C}\left(\overline{\Omega}\right), \tag{3.34}$$

and we have

$$\begin{aligned} 0\le\int_\Omega|\widetilde{v}(x)|^2\,dx &= \int_\Omega|\widetilde{v}(x)-v(t_n,x)+v(t_n,x)|^2\,dx \\ &= \int_\Omega|\widetilde{v}(x)-v(t_n,x)|^2\,dx+\int_\Omega|v(t_n,x)|^2\,dx \qquad (3.35) \\ &\quad +2\int_\Omega|(\widetilde{v}(x)-v(t_n,x))\,v(t_n,x)|\,dx, \end{aligned}$$

we apply Young's inequality, it holds that

$$\int_{\Omega} \left|\left(\widetilde{v}(x) - v(t_n,x)\right) v(t_n,x)\right| dx \le \frac{1}{2}\int_{\Omega}\left(\widetilde{v}(x) - v(t_n,x)\right)^2 + \frac{1}{2}\int_{\Omega}\left(v(t_n,x)\right)^2. \tag{3.36}$$

Combining (3.35) with (3.36) we reach

$$\begin{aligned}
\int_{\Omega}\left|\widetilde{v}(x)\right|^2 dx &\le 2\left[\int_{\Omega}\left|\widetilde{v}(x) - v(t_n,x)\right|^2 dx + \int_{\Omega}\left|v(t_n,x)\right|^2 dx\right] \\
&\le 2\left[\left\|\widetilde{v}(.) - v(t_n,.)\right\|_{\infty}^2 |\Omega| + \int_{\Omega}\left|v(t_n,x)\right|^2 dx\right] \\
&= 2\left[\left\|\widetilde{v}(.) - v(t_n,.)\right\|_{\infty}^2 |\Omega| + \left\|v(t_n,.)\right\|_2\right],
\end{aligned}$$

when t goes to infinity and using (3.32) and (3.33), we get

$$\int_{\Omega}\left|\widetilde{v}(x)\right|^2 dx = \left\|\widetilde{v}(x)\right\|_2 = 0,$$

which implies

$$\widetilde{v}(x) = 0 \quad \text{for all} \quad x \in \Omega$$

using (3.33)

$$\lim_{n\to+\infty} v(t_n,x) = 0 \qquad\qquad \text{in } \mathbb{C}\left(\overline{\Omega}\right)$$

or

$$\lim_{t\to+\infty}\left\|v(t,.)\right\|_{\infty} = 0,$$

the limits (3.31) are true. Similarly, we have

$$\begin{aligned}
0 &\le \int_{\Omega}\left|\widetilde{u}(x)\right|^2 dx = \int_{\Omega}\left|\widetilde{u}(x) - \left(u(t_n,x) - \tfrac{\Lambda}{\mu}\right) + \left(u(t_n,x) - \tfrac{\Lambda}{\mu}\right)\right|^2 dx \\
&= \int_{\Omega}\left|\widetilde{u}(x) - \left(u(t_n,x) - \tfrac{\Lambda}{\mu}\right)\right|^2 dx + \int_{\Omega}\left|\left(u(t_n,x) - \tfrac{\Lambda}{\mu}\right)\right|^2 dx \\
&\quad + 2\int_{\Omega}\left|\widetilde{u}(x) - \left(u(t_n,x) - \tfrac{\Lambda}{\mu}\right)\left(u(t_n,x) - \tfrac{\Lambda}{\mu}\right)\right| dx \\
&\le 2\left[\int_{\Omega}\left|\widetilde{u}(x) - \left(u(t_n,x) - \tfrac{\Lambda}{\mu}\right)\right|^2 dx + \int_{\Omega}\left|\left(u(t_n,x) - \tfrac{\Lambda}{\mu}\right)\right|^2 dx\right] \\
&\le 2\left[\left\|\widetilde{u}(.) - \left(u(t_n,x) - \tfrac{\Lambda}{\mu}\right)\right\|_{\infty}^2 |\Omega| + \int_{\Omega}\left|\left(u(t_n,x) - \tfrac{\Lambda}{\mu}\right)\right|^2 dx\right] \\
&= 2\left[\left\|\widetilde{u}(.) - \left(u(t_n,x) - \tfrac{\Lambda}{\mu}\right)\right\|_{\infty}^2 |\Omega| + \left\|u(t_n,x) - \tfrac{\Lambda}{\mu}\right\|_2\right],
\end{aligned}$$

when t goes to infinity and using (3.32) and (3.34), we get

$$\int_{\Omega}\left|\widetilde{u}(x)\right|^2 dx = \left\|\widetilde{u}(x)\right\|_2 = 0,$$

wich implie

$$\widetilde{u}(x) = 0 \text{ for all } x \in \Omega,$$

using (3.33)

$$\lim_{n \to +\infty} \left(u(t_n, x) - \tfrac{\Lambda}{\mu} \right) = 0 \qquad \text{in } \mathbb{C}\left(\overline{\Omega}\right),$$

or

$$\lim_{t \to +\infty} \left\| u(t_n, x) - \tfrac{\Lambda}{\mu} \right\|_{\infty} = 0.$$

Which proves the Theorem 19. □

3.8 Numerical Examples

In this section, we would like to show by numerical methods that the results of this paper as reported in Sections 3.6 and 3.7 are both sound and practical. We consider the following two examples.

3.8.1 Example 1

Let's consider the following system

$$\begin{cases} f(\xi, \eta) = \xi (1 + \eta^m) e^{\eta^\beta}, \\ g(\xi, \eta) = \xi e^{\eta^\beta}, \; m > \beta - 1 \geq 0, \\ \psi(\eta) = \frac{1}{1+\eta^m}, \end{cases} \tag{3.37}$$

with $\ell = 0$. We will show that the result of the theorem 18 concerning the global existence of the solution is correct. With

$$\begin{array}{ll} m = 1 & \beta = 1 \\ \Lambda = 0.2, & \mu = 1.5, \\ \sigma = 1, & a = 0.17, \\ b = 0.2, & \end{array} \tag{3.38}$$

we get

$$\begin{cases} \dfrac{\partial u}{\partial t} - 0.17\Delta u = 0.2 - u(1+v)e^v - 1.5u, \\ \dfrac{\partial v}{\partial t} - 0.2\Delta v = ue^v - v. \end{cases} \tag{3.39}$$

It is easy to verify that conditions (**A1**)-(**A3**) are satisfied in this example:

1. For (**A1**) we have

$$f(0, \eta) = g(0, \eta) = 0,$$

and

$$g(\xi, 0) = \xi \geq 0.$$

2. For $(\mathbf{A2})$ we have

$$\begin{aligned} g(\xi,\eta) &= \xi e^{\eta} = \frac{1}{1+\eta}\xi(1+\eta)e^{\eta} \\ &\leq \psi(\eta) f(\xi,\eta), \end{aligned}$$

where $\beta = 1 \geq 1$ and $\lim\limits_{\eta\to\infty} \eta^{\beta-1}\frac{1}{1+\eta} = \ell = 0.$

3. For $(\mathbf{A3})$ we have $g(\xi,\eta) = \xi e^{\eta} \leq \phi(\xi) e^{\eta}$.

We assume initial conditions

$$\begin{cases} u(x,0) = 0.3\times(1+0.7\sin(0.15x)), \\ v(x,0) = 1.4\times(1+0.4\cos(0.25x)), \end{cases} \tag{3.40}$$

and zero Neumann boundaries. From (3.4), since $\ell = 0$, it enough that $\|u_0\|_{\infty} < M$. So, we take $M = 0.5$. Theorem 18 asserts that solutions are global in time, which is evident from Figure 3.1, which maps the solutions in the one–dimensional state where we choose $t \in [0,5]$ and $x \in [0,80]$. The solutions are obtained using the implicit finite difference method with $t = 0.01$ and $\triangle t = 2$. The solutions seem to converge towards a stable fixed state.

3.8.2 Example 2

In order to show that the results presented in Section 3.7 regarding the asymptotic behavior of solution hold. In this example we consider the system

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\Delta u = \Lambda - \lambda u\varphi(v) - \mu u, \\ \dfrac{\partial v}{\partial t} - b\Delta v = \lambda u\varphi(v) - \sigma v, \end{cases} \tag{3.41}$$

with the following nonlinearity

$$\varphi(v) = \frac{ve^{v}}{100}. \tag{3.42}$$

It is easy to know that, the derivative of $\varphi(v)$ is given by

$$\varphi'(v) = \frac{1}{100}e^{v}(v+1). \tag{3.43}$$

Conditions **(A1)–(A3)** are easily verified. Also, consider the set of parameters

$$\begin{array}{ll} \Lambda = 0.2, & \lambda = 0.7, \\ \mu = 1.5, & \sigma = 1, \\ a = 0.17, & b = 0.2, \end{array} \tag{3.44}$$

initial conditions

$$\begin{cases} u(x,0) = 2\times(1+0.5\sin(0.2x)), \\ v(x,0) = 1\times(1+0.5\cos(0.2x)), \end{cases} \tag{3.45}$$

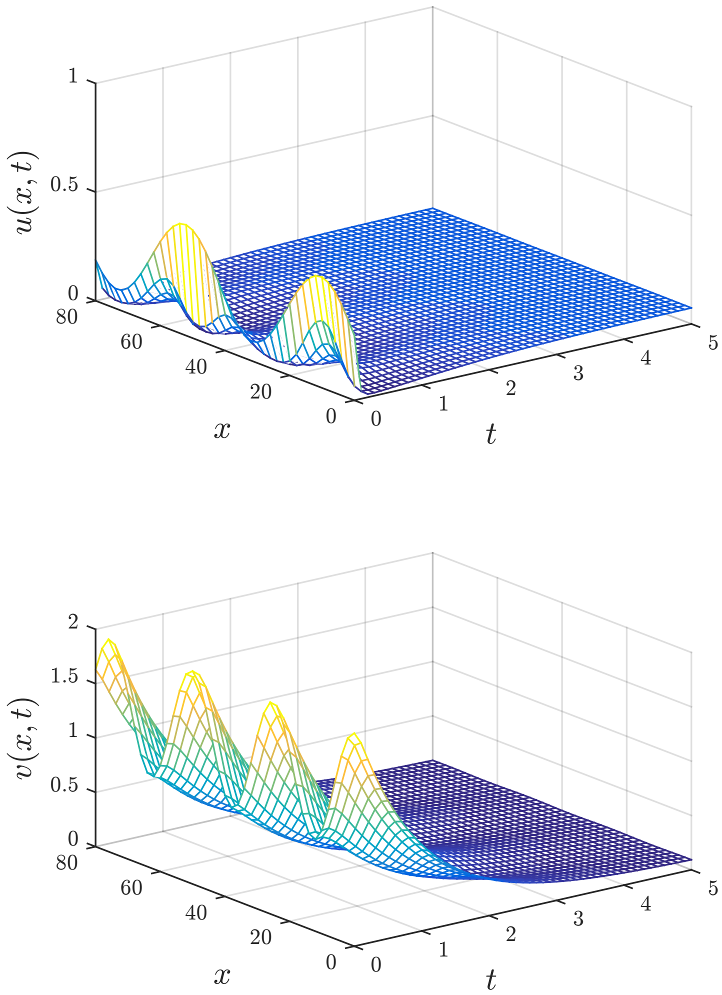


Figure 3.1: Solutions of system (3.37) with parameters (3.38), initial conditions (3.40), and zero Neumann boundaries in the one–dimensional case.

and zero Neumann bounaries. To satisfy (3.4), we choose a value M that is greater than the supremum of $u(x,0)$. We can easily show that $M = 3$ is sufficient. We may choose the parameters

$$\beta = 1,\ \ell = 1,\ \text{and } n = 1, \tag{3.46}$$

which implies to

$$M = 3 < \frac{8ab}{\beta \ell n (a-b)^2} = 302.22. \tag{3.47}$$

For (A4), we can show that it is verified by taking

$$\theta = 4.1 \geq \frac{(a+b)^2}{ab} = 4.0265, \tag{3.48}$$

and satisfying that

$$\sup_{0<\eta\leq M} |\varphi'(\eta)| = 0.8034 \leq \frac{\mu+\sigma}{\lambda\left(\theta\frac{\Lambda}{\mu} + \frac{\Lambda}{\sigma}\right)} = 4.1209. \tag{3.49}$$

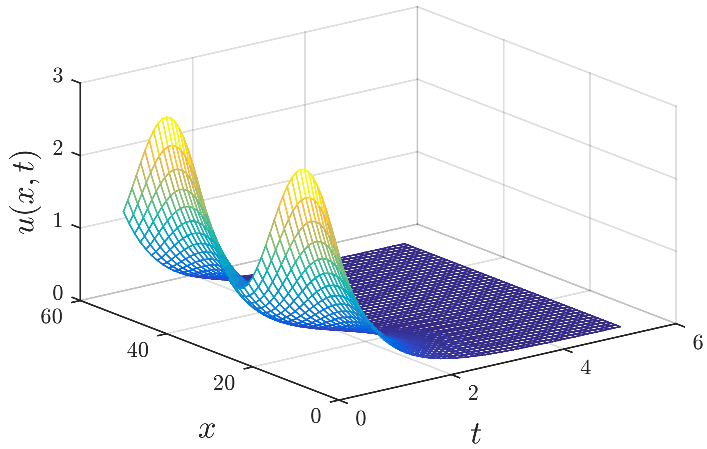
Figure 3.2 shows the spatio–temporal evolution of the solutions. Theoretically, the steady state is given by

$$(u^*, v^*) = \left(\tfrac{\Lambda}{\mu}, 0\right) = \left(\frac{2}{15}, 0\right). \tag{3.50}$$

Solving the system numerically by means of the implicit finite difference method for $t \in [0,5]$ and $x \in [0,50]$ with $\Delta t = 0.01$ and $\Delta t = 1$ yields the solution

$$(u(10,5), v(10,5)) \approx (0.1347, 0.005405) \tag{3.51}$$

at $x = 10$ and $t = 5$. Figure 3.3 shows the time evolution of the solutions at $x = 10$. Clearly, the solutions do converge towards the constant steady state (u^*, v^*).



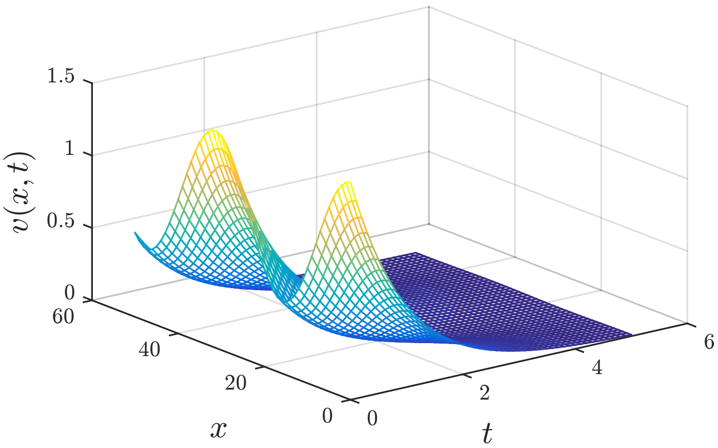


Figure 3.2: Solutions of system (3.41) with parameters (3.44), initial conditions (3.45), and zero Neumann boundaries in the one–dimensional case.

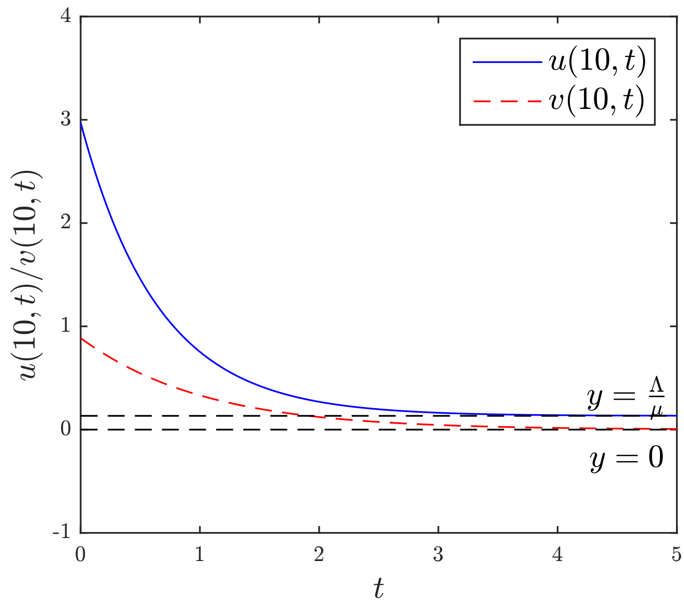


Figure 3.3: Solutions of system (3.41) with parameters (3.44), initial conditions (3.45), and zero Neumann boundaries in the one–dimensional case taken at the single point in space $x = 10$.

Chapter 4

Global and local asymptotic stability of an epidemic reaction-diffusion model with a nonlinear incidence

In this chapter, we consider the following reaction-diffusion epidemic phenomena proposed in Djebara, Abdelmalek and Bendoukha [23], with the nonlinear incidence $u\varphi(v)$, which is an extended version of the SI epidemic model.

In this chapter, we detail the work of Djebara, Douaifia, Abdelmalek and Bendoukha [24] [1]

4.1 System model

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u := \Lambda - \mu u - \lambda u \varphi(v) = F(u,v) & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial v}{\partial t} - d_2 \Delta v := -\sigma v + \lambda u \varphi(v) = G(u,v) & \text{in } \mathbb{R}^+ \times \Omega. \end{cases} \tag{4.1}$$

Throughout this paper, we will use the following general notations, the notation Δ denotes the laplacian operator on Ω and ∇ denotes the gradient operator, where Ω is an open bounded subset of $\mathbb{R}^n$ with piecewise smooth boundary $\partial\Omega$. The constant parameters $d_1, d_2 > 0$ are the diffusion coefficient. We assume the initial conditions

$$u_0(x) = u(x,0), \quad v_0(x) = v(x,0) \qquad \text{in } \Omega, \tag{4.2}$$

where $u_0(x), v_0(x) \in C^2(\Omega) \cap C(\overline{\Omega})$, and homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega, \tag{4.3}$$

with ν being the unit outer normal to $\partial\Omega$. We will also assume that the initial conditions $u_0(x), v_0(x) \in \mathbb{R}_{\geq 0}$. We can consider the system (4.1)–(4.3) as a model describing the transmission of communicable disease through individuals for example HIV/AIDS for instance and the population assumed to be divided into the susceptible and infective

[1] L. Djebara, R. Douaifia, S. Abdelmalek, S. Bendoukha, *Global and local asymptotic stability of an epidemic reaction-diffusion model with a nonlinear incidence*, arXiv:2011.05408v1 [math.AP] 10 Nov (2020).

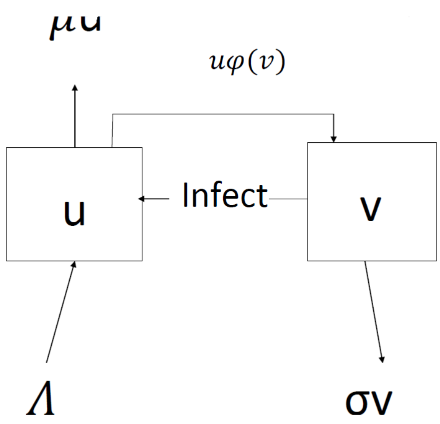


Figure 4.1: A transfer diagram of the proposed system model

classes (see Hamaya [29]), where $u(x,t)$ and $v(x,t)$ denote the non dimensional population densities of susceptible and infective individuals at location x and time t, respectively. The parameter $\Lambda > 0$ denote the growth of different categories of susceptibles whether through birth or migration (the recruitment rate of the population), μ is the natural death rate and λ is the rate at which the susceptible become infectious, σ is the rate at which invectives recover from the disease. We will assume that $\mu > 0$, $\sigma > 0$, and $\lambda > 0$.

The functions $\varphi(v)$ is the incidence function allowing for a nonlinear relation between the two classes of individuals. The incidence function φ is assumed to be a continuously differentiable function on $\mathbb{R}^+$, and to satisfy the following criteria:

$$\varphi(0) = 0, \tag{4.4}$$

and

$$0 < v\varphi'(v) \leq \varphi(v) \quad \text{for all } v > 0. \tag{4.5}$$

4.2 Objectives of This Study

In the present work, we generalised the model of Li et al. [52] to a epidemic model with spatial diffusion (diffusive epidemic model) and we deal with the same model as in [23], where the global existence of solutions to problem (4.1)–(4.3) was established and in this work we will continue with examine the existence of equilibria and the asymptotic stability conditions.

Our paper is organized as follows. In Section 4.1 we will firstly present the proposed system model and identify its main characteristics and the conditions on the parameters. Section 4.4 will define the basic reproductive number R_0 and we examine the existence of equilibrium, local asymptotic stability and instability of the disease-free equilibrium and the endemic equilibrium of the model. In Section 4.5 we will prove that the two steady states of the model the disease-free equilibrium and the endemic equilibrium are globally asymptotically stable using an appropriate Lyapunov functional. Finally, in Section 4.6 we will present some numerical examples.

The aim of this chapter is to study the dynamics of a Reaction-diffusion SI (susceptible-infectious) epidemic model with a nonlinear incidence rate describing the transmission of communicable disease through individuals. We prove that the proposed model has two steady states under one condition. By analysing the eigenvalues and an appropriately constructed Lyapunov functional we establish both locally and globally in the ODE and PDE cases of the non negative constant steady states if the basic reproduction number is greater than unit and of the disease free equilibrium if the basic reproduction number is smaller than or equal to the unit in ODE case. By applying an appropriately constructed Lyapunov functional we give the condition of the global stability in PDE case. Finally, we present some numerical examples illustrating and confirming the analytical results obtained below.

4.3 Previous results for our second work

Infectious diseases are the leading cause of death of living organisms. Historically, a number of authors have moved to research in epidemiology. The goal is to improve and develop treatment with planning and predictions that help stop the spread of the disease and thus reduce mortality.

Hethcote [33] study the following simple model given with the bi-linear and standard incidences $u\varphi(v) = \lambda uv,\ d_1 = d_2 = 0$ a simple model given by

$$\begin{cases} S_t = -\lambda SI + \delta - \delta S, \\ I_t = \lambda SI - (\gamma + \delta) I,\ t \in \mathbb{R}^+, \\ R(t) = 1 - (S(t) + I(t)), \\ S(0) = S_0 > 0,\ I(0) = I_0 > 0, R(0) \geq 0. \end{cases}$$

First, he proved the existence of the disease-free equilibrium $E_0 = (1, 0)$ and if $R_0 \leq 1$ and proved that the region

$$D = \left\{(S, 0) \in \mathbb{R}^+ : 1 \geq S \geq 0\right\}$$

is an asymptotic stability regions for the equilibrium points of model depend on the basic reproduction number, where if $R_0 > 1$, the region D is an asymptotic stability region for $E^* = \left(\frac{1}{R_0}, \delta\frac{(R_0-1)}{\lambda}\right)$ where $R_0 = \frac{\lambda}{\delta+\gamma}$, refer also to, e.g, [7, 44, 11] and [54] as for this case of epidemic models in other situations. The nonlinear incidences of the forms $u\varphi(v)$ was considered in ([14, 8, 52, 35, 10]) but with absence of spatial diffusion. In 1978, [14] Capasso and Serio introduced a saturated incidence $S\varphi(I)$ into epidemic models to study the cholera epidemic spread in Bari in 1973, where they studied the positivity, global existence, uniqueness of the solution and also the asymptotic stability of the following epidemiological model

$$\begin{cases} S_t = -\mu S - S\varphi(I), \\ I_t = S\varphi(I) - \sigma I, \quad \text{in} \quad \mathbb{R}^+, \end{cases}$$

initial conditions of the following type:

$$S(0) = S_0 > 0,\ I(0) = I_0 > 0,\ \text{such that}\ \ S_0 + I_0 = N,$$

where they introduced a saturated incidence $S\varphi(I)$ into epidemic models to study the cholera epidemic spread in Bari, where they studied the positivity, global existence, and uniqueness of the solution of the system and after the asymptotic stability analysis of the equilibrium points, it is obtained an extension of the threshold theory. The nonlinear $\varphi(I)$ is a positive continuous function on $[0, +\infty)$, which satisfies the following conditions:

(i) $\varphi(0)=0, \varphi(I)>0$, for all $v>0$;

(ii) $\exists c \in \mathbb{R}_{+}-\{0\}: \varphi(I) \leq c$;

(iii) $\varphi'(v)$ the derivative of φ, exists and is bounded on any compact interval $[0,+\infty)$, with $\varphi'(0)>0$ and $\varphi(I) \leq \varphi'(0) I$ for all $I>0$.

The unique continuously differentiable solution $Z(t)=(S(t), I(t))$ exists for the system (4.9)–(4.10) at any time $t \in \mathbb{R}_{+}$, and its trajectory is always contained in the triangle

$$\mathcal{D}=\left\{(S, I) \in \mathbb{R}^{+}: I, S \geq 0 \text{ and } \ S+I \leq N\right\}.$$

The only equilibrium point is $(0,0)$ and this point is always asymptotically stable. The authors also suggest the following numerical example in detail $\varphi(I)=\frac{kI}{1+\left(\frac{I}{\alpha}\right)}, k>0, \alpha>0$.

In 1981, Webb [76] established the existence of solutions $(S(t,.), I(t,.), R(t,.))$ to the following SIR models for the diffusive epidemic model in one dimensional spatial region

$$\begin{cases} \dfrac{dS}{dt}-\frac{\partial^2 S}{\partial x^2}=-\lambda SI, \\ \dfrac{dI}{dt}-\frac{\partial^2 I}{\partial x^2}=\lambda SI-\lambda I, \\ \dfrac{dR}{dt}-\frac{\partial^2 R}{\partial x^2}=\lambda I\ , \ \text{ in } [-L, L] \quad \times \ \mathbb{R}^{+}, \end{cases}$$

$$S_0(x)=S(x, 0), I_0(x)=I(x, 0), R_0(x)=R(x, 0) \text{ in } [-L, L],$$

with the homogeneous Neuman boundary conditions

$$S_x(L_{\pm}, t)=I_x(L_{\pm}, t)=R_x(L_{\pm}, t)=0\ ,\ t \geq 0,$$

with $\Lambda=\mu=0$, $d_1=d_2$ and $\varphi(v)=\lambda v$, in one dimensional $(n=1)$ spatial interval $[-L, L]$ region. and analyzed their behavior as time goes to infinity where $(S(t,.), I(t,.), R(t,.)) \rightarrow (S_{\infty}, 0, R_{\infty})$ with S_{∞}, R_{∞} are positive constants functions on $[-L, L]$, they used in their proof the dynamic systems and functional analysis tools, specifically on the theory of linear and nonlinear semi-linear groups in Banach distances and Lyapunov stability techniques for dynamic systems in metric spaces. Also in the same case but with Λ, $\mu>0$ in section three of [41] Kim et al. introduced the same bird system with homogeneous Neumann boundary conditions

$$\begin{cases} S_t-\alpha \Delta S=\Lambda-\beta SI-\mu S, \\ I_t-\alpha \Delta I=\beta SI-(\mu+\delta) I \qquad \text{in } \mathbb{R}^{+} \times \Omega, \end{cases} \tag{4.6}$$

and they proved that this system always has the disease free equilibrium. They investigated the uniform bounds of the solutions, also local and global stabilities of the equilibrium's are discussed, for prove the global stability of the bird system they considered the following Lyapunov function $V(t)=\int_{\Omega}\left[\frac{1}{2}\left(S-\frac{\Lambda}{\mu}\right)^2+\frac{\Lambda}{\mu} I\right] dx$. The authors also showed that the global stability of the model depends on the basic reproductive number. The study of Chinviriyasit [19] have introduced the same results as [41]. In the case when

$n \geq 1$ and $d_i > 0$, $\Lambda = \mu = 0$, with the bi-linear incidence $u\varphi(v) = \lambda uv$, Fitzgibbon et al. [25] in 1995, proposed a criss-cross infection model to describe the spread of FIV (Feline immunodeficiency virus). Consider four state variables u, w, v, z representing population densities of susceptible males, infective males, susceptible females, and infective females, respectively, the criss-cross epidemic model without age structure

$$\begin{cases} u_t - d_1\Delta u = -k_1 uw - k_2 uz, & \text{in } \mathbb{R}^+ \times \Omega, \\ v_t - d_2\Delta v = k_1 uw + k_2 uz - \lambda_1 w, & \text{in } \mathbb{R}^+ \times \Omega, \\ w_t - d_3\Delta w = -k_3 vw - k_4 vz, & \text{in } \mathbb{R}^+ \times \Omega, \\ z_t - d_4\Delta z = k_3 vw + k_4 vz - \lambda_2 z \ , & \Omega \subset \mathbb{R}^n,\ 1 \leq n \leq 3, \end{cases}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} = \frac{\partial z}{\partial \eta} = 0, \qquad \text{on } \mathbb{R}^+ \times \partial\Omega.$$

In 2004, Alexander et al. [8], devloped and analysed the locally and globally asymptotically stable of a simple epidemiological model with a generalized non-linear incidence rate $\varphi(v) = \beta[1 + \varphi_\eta(v)]\, v$ by using the Poincaré index theory. Furthermore, it is shown that the basic reproductive number is independent of the functional $\varphi_\eta(v)$ they studied the dynamics of the following epidemiological model

$$\begin{cases} S_t = (1-p)\Lambda - \beta[1 + \varphi_\nu(I)]\, IS - \mu S, \\ I_t = \beta[1 + \varphi_\nu(I)]\, IS - (\mu + \alpha)\, I, \qquad \text{in} \quad \mathbb{R}^+, \end{cases}$$

where Λ is the rate of recruitment of individual. β is the probability of infection per contact per unit time.

The nonlinear $\varphi_\nu(I) \in C^3(\mathbb{R})$ for $I, \nu \geq 0$ is a non-linear function which satisfies the following assumptions:

(i) $\varphi_\nu(0) = \varphi_0(I) = 0$;

(ii) $\varphi'_\nu(I) > 0$ for all $I > 0$;

(iii) $\varphi''_\nu(I) \leq 0$, for all $I > 0$.

First, they proved the existence of the disease-free equilibrium $E_0 = \left(\frac{(1-p)\Lambda}{\mu}, 0\right)$ which is locally and globally asymptotically stable if $R_0 = \frac{\beta(1-p)\Lambda}{\mu(\mu+\alpha)} < 1$ and unstable if $R_0 > 1$. Also the model has a unique endemic equilibrium if $R_0 = \frac{\beta(1-p)\Lambda}{\mu(\mu+\alpha)} > 1$.

In 2005 [45], Korobeinikov and Maini established the Lyapunov functions for ODE SI and SEIR systems with the incidence rate of the form $\varphi(v)\, g(u)$ and $\Lambda = \mu \geq 0$. Let's look at their model SI

$$\begin{cases} S_t = \mu - h(S)\,\varphi(I) - \mu S, \\ I_t = h(S)\,\varphi(I) - (\delta + \mu)\, I, \qquad \text{in } \mathbb{R}^+. \end{cases}$$

The non linearities $\varphi(I)$, $h(S)$ are continuously differentiable function on $\mathbb{R}_+$ satisfying:

(i) $\varphi(0) = 0 = h(0)$;

(ii) $\frac{h(S)\partial\varphi(I)}{\partial I} > 0$, $\frac{\varphi(I)\partial h(S)}{\partial S} > 0$, for all $I > 0$ and $S > 0$.

(iii) $\frac{h(S)\partial^2\varphi(I)}{\partial I^2} \leq 0$, for all $I > 0$ and $S > 0$.

The condition (iii) ensures that the system has two equilibrium states: an infection-free equilibrium and an positive endemic equilibrium state and if $R_0 > 1$, the endemic equilibrium state E^* of this system is asymptotically stable.

In [46], the same authors generalised the incidence rate to a very general form of incidence rate given by $f(u,v)$ and $\Lambda = \mu \geq 0$. The authors extend the Lyapunov functions constructed in [45] and proved the global stability based of the following mathematical model in epidemiology

$$\begin{cases} S_t = \mu - f(S,I) - \mu S, \text{ in } \mathbb{R}^+, \\ I_t = f(S,I) - \delta I, \text{ in } \mathbb{R}^+. \end{cases}$$

The nonlinear $f(S,I)$ is a positive continuous function on $[0,+\infty)$, which satisfies the following conditions:

(i) $f(0,I) = f(S,0) = 0$, for all $S, I > 0$;

(ii) The positive function $f(S,I)$ is monotonically grows with respect to S and I and is concave with respect to the variable I (that is $\dfrac{\partial^2 f}{\partial I^2} \leq 0$) ;

If $R_0 > 1$, then the system has an unique positive endemic equilibrium state E^* which is globally asymptotically stable.

If $R_0 \leq 1$, then there is no positive endemic equilibrium state, and the infection-free equilibrium state E_0 is globally asymptotically stable, where $R_0 = \frac{\partial f(S_0,I)}{\delta\, \partial T}$.

The nonlinear incidence of the form $k\frac{vu}{1+\delta v}$ proposed by Che et al. in [18], they discussed a Avian influenza model given by

$$\begin{cases} X_t = C - k\frac{vX}{1+\delta Y} - dX & \text{in } \mathbb{R}^+, \\ Y_t = k\frac{YX}{1+\delta Y} - (d+m)Y & \text{in } \mathbb{R}^+, \\ u_t = b - \beta\frac{Yu}{1+\delta Y} + -\alpha u & \text{in } \mathbb{R}^+, \\ v_t = \beta\frac{Yu}{1+\delta Y} - (\alpha+\varepsilon+\gamma)v & \text{in } \mathbb{R}^+, \end{cases}$$

they get the reproduction number R_0, it is the threshold which is endemic or not. If $R_0 \leq 1$ the disease-free equilibrium E_0 is global asymptotic stability which implies that the disease will be extinct. If $R_0 > 1$, the endemic equilibrium E^* is globally asymptotically stable. See also Lahrouz et al.[49].

In 2016, Li et al.[52] they consider an SIR epidemic model with the nonlinear incidence $S\varphi(I)$

$$\begin{cases} S_t = \mu A - \mu S - S\varphi(I), \\ I_t = S\varphi(I) - (\mu+\gamma+\alpha)I, \\ R_t = \gamma I - \mu R, \quad \text{in } \mathbb{R}^+, \end{cases} \tag{4.7}$$

where they investigated the existence of equilibrium and have introduced new technique to prove global asymptotical stability of the endemic equilibrium of the ODE SIS epidemic models with the non-linear incidence rate $\varphi(v) = \lambda u \varphi(v)$, where they constructing a more general Lyapunov function. Since the R variable in SIR model does not appear in the equations S and I, for model (4.7), they only needed to consider subsystem

$$\begin{cases} S_t = \mu A - \mu S - S\varphi(I), \\ I_t = S\varphi(I) - (\mu+\gamma+\alpha)I \quad \text{in } \mathbb{R}^+, \end{cases} \tag{4.8}$$

where the nonlinear $\varphi(I)$ is a real locally Lipschitz function at least on on $[0,+\infty[$, which satisfies the following conditions:

(i) $\varphi(0)=0$, $\varphi(I)>0$, for all $I>0$;

(ii) $\frac{\varphi(I)}{I}$ is continuous and monotonely non-increasing for all $v>0$, and $\beta=\lim\limits_{I\to 0^+}\frac{\varphi(I)}{I}$ exists , $0<c<+\infty$;

(iii) $\int_0^1 \frac{1}{\varphi(I)}dI=+\infty$.

Then it follows that the region $D=\left\{(S,I)\in\mathbb{R}^+ : I,S\geq 0 \text{ and } \; S+I\leq N\right\}$ is a positively invariant attractive set for model (4.8) and they investigated the existence of equilibrium, where when $R_0=\frac{\beta A}{\mu+\gamma+\alpha}\leq 1$, model (4.8) has only the disease-free equilibrium $E_0(A,0)$ and when $R_0>1$, besides the disease-free equilibrium E_0, model (4.8) also by using the Intermediate Value Theorem has a unique endemic equilibrium $E^*=(S^*,I^*)$ where $S^*=\frac{(\mu+\gamma+\alpha)I^*}{\varphi(I^*)}$.

In [48] Kuniya et al. they concerned with the following diffusive SI epidemic model

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t}=b(x)-\mu(x)S-\beta(x)SI, \\ \dfrac{\partial I(t,x)}{\partial t}-k_i\triangle I(t,x)=\beta(x)SI-(\mu(x)+\gamma(x))I, \end{cases}$$

in $\mathbb{R}^+\times\Omega$, with homogeneous Neumann boundary condition:

$$\frac{\partial S}{\partial \nu}=\frac{\partial I}{\partial \nu}=0, \qquad \text{on } \mathbb{R}^+\times\partial\Omega,$$

in two cases $(d_1=0,\ d_2>0)$ and $(d_1>0,\ d_2=0)$, is assumed that all parameters can be spatially heterogeneous, where $\varphi(v)=\lambda(x)v$ and $\sigma(.),\mu(.),\gamma(.),\lambda(.)\in C\left(\overline{\Omega};\mathbb{R}\right)$ are all strictly positive and the globally asymptotically stable of $E_0=\left(\frac{b(x)}{\beta(x)},0\right)$ and $E^*=(S^*(x),I^*(x))$ in both cases, the authors obtained that the standard threshold dynamical behaviors, depends on the basic reproductive number.

It 2017, Jia and Qin [35] investigated the stability of the disease-free equilibrium and the unique endemic equilibrium for the following SITAR epidemiological model, with general nonlinear incidence rate $u\varphi(v)$ and treatment

$$\begin{cases} S_t=\Lambda-(d+\mu_1)S-S\varphi(I), \\ I_t=-(d+k_1+k_2)I+\alpha_1 T+S\varphi(I), \\ T_t=k_2 I-nT, \\ A_t=k_1 I+\alpha_2 T-(d+\delta_1)A, \\ R_t=\mu_1 S-dR, \quad \text{in} \quad \mathbb{R}^+, \end{cases}$$

where $\varphi(I)$ satisfies

(i) $\varphi(0)=0$, $\varphi'(I)>0$, $\varphi''(I)\leq 0$, for all $I\geq 0$;

(ii) $\lim\limits_{I\to 0^+}\frac{\varphi(I)}{I}=\beta$; when $0<\beta<+\infty$.

4.4 Preliminary Properties of the Model

Let us assume that the initial conditions $(u_0, v_0) \in \mathbb{R}^2_{\geq 0}$. Note that for $(u, v) \in \mathbb{R}^2_{\geq 0}$, we have

$$F(0, v) = \Lambda > 0,$$
$$G(u, 0) = u\varphi(0) = 0.$$

Which makes the function $(F, G)^T$ essentially nonnegative. Hence, the nonnegative quadrant $\mathbb{R}^2_{\geq 0}$ is an invariant set [see [27, Proposition 2.1] and [61, page 288] Proposition 2.1]. By dropping the diffusion terms, the proposed system reduces to the following system of ordinary differential equations:

$$\begin{cases} \frac{d}{dt}u\left(t\right) = F\left(u, v\right), \quad \text{in } \left(0, +\infty\right) \\ \frac{d}{dt}v\left(t\right) = G\left(u, v\right) \quad \text{in } \left(0, +\infty\right), \end{cases} \tag{4.9}$$

$$u\left(0\right) = u_0 \geq 0, v\left(0\right) = v_0 \geq 0. \tag{4.10}$$

In the following subsections, we de ne an invariant region for the system, identify the system's equilibria and their relation to the basic reproduction number R_0, establish the global existence of solutions in time, and investigate the local stability of the system in the ODE and PDE scenarios.

4.4.1 Invariant Regions

Throughout this paper, we let $N = u + v$ the total population size and $\sigma_0 = \min\left(\sigma, \mu\right)$. We also define the region

$$\mathcal{D} = \left\{(u, v) : u, v \geq 0 \textit{ and } u + v \leq \frac{\Lambda}{\sigma_0}\right\}.$$

The following proposition shows that $\mathcal{D}$ is an invariant region of system (4.9)–(4.10).

Proposition 9 *The region $\mathcal{D}$ is non-empty, attracting and positively invariant.*

proof. We start by summing the equations of system (4.9)–(4.10), which yields

$$\frac{d}{dt}N\left(t\right) = u_t + v_t \leq \Lambda - \sigma_0 N.$$

This implies that

$$N\left(t\right) \leq \frac{\Lambda}{\sigma_0}\left(1 - e^{-\sigma_0 t}\right) + N_0 e^{-\sigma_0 t}.$$

Substituting the value of N yields

$$\left(u + v\right)\left(t\right) \leq \frac{\Lambda}{\sigma_0}\left(1 - e^{-\sigma_0 t}\right) + \left(u + v\right)\left(0\right)e^{-\sigma_0 t}, \qquad \text{for } t \geq 0.$$

If the initial states satisfy $\left(u + v\right)\left(0\right) \leq \frac{\Lambda}{\sigma_0}$, then $\left(u + v\right)\left(t\right) \leq \frac{\Lambda}{\sigma_0}$ and

$$\limsup_{t\to\infty} N\left(t\right) \leq \frac{\Lambda}{\sigma_0}.$$

As a result, region $\mathcal{D}$ is positively invariant and attracting within $\mathbb{R}^2_{\geq 0}$. Therefore, it is sufficient to consider the dynamics of the model within $\mathcal{D}$ as $\mathcal{D}$ is the biologically feasible region of the system where the existence and uniqueness results hold for the system. □

4.4.2 Existence of Equilibrium's and the Basic Reproduction Number R_0

Existence of Equilibrium

In this section, we aim to show the existence of equilibrium solutions for (4.9)–(4.10) and calculate the basic reproduction number.

Theorem 20 *Subject to conditions* (4.4), (4.5). *The system* (4.9)–(4.10) *has a single disease-free equilibrium* $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$. *If* $R_0 = \frac{\lambda}{\sigma}\frac{\Lambda}{\mu}\varphi'(0) > 1$, *then the system* (4.9)–(4.10) *has two distinct equilibrium, namely: The disease-free equilibrium (DFE)* E_0 *and a positive endemic equilibrium (EE)* $E^* = (u^*, v^*) \in \mathbb{R}^2_{\geq 0}$.

proof. The steady states of model (4.9)–(4.10) satisfy

$$\begin{cases} F(u,v) = \Lambda - \lambda u\varphi(v) - \mu u = 0 \\ G(u,v) = \lambda u\varphi(v) - \sigma v = 0. \end{cases} \tag{4.11}$$

If $u = 0$, it is easy to see that the system has no equilibrium. On the other hand, only equilibrium is found for $v = 0$ and that is $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$. Next, we study the existence conditions of an endemic steady state in the case $v > 0$. From the second part of (4.11), and because $\sigma > 0$ and, $\varphi(v) > 0$, we obtain

$$u = \frac{\sigma v}{\lambda\varphi(v)}.$$

Substituting this into the first equation yields

$$h(v) = 0, \qquad \text{for any } v > 0. \tag{4.12}$$

Where

$$h(v) = \frac{\Lambda\lambda}{\mu\sigma}\frac{\varphi(v)}{v} - \frac{\sigma\lambda}{\mu\sigma}\varphi(v) - 1,$$

is continuous for any $v > 0$ and that

$$\lim_{v\to 0} h(v) = \lim_{v\to 0}\frac{\Lambda\lambda}{\mu\sigma}\frac{\varphi'(v)}{1} - \frac{\sigma\lambda}{\mu\sigma}\lim_{v\to 0}\varphi(v) - 1,$$

it is clear that

$$\lim_{v\to 0} h(v) = \lim_{v\to 0}\frac{\Lambda\lambda}{\mu\sigma}\varphi'(0) - 1 = R_0 - 1, \tag{4.13}$$

where R_0 is the basic reproduction number to be identified next. Using $\sigma_0 = \min(\sigma, \mu)$, we have

$$\lim_{v\to\frac{\Lambda}{\sigma_0}} h(v) = h\left(\frac{\Lambda}{\sigma_0}\right) = \underbrace{\frac{\lambda(\sigma_0 - \sigma)}{\mu\sigma}\varphi\left(\frac{\Lambda}{\sigma_0}\right)}_{\leq 0} - 1 < 0. \tag{4.14}$$

Hence for $R_0 > 1$,

$$\lim_{v\to 0} h(v) \times h\left(\frac{\Lambda}{\sigma_0}\right) = (R_0 - 1)h\left(\frac{\Lambda}{\sigma_0}\right) < 0.$$

By applying the intermediate value theorem, there exists a real $v^* \in \left(0, \frac{\Lambda}{\sigma_0}\right)$ such that (4.12) holds. Using (4.5) we can show that the function h is monotonically decreases for all $v > 0$ as

$$\frac{dh}{dv}(v) = \frac{\Lambda\lambda\left[v\varphi'(v) - \varphi(v)\right] - \sigma\lambda v^2\varphi'(v)}{\mu\sigma v^2} < 0,$$

then, there exists a unique real $v^* \in \left(0, \frac{\Lambda}{\sigma_0}\right)$ such that $h(v^*) = 0$, which implie the existence of a unique $u^* = \frac{\sigma v^*}{\lambda\varphi(v^*)}$.

Note that in $\left(\frac{\Lambda}{\sigma_0}, +\infty\right)$ the second equation of (4.11) has note solution because

$$\max_{v\in\left(\frac{\Lambda}{\sigma_0},+\infty\right)} h(v) \leq h\left(\frac{\Lambda}{\sigma_0}\right) < 0.$$

This completes the proof. □

Remark 11 *If $R_0 < 1$ and φ is a non negative and continuously differentiable function on $[0, +\infty)$, then the only equilibrium is $E_0 = \left(0, \frac{\Lambda}{\mu}\right)$. If $R_0 > 1$ and φ is a non negative and continuously differentiable function on $[0, +\infty)$ satisfy the asumptions (4.5), then there are two equilibrium $E_0 = \left(0, \frac{\Lambda}{\mu}\right) \in \partial\mathcal{D}$ and $E^* = (u^*, v^*) \in \mathring{D}$.*

The Basic Reproduction Number $\mathbf{R}_0$

In general, the basic reproduction number is represented by R_0.

In the previous proof, we used the reproduction number R_0 of (4.9)-(4.10). It is now time to identify its value by means of the next generation matrix method formulated in [67] and described further in [72, Lemma 1 page 32].

First, we can write the system (4.9)-(4.10) as

$$\begin{aligned}\begin{pmatrix} v_t \\ u_t \end{pmatrix} &= \begin{pmatrix} \lambda u\varphi(v) - \sigma v \\ \Lambda - \lambda u\varphi(v) - \mu u \end{pmatrix} \\ &= \begin{pmatrix} \lambda u\varphi(v) \\ 0 \end{pmatrix} - \begin{pmatrix} \sigma v \\ -\Lambda + \lambda u\varphi(v) + \mu u \end{pmatrix}.\end{aligned}$$

The Jacobian matrices corresponding to vectors $\begin{pmatrix} \lambda u\varphi(v) \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \sigma v \\ -\Lambda + \lambda u\varphi(v) + \mu u \end{pmatrix}$ at the disease-free equilibrium $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$, are given, respectively, by

$$\begin{pmatrix} \frac{\lambda\Lambda}{\mu}\varphi'(0) & 0 \\ 0 & 0 \end{pmatrix} := \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} \sigma & 0 \\ \frac{\lambda\Lambda}{\mu}\varphi'(0) & \mu \end{pmatrix} := \begin{pmatrix} V & 0 \\ V_1 & V_2 \end{pmatrix},$$

since, the basic reproduction number R_0 is the spectral radius of the next generation matrix

$$K = SV^{-1} = \left(\frac{\lambda\Lambda}{\mu}\varphi'(0)\right)(\sigma)^{-1} = \frac{\lambda\Lambda}{\mu\sigma}\varphi'(0),$$

which is given by

$$R_0 = \rho\left(FV^{-1}\right) = \frac{\Lambda\lambda}{\mu\sigma}\varphi'(0).$$

4.4.3 The Local ODE Asymptotic Stability

Now that we have identified two constant steady states for system (4.9)–(4.10) namely $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$ and $E^* = (u^*, v^*)$, we move to study their local asymptotic stability as described in the following theorem.

Theorem 21 *Assuming that the incidence function φ satisfies (4.4)- (4.5), the following statements hold for system (4.9)–(4.10):*

(i) If $R_0 < 1$, the disease-free equilibrium solution (DFE) $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$ is the only steady state of the system and is locally asymptotically stable.

(ii) If $R_0 > 1$, E_0 is unstable and the positive constant endemic steady state $E^* = (u^*, v^*)$ is locally asymptotically stable.

proof. To prove the local asymptotic stability of the constant steady states, we make advantage of the Jacobian matrix, which may be given by

$$\begin{aligned} J(u,v) &= \begin{pmatrix} F_u(u,v) & F_v(u,v) \\ G_u(u,v) & G_v(u,v) \end{pmatrix} \\ &= \begin{pmatrix} [\Lambda - \lambda u\varphi(v) - \mu u]_u & [\Lambda - \lambda u\varphi(v) - \mu u]_v \\ [\lambda u\varphi(v) - \sigma v]_u & [\lambda u\varphi(v) - \sigma v]_v \end{pmatrix} \end{aligned}$$

$$J(u,v) = \begin{pmatrix} -\lambda\varphi(v) - \mu & -\lambda u\acute{\varphi}(v) \\ \lambda\varphi(v) & \lambda u\acute{\varphi}(v) - \sigma \end{pmatrix}. \tag{4.15}$$

The local stability of $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$ with (4.4) in mind yields

$$\begin{aligned} J(E_0) &= \begin{pmatrix} -\lambda\varphi(0) - \mu & -\lambda\frac{\Lambda}{\mu}\varphi'(0) \\ \lambda\varphi(0) & \lambda\frac{\Lambda}{\mu}\varphi'(0) - \sigma \end{pmatrix} \\ &= \begin{pmatrix} -\mu & -\sigma R_0 \\ 0 & \sigma(R_0 - 1) \end{pmatrix}. \end{aligned} \tag{4.16}$$

First case ($R_0 < 1$)

To determine the local stability of $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$, rests on the nature of the eigenvalues corresponding to $J(E_0)$. The eigenvalues can be easily shown to be $\lambda_1 = -\mu < 0$, $\lambda_2 = \lambda\frac{\Lambda}{\mu}\varphi'(0) - \sigma = \sigma(R_0 - 1)$, It is easy to see that $\lambda_2 < 0$ if $R_0 < 1$, leading to the asymptotic stability of E_0.

Second case ($R_0 > 1$)

The equilibrium E_0 is clearly unstable but the system possesses a positive endemic equilibrium $E^* = (u^*, v^*)$ where $u^*, v^* > 0$. Since E^* satisfies (4.11), we have

$$\begin{cases} \Lambda = & \lambda u^*\varphi(v^*) + \mu u^*, \\ \sigma = & \frac{\lambda u^*\varphi(v^*)}{v^*}. \end{cases} \tag{4.17}$$

Evaluating the Jacobian matrix (4.15) at $E^* = (u^*, v^*)$ yields

$$J(u^*, v^*) = \begin{pmatrix} -\lambda\varphi(v^*) - \mu & -\lambda u^*\acute{\varphi}(v^*) \\ \lambda\varphi(v^*) & \lambda u^*\acute{\varphi}(v^*) - \sigma \end{pmatrix},$$

which has trace

$$tr(J(u^*, v^*)) = -[\lambda\varphi(v^*) + \mu] - \sigma + \lambda u^* \varphi'(v^*),$$

using (4.17) and (4.5), this can be rewritten as

$$tr(J(u^*, v^*)) = -\frac{\Lambda}{u^*} - \lambda u^* \left[\frac{\varphi(v^*)}{v^*} - \varphi'(v^*)\right] < 0 .$$

The determinant of the Jacobian may be given by

$$\det(J(u^*, v^*)) = \lambda\sigma\varphi(v^*) + \mu\sigma - \mu\lambda u^* \acute{\varphi}(v^*),$$

using (4.17) we obtain

$$\det(J(u^*, v^*)) = \lambda \frac{\lambda u^* \varphi(v^*)}{v^*} \varphi(v^*) + \mu\lambda u^* \left[\frac{\varphi(v^*)}{v^*} - \acute{\varphi}(v^*)\right],$$

and from (4.5), we obtain

$$\det(J(u^*, v^*)) > 0.$$

Hence, the equilibrium $E^* = (u^*, v^*)$ is locally asymptotically stable. This concludes the proof of the Theorem. □

Remark 12 *Any infectious disease is capable of invading a population and many epidemiological models contain a Disease Free Equilibrium (DFE) where the disease is not present in the population. These models usually contain a parameter, which is the basic reproduction number R_0. If it is $R_0 < 1$, then DFE is locally asymptotically stable, disease will not spread, but if $R_0 > 1$, DFE is unstable and disease will spread throughout the population.*

4.4.4 Global Existence of Solutions

Assuming that the function φ satisfies conditions (4.4) and (4.5), the following proposition establishes that solutions of system (4.1)–(4.3) exist globally in time and are bounded by a parameter dependent constant.

First, however, we state a lemma that was developed in [5] and which will become useful later on.

Lemma 8 *Condition* (4.5) *imply that*

$$0 < \frac{\varphi(v)}{v} \leq \varphi'(0) \quad \textit{for all } v > 0. \tag{4.18}$$

proof. Clearly, if (4.5) is satisfied, we have

$$\left(\frac{\varphi(v)}{v}\right)' = \frac{v(\varphi(v))' - \varphi(v)}{v^2} \leq 0.$$

Therefore, $\frac{\varphi(v)}{v}$ is a decreasing function. Now, for some $\zeta \in (0, v)$, we have

$$\frac{\varphi(v)}{v} \leq \frac{\varphi(\zeta)}{\zeta},$$

it follows that
$$\frac{\varphi(v)}{v} \leq \lim_{\zeta \to 0} \frac{\varphi(\zeta)}{\zeta},$$
which yields (4.18). □

Proposition 10 *For any nonnegative initial data* $(u_0, v_0) \in \mathbb{C}(\overline{\Omega}) \times \mathbb{C}(\overline{\Omega})$*, the nonnegative solution* $(u(t,x), v(t,x))$ *of system (4.1)–(4.3) exists uniquely and globally in time. In addition, there exists a positive constant* $\mathcal{C}(u_0, v_0, \Lambda, \mu, \lambda, \sigma)$ *such that* $\forall t > 0$,
$$\|u(t,.)\|_{L^\infty(\Omega)} + \|v(t,.)\|_{L^\infty(\Omega)} \leq \mathcal{C} \ . \tag{4.19}$$
Furthermore, there exists a positive constant $\tilde{\mathcal{C}}(\Lambda, \mu, \lambda, \sigma)$ *such that* $\forall t \geq T$ *for some large* $T > 0$,
$$\|u(t,.)\|_{L^\infty(\Omega)} + \|v(t,.)\|_{L^\infty(\Omega)} \leq \tilde{\mathcal{C}}. \tag{4.20}$$

proof. Let us define $\mathcal{X} = \mathcal{Y} \times \mathcal{Y}$ with $\mathcal{Y} = \mathbb{C}(\overline{\Omega})$. Given $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, the norm on $\mathcal{X}$ is defined as
$$\|\psi\| = \|\psi_1\|_{\mathcal{Y}} + \|\psi_2\|_{\mathcal{Y}} .$$
Hence, $(\mathcal{X}, \|.\|)$ is a Banach space. Let us also define the unbounded operator
$$\tilde{A} : \mathcal{D}\left(\tilde{A}\right) \subset \mathcal{X} \to \mathcal{X}$$
such that
$$\tilde{A}\begin{pmatrix} u \\ v \end{pmatrix} = \left(\tilde{A}_1 u, \tilde{A}_2 v\right)^T = (d_1 \Delta u, d_2 \Delta v)^T,$$
and
$$D\left(\tilde{A}\right) = D\left(\tilde{A}_1\right) \times D\left(\tilde{A}_2\right),$$
with
$$D\left(\tilde{A}_i\right) = \left\{\varphi \in \mathbb{C}^2(\Omega) \times \mathbb{C}^1\left(\overline{\Omega}\right) : \ \tilde{A}_i \varphi \in \mathbb{C}\left(\overline{\Omega}\right), \ \frac{\partial \varphi}{\partial \eta} = 0 \text{ on } \partial\Omega\right\}, \quad \text{for } \ i = 1, 2.$$
We may write system (4.1)–(4.3) in abstract form in $\mathcal{X}$ as
$$\begin{cases} \frac{dU}{dt}(t) = AU(t) + \Upsilon(U(t)), \ t > 0, \\ U(0) = U_0 \in \mathcal{X}, \end{cases}$$
where $U(t) = (u(t,.), v(t,.))^T$, $\Upsilon(U(t)) = (F(U(t)), G(V(t)))^T$ and $A = diag(A_1, A_2)$, in which A_i is the closure of $\tilde{A}_i$ in $\mathcal{Y}$ for $i = 1, 2$. The operator A is the infinitesimal generator of an analytical semi group of bounded linear operators $\{T(t)\}_{t \geq 0}$ on $\mathcal{X}$, [70]. Since Υ is locally Lipschitz in $\mathcal{X}$, it follows that system (4.1)–(4.3) admits a unique local solution $(u(t,x), v(t,x))$ for $t \in [0, T_{\max})$ and $x \in \overline{\Omega}$ where $T_{\max}$ is the maximal existence time, see [58]. Let us now consider the case $(t,x) \in (0, T_{\max}) \times \Omega$, which can be formulated as
$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u = \Lambda - \mu u - \lambda u \varphi(v), \text{ in } (0, T_{\max}) \times \Omega, \\ u(0,x) = u_0(x), \qquad \text{on } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, \qquad (0, T_{\max}) \times \partial\Omega. \end{cases} \tag{4.21}$$

We can easily observe that an upper solution exists for (4.21) for any nonnegative function u and v. This upper solution is given by

$$N_1 := \max\left\{\frac{\Lambda}{\mu}, \|u_0\|_{\mathbb{C}(\overline{\Omega})}\right\}.$$

Using the comparison principle from [30], we obtain $u(t;x) \leq N_1$ in $[0, T_{\max}) \times \overline{\Omega}$, and consequently, u is uniformly bounded in $[0, T_{\max}) \times \overline{\Omega}$. Integrating the equations of (4.1) over Ω and taking the sum of the resulting two identities yields

$$\begin{aligned} \frac{d}{dt}\int_{\Omega}(u(t,x)+v(t,x))\,dx &= \Lambda\,|\Omega| - \int_{\Omega}(\mu u(t,x)+\sigma v(t,x))\,dx, \\ &\leq \Lambda\,|\Omega| - \sigma_0\int_{\Omega}(u(t,x)+v(t,x))\,dx, \end{aligned} \tag{4.22}$$

where $\sigma_0 = \min\{\sigma, \mu\}$. From the well-known Gronwall's inequality, we have for $t \in (0, T_{\max})$,

$$\int_{\Omega}(u(t,x)+v(t,x))\,dx \leq N_2, \tag{4.23}$$

where $N_2 > 0$. Hence, for $t \in (0, T_{\max})$,

$$\int_{\Omega} v(t,x)dx \leq N_2.$$

Following the footsteps of [9, Theorem 3.1] and using the v-equation, we conclude $\exists N_3$ depending on N_2 such that $v(t,x) \leq N_3$ over $[0, T_{\max}) \times \overline{\Omega}$. Therefore, v is also uniformly bounded in $[0, T_{\max}) \times \overline{\Omega}$. Using the standard theory of semi linear parabolic systems described in [32], we deduce $T_{\max} = \infty$. When $T_{\max} = \infty$, system (4.21) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1\Delta u = \Lambda - \mu u - \lambda u\varphi(v) \leq \Lambda - \mu u, & \text{in } (0,\infty)\times\Omega, \\ u(0,x) = u_0(x) \leq \|u_0\|_{\mathbb{C}(\overline{\Omega})}, & \text{on } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{in } (0,\infty)\times\partial\Omega. \end{cases} \tag{4.24}$$

By means of the comparison principle we get $u(t,x) \leq \omega(t)$ for $t \in [0,\infty)$, where

$$\omega(t) = \|u_0\|_{\mathbb{C}(\overline{\Omega})} e^{-\mu t} + \left(\frac{\Lambda}{\mu}\right)\left(1 - e^{-\mu t}\right),$$

is the unique solution of the problem

$$\begin{cases} \dfrac{d\omega}{dt} = \Lambda - \mu\omega, & t > 0 \\ \omega(0) = \|u_0\|_{\mathbb{C}(\overline{\Omega})}. \end{cases} \tag{4.25}$$

Consequently, for $x \in \overline{\Omega}$, we have

$$u(t,x) \leq \omega(t) \overset{t\to\infty}{\to} \frac{\Lambda}{\mu}.$$

Therefore, we have an upper bound for $\|u(t,.)\|_{\mathbb{L}^\infty(\Omega)}$ independent of the initial conditions given a sufficiently large t. By applying [59, Lemma 3.1] we find that $\|v(t,.)\|_{\mathbb{L}^\infty(\Omega)}$ is also bounded by a positive constant independent of the initial conditions for a large enough t.

□

Remark 13 *From Lemma8, we obtain*

$$\varphi(v) < \varphi'(0)\, e^{v}, \quad \textit{for all } v \in (0, +\infty). \tag{4.26}$$

According to the previous result, the solution of system (4.1)–(4.3) exists uniquely and globally for any nonnegative initial data u_0, $v_0 \in \mathbb{L}^{\infty}(\Omega)$ satisfying $\|u_0\|_{\infty} < \frac{8d_1 d_2}{n(d_1-d_2)^2}$ with $d_1 \neq d_2$

(following the work in [23, Theorem 3.4]).

4.4.5 The Local PDE Asymptotic Stability

We have already established sufficient conditions for the local asymptotic stability of system (4.9)–(4.10) in the ODE scenario. Let us now examine the more general PDE case (4.1)–(4.3).

Theorem 22 *Assuming that the incidence function φ satisfies* (4.4)- (4.5)*, the following statements hold for system* (4.9)–(4.10)*.*

(i) If $R_0 < 1$, the disease-free equilibrium solution $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$ is locally asymptotically stable.

(ii) If $R_0 > 1$, the positive constant endemic steady equilibrium $E^* = (u^*, v^*)$ is locally asymptotically stable.

proof. (i) In the presence of diffusion, the equilibrium point $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$ satisfies

$$\begin{cases} d_1 \Delta u + \Lambda - \lambda u^* \varphi(v^*) - \mu u^* = 0, \\ d_2 \Delta v + \lambda u^* \varphi(v^*) - \sigma v^* = 0, \end{cases}$$

with Neumann boundaries

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on} \qquad \mathbb{R}^+ \times \partial\Omega.$$

We denote the indefinite sequence of positive eigenvalues for the Laplacian operator Δ over Ω, with Neumann boundary conditions by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \nearrow^{+\infty}$. Note that the first eigenfunction is a constant which is why the corresponding eigenvalue is equal to zero. The corresponding sequence of eigenfunctions is denoted by $(\Phi_{ij})_{j=\overline{1,m_i}}$, where $m_i \geq 1$ is the algebraic multiplicity of λ_i. These functions are the solutions of

$$\begin{cases} -\Delta \Phi_{ij} = \lambda_i \Phi_{ij} \ \text{ in } \Omega, \\ \dfrac{\partial \Phi_{ij}}{\partial \nu} = 0 \ \text{ on } \partial\Omega. \end{cases}$$

The eigenfunctions are normalized according to

$$\|\Phi_{ij}\|_2 = \int_{\Omega} \Phi_{ij}^2(x)\, dx = 1.$$

The set of eigenfunctions $\left\{\Phi_{ij} : i \geq 0, j = \overline{1, m_i}\right\}$ forms a complete orthonormal basis in $L^2(\Omega)$. In order to establish the local asymptotic stability of the steady states, we must examine all the eigenvalues of the linearizing operator and if they all have negative real parts, then the solution is locally asymptotically stable. The linearizing operator may be given by

$$\mathcal{L}(E_0) = \begin{pmatrix} d_1\Delta - \mu & -\lambda\frac{\Lambda}{\mu}\varphi'(0) \\ 0 & d_2\Delta + \lambda\frac{\Lambda}{\mu}\varphi'(0) - \sigma \end{pmatrix}.$$

Using the same method from [2] the stability of E_0 reduces to examining the eigenvalues of the matrices

$$J_i(E_0) = \begin{pmatrix} -d_1\lambda_i - \mu & -\lambda\frac{\Lambda}{\mu}\varphi'(0) \\ 0 & -d_2\lambda_i + \lambda\frac{\Lambda}{\mu}\varphi'(0) - \sigma \end{pmatrix} \text{ for all } i \geq 0,$$

which are given for all $i \geq 0$ by

$$\begin{cases} r_{i1} = -d_1\lambda_i - \mu, \\ r_{i2} = -d_2\lambda_i + \lambda\frac{\Lambda}{\mu}\varphi'(0) - \sigma. \end{cases}$$

Since the Laplacian eigenvalues are positive and in ascending order, both r_{i1}, r_{i2} clearly have negative real parts for $R_0 < 1$, leading to the local stability of $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$ is locally stable.

(ii) The second steady state $E^* = (u^*, v^*)$ satisfies

$$\begin{cases} d_1\Delta u + \Lambda - \lambda u^*\varphi(v^*) - \mu u^* = 0, \\ d_2\Delta v + \lambda u^*\varphi(v^*) - \sigma v^* = 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \mathbb{R}^+ \times \partial\Omega. \end{cases}$$

Hence, the stability of E^* rests on the negativity of the real parts of the eigenvalues of matrices

$$\mathcal{L}(E^*) = \begin{pmatrix} d_1\Delta - \lambda\varphi(v^*) - \mu & -\lambda u^*\acute{\varphi}(v^*) \\ \lambda\varphi(v^*) & d_2\Delta + \lambda u^*\acute{\varphi}(v^*) - \sigma \end{pmatrix}.$$

The stability of E^* reduces to examining the eigenvalues of the following matrices

$$J_i(E^*) = \begin{pmatrix} -d_1\lambda_i - \lambda\varphi(v^*) - \mu & -\lambda u^*\acute{\varphi}(v^*) \\ \lambda\varphi(v^*) & -d_2\lambda_i + \lambda u^*\acute{\varphi}(v^*) - \sigma \end{pmatrix}, \text{ for all } i \geq 0.$$

which is guaranteed if the trace and determinant of $J_i(E^*)$ is given by

$$\mathrm{tr}(J_i(E^*)) < 0, \ \det(J_i(E^*)) > 0, \text{ for all } i \geq 0.$$

The trace of $J_i(E^*)$ is given by

$$\begin{aligned} \mathrm{tr}(J_i(E^*)) &= -d_1\lambda_i - d_2\lambda_i - \lambda\varphi(v^*) - \mu \\ &\quad +\lambda u^*\acute{\varphi}(v^*) - \sigma \\ &= -\lambda_i(d_1 + d_2) + tra(J(u^*, v^*)). \end{aligned}$$

Since $\mathrm{tr}(J(u^*, v^*)) < 0$, it follows that $\mathrm{tr}(J_i(E^*)) < 0$ for all $i \geq 0$. The determinant is given by

$$\begin{aligned} \det(J_i(E^*)) &= d_1d_2\lambda_i^2 + \lambda_i\left[-d_1\lambda u^*\acute{\varphi}(v^*) + d_1\sigma + \lambda\varphi(v^*)d_2 + \mu d_2\right] \\ &\quad +\lambda\sigma\varphi(v^*) + \mu\sigma - \mu\lambda u^*\acute{\varphi}(v^*), \\ &= d_1d_2\lambda_i^2 + \lambda_i H_0 + \det(J(u^*, v^*)) \text{ for all } i \geq 0, \end{aligned}$$

where

$$H_0 = -d_1 \lambda u^* \acute{\varphi}(v^*) + d_1 \sigma + \lambda \varphi(v^*) d_2 + \mu d_2,$$

using (4.5) and (4.17), it holds that

$$\begin{aligned} H_0 &\geq -d_1 \lambda u^* \frac{\varphi(v^*)}{v^*} + d_1 \frac{\lambda u^* \varphi(v^*)}{v^*} + \lambda \varphi(v^*) d_2 + \mu d_2, \\ &= d_2 (\lambda \varphi(v^*) + \mu), \\ &= d_2 \frac{\Lambda}{u^*} > 0, \end{aligned}$$

which leads to

$$\det(J_i(E^*)) = d_1 d_2 \lambda_i^2 + \lambda_i H_0 + \det(J(u^*, v^*)) > 0 \text{ for all } i \geq 0.$$

Hence, E^* is locally asymptotically stable. □

4.5 Global Asymptotic Stability

Next, we study the global asymptotic stability of the two steady states E_0 and E^*. The global stability depends on the reproduction number R_0, which is why we have decided to treat the scenarios $R_0 < 1$ and $R_0 > 1$ separately.

4.5.1 Global Asymptotic Stability with $R_0 < 1$

$E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$ is globally asymptotically stable if there is a continuous differentiable function $\mathcal{V}(u, v)(t)$ satisfying the following

(i) $\mathcal{V}\left(\frac{\Lambda}{\mu}, 0\right) = 0,$

(ii) $\mathcal{V}(u, v)(t) > 0$, for $(u, v) \neq \left(\frac{\Lambda}{\mu}, 0\right),$

(iii) $\frac{d}{dt}\mathcal{V}(u, v)(t) < 0$, for $(u, v) \neq \left(\frac{\Lambda}{\mu}, 0\right).$

For the global asymptotic stability of the disease free equilibrium solution, we consider the following candidate Lyapunov function

$$\mathcal{V}_\theta(t) = \int_\Omega \left[uv + \frac{\theta}{2}\left(u - \frac{\Lambda}{\mu}\right)^2 + \frac{1}{2} v^2 + \frac{\Lambda}{\sigma} v \right] dx, \quad \text{where } \theta > 0.$$

Our goal in this subsection is to use a condition that is weaker than the condition used in our previous work in Djebara et al. [23] to extend the results of global asymptotic stability.

Theorem 23 *Assuming that* (4.5) *holds, if* $R_0 < 1$, *then* E_0 *is a globally asymptotically stable disease-free equilibrium for the system* (4.1)-(4.3) *under the following assumption*

$$\varphi'(0) \leq \frac{\mu + \sigma}{\lambda\left(\theta \frac{\Lambda}{\mu} + \frac{\Lambda}{\sigma}\right)}, \tag{4.27}$$

with

$$\theta \geq \frac{(d_1 + d_2)^2}{4 d_1 d_2}. \tag{4.28}$$

proof. To prove that $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$ is globally asymptotically stable, we need to establish that $\mathcal{V}_\theta(t)$ is a Lyapunov function. First, $\mathcal{V}_\theta(t)$ is positive definite. Now evaluating its derivative with respect to time, and for this we have to distinguish two cases.

The ODE Case

Let $(u, v)(t)$ be a solution of (4.9)–(4.10) and let consider $\mathcal{V}_\theta$ that is defined by

$$\mathcal{V}_\theta(t) = uv + \frac{\theta}{2}\left(u - \frac{\Lambda}{\mu}\right)^2 + \frac{1}{2}v^2 + \frac{\Lambda}{\sigma}v, \quad \text{where } \theta > 0.$$

$\mathcal{V}_\theta(t)$ is a non-negative ($\mathcal{V}_\theta(t) > 0$ for all $(u, v) \neq \left(\frac{\Lambda}{\mu}, 0\right)$), continuous differentiable function and $\mathcal{V}_\theta\left(\frac{\Lambda}{\mu}, 0\right) = 0$. Now to prove that the disease-free equilibrium $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$ is globally asymptotically stable, we need to establish that $\frac{d}{dt}\mathcal{V}_\theta(u, v)(t) < 0$, for $(u, v) \neq \left(\frac{\Lambda}{\mu}, 0\right)$. First, we differentiate $\mathcal{V}_\theta(t)$ with respect to time

$$\frac{d}{dt}\mathcal{V}_\theta(t) = \left(\frac{du}{dt}v + u\frac{dv}{dt}\right) + \theta\left(u - \frac{\Lambda}{\mu}\right)\frac{du}{dt} + v\frac{dv}{dt} + \frac{\Lambda}{\sigma}\frac{dv}{dt},$$

substituting $\frac{du}{dt}, \frac{dv}{dt}$ with its values, we get

$$\frac{d}{dt}\mathcal{V}_\theta(t) = J_1 + J_2 + J_3,$$

where

$$\begin{aligned} J_1 &= (\Lambda - \lambda u\varphi(v) - \mu u)v + u(\lambda u\varphi(v) - \sigma v) \\ &= \Lambda v - \lambda uv\varphi(v) + \lambda u^2\varphi(v) - (\sigma + \mu)uv, \end{aligned}$$

$$\begin{aligned} J_2 &= \theta\left(u - \frac{\Lambda}{\mu}\right)(\Lambda - \mu u - \lambda u\varphi(v)) \\ &= -\mu\theta\left(u - \frac{\Lambda}{\mu}\right)^2 - \lambda\theta u^2\varphi(v) + \lambda\theta\frac{\Lambda}{\mu}u\varphi(v), \end{aligned}$$

and

$$\begin{aligned} J_3 &= v(\lambda u\varphi(v) - \sigma v) + \frac{\Lambda}{\sigma}(\lambda u\varphi(v) - \sigma v) \\ &= \lambda uv\varphi(v) - \sigma v^2 + \lambda\frac{\Lambda}{\sigma}u\varphi(v) - \Lambda v. \end{aligned}$$

Finally, adding J_1, J_2, J_3 it is seen that

$$\begin{aligned} \frac{d}{dt}\mathcal{V}_\theta(t) &= \Lambda v - \Lambda v - \lambda uv\varphi(v) + \lambda uv\varphi(v) + \lambda(1 - \theta)u^2\varphi(v) \\ &\quad + \lambda\theta\frac{\Lambda}{\mu}u\varphi(v) + \lambda\frac{\Lambda}{\sigma}u\varphi(v) - (\sigma + \mu)uv \\ &\quad - \mu\theta\left(u - \frac{\Lambda}{\mu}\right)^2 - \sigma v^2. \\ &= I + J, \end{aligned}$$

where

$$I = \lambda(1 - \theta)u^2\varphi(v) - \mu\theta\left(u - \frac{\Lambda}{\mu}\right)^2 - \sigma v^2$$

and

$$J = \lambda \left(\theta \frac{\Lambda}{\mu} + \tfrac{\Lambda}{\sigma} \right) u\varphi(v) - (\sigma + \mu) uv,$$

if we choose $\theta \geq 1$, we obtain

$$I \leq 0, \tag{4.29}$$

Now, let us examine the second part of the derivative $\frac{d}{dt}\mathcal{V}_\theta(t)$. Applying Lemma8 yields

$$J \leq \left[\lambda \left(\theta \tfrac{\Lambda}{\mu} + \tfrac{\Lambda}{\sigma} \right) \varphi'(0) - (\mu + \sigma) \right] uv, \tag{4.30}$$

for $R_0 < 1$, we have

$$\varphi'(0) < \frac{\mu\sigma}{\Lambda\lambda}, \tag{4.31}$$

combining (4.30) and (4.31), yields

$$J \leq \left[\lambda \left(\theta \tfrac{\Lambda}{\mu} + \tfrac{\Lambda}{\sigma} \right) \frac{\mu\sigma}{\Lambda\lambda} - (\mu + \sigma) \right] uv,$$

and

$$\begin{aligned}
\lambda \left(\theta \tfrac{\Lambda}{\mu} + \tfrac{\Lambda}{\sigma} \right) \frac{\mu\sigma}{\Lambda\lambda} - (\mu + \sigma) &= \left(\frac{\theta}{\mu} + \frac{1}{\sigma} \right) \mu\sigma - (\mu + \sigma) \\
&= (\theta\sigma + \mu) - (\mu + \sigma) \\
&= (\theta - 1)\sigma \leq 0 \text{ if } \theta \leq 1.
\end{aligned}$$

Then

$$J \leq \left[\lambda \left(\theta \tfrac{\Lambda}{\mu} + \tfrac{\Lambda}{\sigma} \right) \frac{\mu\sigma}{\Lambda\lambda} - (\mu + \sigma) \right] uv \leq 0 \text{ if } \theta \leq 1. \tag{4.32}$$

From (4.32) and (4.29), we have if $\theta = 1$

$$\frac{d}{dt}\mathcal{V}_\theta(t) = I + J \leq 0.$$

Hence $\frac{d}{dt}\mathcal{V}_\theta(t) = 0$ only at the equilibrium point E_0, because

$$\begin{aligned}
\frac{d}{dt}\mathcal{V}_\theta\left(\tfrac{\Lambda}{\mu}, 0 \right)(t) &= \lambda(1 - \theta) \tfrac{\Lambda}{\mu}^2 \varphi(0) - \mu\theta \left(\tfrac{\Lambda}{\mu} - \tfrac{\Lambda}{\mu} \right)^2 - \sigma.0^2 + \\
+\lambda \left(\theta \frac{\Lambda}{\mu} + \tfrac{\Lambda}{\sigma} \right) \tfrac{\Lambda}{\mu}\varphi(0) - (\sigma + \mu) \tfrac{\Lambda}{\mu}.0 &= 0.
\end{aligned}$$

Therefore, by Lyapunov's direct method $\frac{d}{dt}\mathcal{V}_1(t)$ is negative semi definite and the largest invariant set is reduced to the disease-free equilibrium $E_0 = \left(\tfrac{\Lambda}{\mu}, 0 \right)$ and E_0 is attractive point, we conclude that E_0 is a globally asymptotically stable equilibrium point if $R_0 < 1$.

The PDE Case

To prove that $E_0 = \left(\tfrac{\Lambda}{\mu}, 0 \right)$ is globally asymptotically stable, we must show that $\mathcal{V}_\theta(t)$ is a Lyapunov function. The positive definiteness of $\mathcal{V}_\theta(t)$ is evident. Evaluating its derivative with respect to time gives

$$\frac{d}{dt}\mathcal{V}_\theta(t) = \int_\Omega \left(\tfrac{\partial u}{\partial t} v + u \tfrac{\partial v}{\partial t} \right) dx + \theta \int_\Omega \left(u - \tfrac{\Lambda}{\mu} \right) \tfrac{\partial u}{\partial t} dx + \int_\Omega v \tfrac{\partial v}{\partial t} dx + \tfrac{\Lambda}{\sigma} \int_\Omega \tfrac{\partial v}{\partial t} dx.$$

Substituting the partial derivatives $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ with their respective values from (4.1) and applying Green's formula with the assumed Neumann boundaries leads to

$$\frac{d}{dt}\mathcal{V}_{\theta}(t) = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_{\Omega} (d_1 \Delta u + \Lambda - \lambda u \varphi(v) - \mu u)\, v + \int_{\Omega} u\,(d_2 \Delta v + \lambda u \varphi(v) - \sigma v)\, dx \\ &= -(d_1 + d_2)\int_{\Omega} \nabla u \nabla v dx + \Lambda \int_{\Omega} v dx - \lambda \int_{\Omega} uv\varphi(v)\, dx + \lambda \int_{\Omega} u^2 \varphi(v)\, dx \\ &\quad - (\sigma + \mu)\int_{\Omega} uv dx, \end{aligned}$$

$$\begin{aligned} I_2 &= \theta \int_{\Omega} \left(u - \tfrac{\Lambda}{\mu}\right)(d_1 \Delta u + \Lambda - \lambda u \varphi(v) - \mu u)\, dx \\ &= -d_1\theta \int_{\Omega} |\nabla u|^2 dx + \theta \int_{\Omega} \left(u - \tfrac{\Lambda}{\mu}\right)\left(-\mu\left(u - \frac{\Lambda}{\mu}\right) - \lambda u \varphi(v)\right) dx \\ &= -d_1\theta \int_{\Omega} |\nabla u|^2 dx - \mu\theta \int_{\Omega} \left(u - \tfrac{\Lambda}{\mu}\right)^2 dx - \lambda\theta \int_{\Omega} u^2 \varphi(v)\, dx + \lambda\theta \frac{\Lambda}{\mu}\int_{\Omega} u\varphi(v)\, dx \end{aligned}$$

and

$$\begin{aligned} I_3 &= \int_{\Omega} v \tfrac{\partial v}{\partial t} dx + \tfrac{\Lambda}{\sigma}\int_{\Omega} \tfrac{\partial v}{\partial t} dx \\ &= -d_2 \int_{\Omega} |\nabla v|^2 dx + \lambda \int_{\Omega} uv\varphi(v)\, dx - \sigma \int_{\Omega} v^2 dx \\ &\quad + \lambda \tfrac{\Lambda}{\sigma}\int_{\Omega} u\varphi(v)\, dx - \Lambda \int_{\Omega} v dx. \end{aligned}$$

Taking the sum of terms I_1, I_2 and I_3 yields

$$\begin{aligned} \frac{d}{dt}\mathcal{V}_{\theta}(t) &= -(d_1 + d_2)\int_{\Omega} \nabla u \nabla v dx + \Lambda \int_{\Omega} v dx - \lambda \int_{\Omega} uv\varphi(v)\, dx \\ &\quad + \lambda \int_{\Omega} u^2\varphi(v)\, dx - (\sigma + \mu)\int_{\Omega} uv dx - d_1\theta \int_{\Omega} |\nabla u|^2 dx \\ &\quad - \mu\theta \int_{\Omega} \left(u - \tfrac{\Lambda}{\mu}\right)^2 dx - \lambda\theta \int_{\Omega} u^2\varphi(v)\, dx + \lambda\theta\frac{\Lambda}{\mu}\int_{\Omega} u\varphi(v)\, dx \\ &\quad - d_2 \int_{\Omega} |\nabla v|^2 dx + \lambda \int_{\Omega} uv\varphi(v)\, dx - \sigma \int_{\Omega} v^2 dx \\ &\quad + \lambda\tfrac{\Lambda}{\sigma}\int_{\Omega} u\varphi(v)\, dx - \Lambda \int_{\Omega} v dx \\ &= I + J, \end{aligned}$$

where

$$I = -d_1\theta \int_{\Omega} |\nabla u|^2 dx - (d_1 + d_2)\int_{\Omega} \nabla u \nabla v dx - d_2 \int_{\Omega} |\nabla v|^2 dx,$$

and

$$\begin{aligned} J &= \lambda(1-\theta)\int_\Omega u^2\varphi(v)\,dx + \lambda\left(\theta\frac{\Lambda}{\mu}+\tfrac{\Lambda}{\sigma}\right)\int_\Omega u\varphi(v)\,dx \\ &\quad -(\sigma+\mu)\int_\Omega uvdx - \mu\theta\int_\Omega\left(u-\tfrac{\Lambda}{\mu}\right)^2dx - \sigma\int_\Omega v^2dx. \end{aligned}$$

The first term may be written as

$$I = -\int_\Omega Q(\nabla u,\nabla v)\,dx,$$

where Q is a quadratic form with respect to ∇u and ∇v, i.e.

$$Q(\nabla u,\nabla v) = d_1\theta\left|\nabla u\right|^2 + (d_1+d_2)\nabla u\nabla v + d_2\left|\nabla v\right|^2,$$

which is non-negative because θ, d_1 and d_2 satisfies $\theta \geq \frac{(d_1+d_2)^2}{4d_1d_2}$ and $d_1\theta > 0$ is chosen such that the discriminant D of Q given by

$$D = (d_1+d_2)^2 - 4(d_1\theta)(d_2) = (d_1+d_2)^2 - 4d_1d_2\theta \leq 0,$$

is negative ($D \leq 0$), from which we obtain

$$I \leq 0. \tag{4.33}$$

On the other hand, using the inequality $\theta \geq \frac{(d_1+d_2)^2}{4d_1d_2} \geq 1$, we have

$$\begin{aligned} J &< \lambda\left(\theta\frac{\Lambda}{\mu}+\tfrac{\Lambda}{\sigma}\right)\int_\Omega u\varphi(v)\,dx - (\sigma+\mu)\int_\Omega uvdx \\ &\quad -\mu\theta\int_\Omega\left(u-\tfrac{\Lambda}{\mu}\right)^2dx - \sigma\int_\Omega v^2dx. \end{aligned}$$

Applying Lemma8 yields

$$\begin{aligned} \frac{d}{dt}\mathcal{V}_\theta(t) &< \int_\Omega\left[\lambda\left(\theta\tfrac{\Lambda}{\mu}+\tfrac{\Lambda}{\sigma}\right)\varphi'(0) - (\sigma+\mu)\right]uvdx \\ &\quad -\mu\theta\int_\Omega\left(u-\tfrac{\Lambda}{\mu}\right)^2dx - \sigma\int_\Omega v^2dx. \end{aligned}$$

Assuming (4.27) holds, the derivative $\frac{d}{dt}\mathcal{V}_\theta(t) \leq 0$ for all $t \geq 0$ with $\mathcal{V}_\theta(t) = 0$ being true when $(u,v) = \left(\frac{\Lambda}{\mu},0\right)$. Finally, by Lyapunov's direct method E_0 is globally asymptotically stable. □

4.5.2 Global Asymptotic Stability with $R_0 > 1$

First, however, let us state a necessary lemma taken from [68] that will aid with the proofs to come.

Lemma 9 *Given that φ satisfies criterion* (4.5) *and*

$$L(x) = x - 1 - \ln(x), \text{ for } x > 0, \tag{4.34}$$

the inequality

$$L\Big(\frac{\varphi(v)}{\varphi(v^*)}\Big) \leq L\big(\frac{v}{v^*}\big). \tag{4.35}$$

where v^* is the second component of the equilibrium point E^*, holds.

proof. Using (4.5), we have $\frac{\varphi(v)}{v}$ is a decreasing function for all $v > 0$. We may separate the proof into two regions:

1. The first region is $v \geq v^*$. Since $\frac{\varphi(v)}{v}$ is a decreasing function, we have

$$\frac{\varphi(v)}{\varphi(v^*)} \leq \frac{v}{v^*}.$$

Note that (4.5) implies that φ is non decreasing, which leads to

$$\varphi(v) \geq \varphi(v^*),$$

and, consequently,

$$1 \leq \frac{\varphi(v)}{\varphi(v^*)} \leq \frac{v}{v^*}.$$

Since L is increasing for all $x > 1$, follows

$$L\Big(\frac{\varphi(v)}{\varphi(v^*)}\Big) \leq L\big(\frac{v}{v^*}\big)$$

for all $v \geq v^*$.

2. The second region is $0 < v < v^*$. Again, since $\frac{\varphi(v)}{v}$ is a decreasing function, we have

$$\frac{\varphi(v)}{\varphi(v^*)} > \frac{v}{v^*},$$

and given the non-decreasing nature of φ, we end up with

$$\varphi(v) < \varphi(v^*).$$

As a result, we get

$$1 > \frac{\varphi(v)}{\varphi(v^*)} > \frac{v}{v^*} > 0.$$

Since L is decreasing for $0 < x < 1$, which implies (4.35) holds.

□

Lemma 10 *Let* $L(x) = x - 1 - \ln(x)$*, for* $x > 0$*, then* $L(x) \geq 0$ *for all* $x > 0$ *and* $\left[L\left(\frac{u}{u^*}\right)\right]' = \frac{1}{u^*}\left(1 - \frac{u^*}{u}\right)$.

Since we can rewritten in the forms

$$\left(1 - \frac{u^*}{u}\right)\left(1 - \frac{u}{u^*}\right) = -L\left(\frac{u}{u^*}\right) - L\left(\frac{u^*}{u}\right),$$

and

$$\begin{aligned} &\left(\frac{u\varphi(v)}{u^*\varphi(v^*)} - \frac{v}{v^*}\right)\left(1 - \frac{v^*}{v}\right) + \left(1 - \frac{u\varphi(v)}{u^*\varphi(v^*)}\right)\left(1 - \frac{u^*}{u}\right) \\ = \; & -L\left(\frac{u^*}{u}\right) + L\left(\frac{\varphi(v)}{\varphi(v^*)}\right) - L\left(\frac{v}{v^*}\right) - L\left(\frac{u\varphi(v)v^*}{u^*\varphi(v^*)v}\right). \end{aligned}$$

proof. First, note that

$$\begin{aligned} L'(x) &= 1 - \frac{1}{x}, \text{ for } x > 0 \\ &= \frac{x-1}{x}, \text{ for } x > 0, \end{aligned}$$

which implie $L(x) \geq 0$ for $x > 0$ and

$$\left[L\left(\frac{u}{u^*}\right)\right]' = L'\left(\frac{u}{u^*}\right)\frac{1}{u^*} = \left(1 - \frac{u^*}{u}\right)\frac{1}{u^*}.$$

$$\begin{aligned} \left(1 - \frac{u^*}{u}\right)\left(1 - \frac{u}{u^*}\right) &= 1 - \frac{u}{u^*} - \frac{u^*}{u} + 1 \\ &= -\frac{u}{u^*} - \frac{u^*}{u} + 2 + \ln u - \ln u + \ln u^* - \ln u^* \\ &= -\left(\frac{u}{u^*} - \ln\frac{u}{u^*} - 1\right) - \left(\frac{u^*}{u} - \ln\frac{u^*}{u} - 1\right). \end{aligned}$$

Now we can write

$$J_1 = \left(1 - \frac{u^*}{u}\right)\left(1 - \frac{u}{u^*}\right) = -L\left(\frac{u}{u^*}\right) - L\left(\frac{u^*}{u}\right),$$

and

$$\begin{aligned} J_2 &= \left(\frac{u\varphi(v)}{u^*\varphi(v^*)} - \frac{v}{v^*}\right)\left(1 - \frac{v^*}{v}\right) + \left(1 - \frac{u\varphi(v)}{u^*\varphi(v^*)}\right)\left(1 - \frac{u^*}{u}\right) \\ &= \frac{u\varphi(v)}{u^*\varphi(v^*)} - \frac{v}{v^*} - \frac{u\varphi(v)v^*}{u^*\varphi(v^*)v} + 1 + 1 - \frac{u^*}{u} - \frac{u\varphi(v)}{u^*\varphi(v^*)} + \frac{\varphi(v)}{\varphi(v^*)}, \end{aligned}$$

because

$$\begin{aligned} &-\ln u + \ln u + \ln u^* - \ln u^* - \ln\varphi(v) + \ln\varphi(v) + \\ &\ln\varphi(v^*) - \ln\varphi(v^*) + \ln v - \ln v^* - \ln v + \ln v^* = 0, \end{aligned}$$

we can write

$$\begin{aligned}
J_2 &= -\frac{v}{v^*} - \frac{u\varphi(v)v^*}{u^*\varphi(v^*)v} + 1 + 1 - \frac{u^*}{u} + \frac{\varphi(v)}{\varphi(v^*)} + \\
&\quad (\ln u^* - \ln u) - (\ln\varphi(v) - \ln\varphi(v^*)) + (\ln v - \ln v^*) \\
&\quad + \ln u + \ln\varphi(v) + \ln v^* - (\ln u^* + \ln\varphi(v^*) + \ln v) + 1 - 1 \\
&= -\left(\frac{u^*}{u} - \ln\frac{u^*}{u} - 1\right) + \left(\frac{\varphi(v)}{\varphi(v^*)} - \ln\frac{\varphi(v)}{\varphi(v^*)} - 1\right) \\
&\quad -\left(\frac{v}{v^*} - \ln\frac{v}{v^*} - 1\right) - \left(\frac{u\varphi(v)v^*}{u^*\varphi(v^*)v} - \ln\frac{u\varphi(v)v^*}{u^*\varphi(v^*)v} - 1\right) \\
&= -L\left(\frac{u^*}{u}\right) + L\left(\frac{\varphi(v)}{\varphi(v^*)}\right) - L\left(\frac{v}{v^*}\right) - L\left(\frac{u\varphi(v)v^*}{u^*\varphi(v^*)v}\right).
\end{aligned}$$

□

Theorem 24 *Assuming that u_0, $v_0 > 0$ and (4.5) holds, if $R_0 > 1$, then $E^* = (u^*, v^*)$ is a globally asymptotically stable endemic steady-state for system (4.1)-(4.3).*

proof. For this proof, we consider the candidate Lyapunov function created by Kouachi and Kirane in [43]

$$\mathcal{V}(t) = \int_\Omega \left[u^* L\left(\frac{u}{u^*}\right) + v^* L\left(\frac{v}{v^*}\right)\right] dx,$$

which is a positive definite and continuously differentiable function. Differentiating $\mathcal{V}(t)$ with respect to time yields

$$\begin{aligned}
\frac{d}{dt}\mathcal{V}(t) &= \int_\Omega \left[u^*\left[L\left(\frac{u}{u^*}\right)\right]'\frac{\partial u}{\partial t} + v^*\left[L\left(\frac{v}{v^*}\right)\right]'\frac{\partial v}{\partial t}\right] dx \\
&= \int_\Omega \left[u^*\frac{1}{u^*}\left(1 - \frac{u^*}{u}\right)\frac{\partial u}{\partial t} + v^*\frac{1}{v^*}\left(1 - \frac{v^*}{v}\right)\frac{\partial v}{\partial t}\right] dx
\end{aligned}$$

substituting $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ with its values, using Green's formula and the Neumann boundary conditions, we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{V}(t) &= d_1\int_\Omega \left(1 - \frac{u^*}{u}\right)\Delta u dx + \Lambda\int_\Omega\left(1 - \frac{u^*}{u}\right)dx - \lambda\int_\Omega\left(1 - \frac{u^*}{u}\right)u\varphi(v)\,dx \\
&\quad -\mu\int_\Omega\left(1 - \frac{u^*}{u}\right)udx + d_2\int_\Omega\left(1 - \frac{v^*}{v}\right)\Delta v dx \\
&\quad +\lambda\int_\Omega\left(1 - \frac{v^*}{v}\right)u\varphi(v)\,dx - \sigma\int_\Omega\left(1 - \frac{v^*}{v}\right)vdx.
\end{aligned}$$

Similar to the previous scenario, we apply Green's formula with Neumann boundaries to expand the derivative to

$$\begin{aligned}
\frac{d}{dt}\mathcal{V}(t) &= -d_1\int_\Omega \frac{u^*}{u^2}|\nabla u|^2 dx + \Lambda\int_\Omega\left(1 - \frac{u^*}{u}\right)dx - \lambda\int_\Omega\left(1 - \frac{u^*}{u}\right)u\varphi(v)\,dx \\
&\quad -\mu\int_\Omega\left(1 - \frac{u^*}{u}\right)udx - d_2\int_\Omega\frac{v^*}{v^2}|\nabla v|^2 dx \\
&\quad +\lambda\int_\Omega\left(1 - \frac{v^*}{v}\right)u\varphi(v)\,dx - \sigma\int_\Omega\left(1 - \frac{v^*}{v}\right)vdx \\
&= I + J,
\end{aligned}$$

where

$$I=-\left[\int_{\Omega}d_1\frac{u^*}{u^2}\left|\nabla u\right|^2+d_2\frac{v^*}{v^2}\left|\nabla v\right|^2\right]dx\leq 0,$$

and

$$J=\int_{\Omega}\left(1-\frac{u^*}{u}\right)\left[\Lambda-\lambda u\varphi\left(v\right)-\mu u\right]dx+\int_{\Omega}\left(1-\frac{v^*}{v}\right)\left[\lambda u\varphi\left(v\right)-\sigma v\right]dx.$$

Replacing Λ by $\lambda u^*\varphi\left(v^*\right)+\mu u^*$, and σ by $\frac{\lambda u^*\varphi(v^*)}{v^*}$ and simplifying the resulting equation leads to

$$\begin{aligned}J &= \int_{\Omega}\mu u^*\left(1-\frac{u}{u^*}\right)\left(1-\frac{u^*}{u}\right)dx\\ &\quad +\lambda\int_{\Omega}u^*\varphi\left(v^*\right)\left[\left(\frac{u\varphi\left(v\right)}{u^*\varphi\left(v^*\right)}-\frac{v}{v^*}\right)\left(1-\frac{v^*}{v}\right)+\left(1-\frac{u\varphi\left(v\right)}{u^*\varphi\left(v^*\right)}\right)\left(1-\frac{u^*}{u}\right)\right]dx\\ &= \int_{\Omega}\mu u^*J_1dx+\lambda\int_{\Omega}u^*\varphi\left(v^*\right)J_2dx\end{aligned}$$

where

$$J_1=\left(1-\frac{u^*}{u}\right)\left(1-\frac{u}{u^*}\right),$$

and

$$J_2=\frac{\varphi\left(v\right)}{\varphi\left(v^*\right)}-\frac{v}{v^*}-\frac{u\varphi\left(v\right)v^*}{u^*\varphi\left(v^*\right)v}-\frac{u^*}{u}+2.$$

Applying Lemma10, J_1 and J_2 can be rewritten in the forms

$$J_1=-L\left(\frac{u}{u^*}\right)-L\left(\frac{u^*}{u}\right),$$

and

$$J_2=-L\left(\frac{u^*}{u}\right)+L\left(\frac{\varphi\left(v\right)}{\varphi\left(v^*\right)}\right)-L\left(\frac{v}{v^*}\right)-L\left(\frac{u\varphi\left(v\right)v^*}{u^*\varphi\left(v^*\right)v}\right),$$

it follows that

$$J=\mu u^*\int_{\Omega}\left[-L\left(\tfrac{u}{u^*}\right)-L\left(\tfrac{u^*}{u}\right)\right]dx$$

$$+\lambda u^*\varphi\left(v^*\right)\int_{\Omega}\left[-L\left(\tfrac{u^*}{u}\right)+L\left(\tfrac{\varphi(v)}{\varphi(v^*)}\right)-L\left(\tfrac{v}{v^*}\right)-L\left(\tfrac{u\varphi(v)v^*}{u^*\varphi(v^*)v}\right)\right]dx$$

Taking advantage of the established positivity of L and applying Lemma9 we end up with

$$J=-\mu u^*\int_{\Omega}\left[L\left(\tfrac{u}{u^*}\right)+L\left(\tfrac{u^*}{u}\right)\right]dx-\lambda u^*\varphi\left(v^*\right)\int_{\Omega}\left[L\left(\tfrac{u^*}{u}\right)+L\left(\tfrac{u\varphi(v)v^*}{u^*\varphi(v^*)v}\right)\right]dx$$

$$+\lambda u^*\varphi\left(v^*\right)\int_{\Omega}\left[L\left(\tfrac{\varphi(v)}{\varphi(v^*)}\right)-L\left(\tfrac{v}{v^*}\right)\right]dx\leq 0,$$

is negative and from (4.33) it is clear that $\frac{d}{dt}\mathcal{V}\left(t\right)=I+J\leq 0$, hence $\mathcal{V}$ is non-increasing in time and

$$\begin{aligned}\frac{d}{dt}\mathcal{V}\left(t\right) &= -d_1\int_{\Omega}\frac{u^*}{u^2}\left|\nabla u\right|^2dx+\Lambda\int_{\Omega}\left(1-\frac{u^*}{u}\right)dx\\ &\quad -\lambda\int_{\Omega}\left(1-\frac{u^*}{u}\right)u\varphi\left(v\right)dx-\mu\int_{\Omega}\left(1-\frac{u^*}{u}\right)udx\\ &\quad -d_2\int_{\Omega}\frac{v^*}{v^2}\left|\nabla v\right|^2dx+\lambda\int_{\Omega}\left(1-\frac{v^*}{v}\right)u\varphi\left(v\right)dx-\sigma\int_{\Omega}\left(1-\frac{v^*}{v}\right)vdx,\end{aligned}$$

substituting in this the equilibrium $E^* = (u^*, v^*)$, which leads to

$$
\begin{aligned}
\frac{d}{dt}\mathcal{V}(t) &= -d_1 \int_\Omega \frac{u^*}{u^{*2}} |\nabla u^*|^2 \, dx + \Lambda \int_\Omega \left(1 - \frac{u^*}{u^*}\right) dx \\
&\quad -\lambda \int_\Omega \left(1 - \frac{u^*}{u^*}\right) u^* \varphi(v) \, dx - \mu \int_\Omega \left(1 - \frac{u^*}{u^*}\right) u dx \\
&\quad -d_2 \int_\Omega \frac{v^*}{v^{*2}} |\nabla v^*|^2 \, dx + \lambda \int_\Omega \left(1 - \frac{v^*}{v^*}\right) u^* \varphi(v^*) \, dx - \sigma \int_\Omega \left(1 - \frac{v^*}{v^*}\right) v^* dx \\
&= 0,
\end{aligned}
$$

Therefore, by Lyapunov's direct method's, the endemic stationary state E^* solution is globally asymptotically stable.We can show that $\frac{d}{dt}\mathcal{V}(t) \leq 0$ and, consequently, $\mathcal{V}$ is non-increasing in time with $\mathcal{V}(t) = 0$ only at the steady state $E^* = (u^*, v^*)$. The global asymptotic stability of E^* follows from Lyapunov's direct method's (See also [3]). $\square$

4.5.3 Important Remarks

It is important to note that, for system (4.9)–(4.10) in ODE case, we have the following results:

(i) For $R_0 < 1$, the disease-free equilibrium E_0 is globally asymptotically stable. This easy to establish as $d_1 = d_2 = 0$ and thus suffices to choose $\theta = 1$ in $\mathcal{V}_\theta(t)$.

(ii) For $R_0 > 1$ and provided that condition (4.5) is satisfied, the endemic equilibrium $E^* = (u^*, v^*)$ is globally asymptotically stable.

4.6 Applications and Numerical Examples

In this section, we aim to use numerical solutions to confirm the validity of the theoretical analysis in our study by looking at a numerical example and one of the three examples in the thesis.. We use consider an incidence function of the form $u\varphi(v)$, satisfying (4.5) and assess the local and global asymptotic stability of the disease-free equilibrium E_0 when $R_0 < 1$ and the endemic equilibrium E^* when $R_0 > 1$. Focus will be on the main results of the paper as reported in Theorem 23 (condition (4.27)) and Theorem24.

In the following, we examine three different numerical examples.

4.6.1 First Example

In this first example, we consider the function

$$\varphi(v) = \alpha v \text{ for all } \alpha > 0.$$

The resulting problem is given by

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - d_1 \Delta u = -\lambda \alpha u v + \Lambda - \mu u & \text{in } (0, \infty) \times \Omega, \\
\dfrac{\partial v}{\partial t} - d_2 \Delta v = \lambda \alpha u v - \sigma v & \text{in } (0, \infty) \times \Omega, \\
u(0, x) = u_0(x), \quad v(0, x) = v_0(x) & \text{on } \Omega, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & \text{on } (0, \infty) \times \partial\Omega,
\end{cases}
\tag{4.36}
$$

Table 4.1: Simulation parameters for example 1: system (4.36).

	Set	Figure	u_0	v_0	d_1	d_2	λ	α	σ	Λ	μ	R_0
ODE	Set 1	4.2(a)	6	$\frac{3}{2}$	$-$	$-$	$\frac{1}{3}$	3	2	8	1	4
case	Set 2	4.2(b)	6	$\frac{3}{2}$	$-$	$-$	2	$\frac{1}{3}$	$\frac{3}{2}$	6	4	0.8333
PDE	Set 1	4.3	$4+\frac{\cos(x)}{10}$	$5+\frac{\sin(x)}{10}$	3	$\frac{5}{4}$	$\frac{1}{3}$	3	2	8	1	4
case	Set 2	4.4	$4+\frac{\cos(x)}{10}$	$5+\frac{\sin(x)}{10}$	3	$\frac{5}{4}$	2	$\frac{1}{3}$	$\frac{3}{2}$	6	4	0.8333

which is a special case of system (4.1)–(4.3). In fact, system (4.36) is identical to the bird system proposed in Section 3 of [41] and in [19] but with $d_1 = d_2$. The PDE and ODE scenarios of this model were treated earlier in [44] and [78], respectively. Conditions (4.4) and are clearly satisfied as

$$\begin{cases} \varphi(0) = 0, \\ \varphi'(v) = \alpha > 0, \\ \varphi'(0) = \alpha, \\ \alpha v = v\varphi'(v) \leq \varphi(v) = \alpha v. \end{cases}$$

System (4.36) possesses two constant steady states $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$ and for $v \neq 0$, we have

$$E^* = \left(\frac{\sigma v^*}{\lambda\varphi(v^*)}, v^*\right) = \left(\frac{\sigma}{\lambda\alpha}, \frac{\Lambda}{\sigma} - \frac{\mu}{\lambda\alpha}\right) = \left(\frac{\sigma}{\lambda\alpha}, \mu\sigma(R_0 - 1)\right).$$

Note that the second steady state E^* exists only when the reproduction number $R_0 = \frac{\lambda\Lambda}{\mu\sigma}\varphi'(0) = \frac{\lambda\Lambda}{\mu\sigma}\alpha > 1$ and is globally asymptotically stable. Also, note that the first steady state E_0 is globally asymptotically stable unconditionally in the ODE case and subject to

$$\frac{\mu}{\Lambda}\left(\frac{\mu+\sigma}{\lambda\alpha} - \frac{\Lambda}{\sigma}\right) \geq \theta \geq \frac{(d_1 + d_2)^2}{4d_1d_2},$$

when $d_1 \neq d_2$, and $\theta = 1$ when $d_1 = d_2$ in the PDE case. As detailed in Table 1, we use different sets of parameters to obtain numerical solutions in the ODE and PDE. Note that throughout the PDE simulations, we assume a single spatial dimension with $\Omega = (0, 10)$.

The following is a description of the results:

- Figure 4.2 shows the solutions in the ODE case subject to sets 1 and 2, with $R_0 = 4$ and $R_0 = 0.8333$, respectively. In the first case, as $R_0 > 1$, $E^* = (2, 3)$ is globally asymptotically stable. In the second case, $R_0 < 1$ and $E_0 = (\frac{3}{2}, 0)$ is globally asymptotically stable.

- Figure 4.3 depicts the solution in the PDE case subject to parameter set 1, where $R_0 = 4 > 1$, which by Theorem24 means that $E^* = (2, 3)$ is globally asymptotically stable.

- Figure 4.4 depicts the solution in the PDE case subject to parameter set 2, where $R_0 = 0.8333 < 1$. By Theorem23 and given $\theta \in \left[\frac{289}{240}, \frac{17}{6}\right]$, $E_0 = (\frac{3}{2}, 0)$ is globally asymptotically stable.

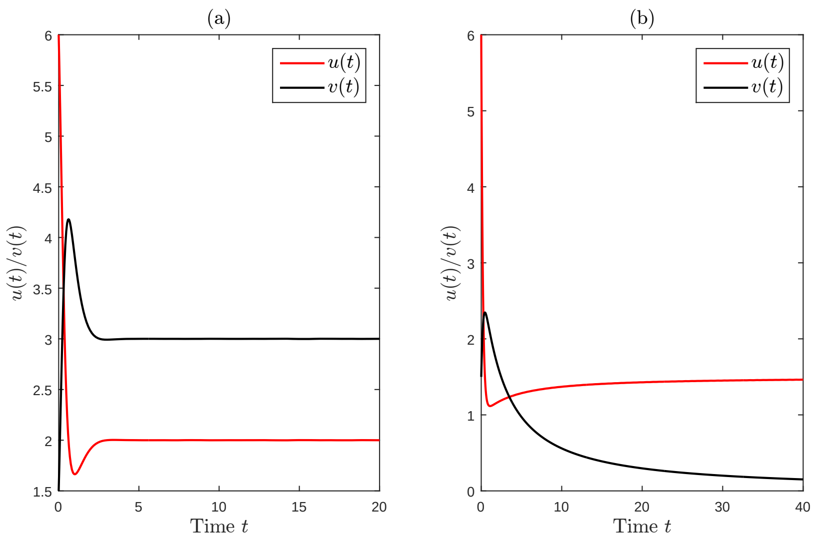


Figure 4.2: Numerical solutions of system (4.36) (ODE case) subject to the first and second sets of parameters.

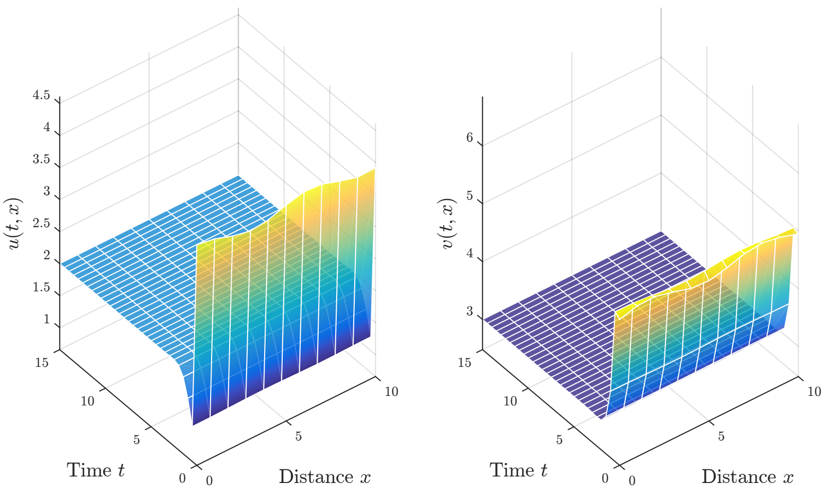


Figure 4.3: Numerical solutions of system (4.36) subject to the first set of parameters.

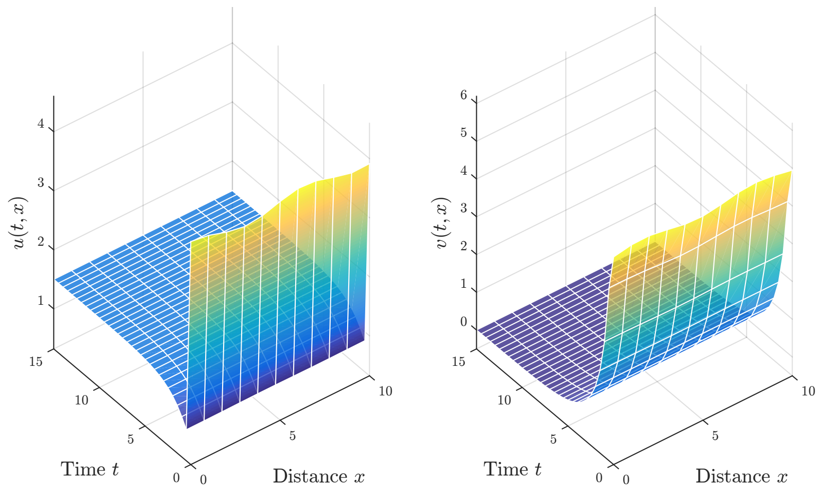


Figure 4.4: Numerical solutions of system (4.36) subject to the second set of parameters.

4.6.2 Second Example

The second numerical example which we are interested in the PDE extension of the ODE SIR model studied in [49], which is a special case of (4.1)–(4.3) with $\varphi(v) = \frac{\alpha v}{1+kv}, \alpha > 0$ and $k \geq 0$. The resulting system is described by

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u = -\lambda \frac{\alpha uv}{1+kv} + \Lambda - \mu u & \text{in } (0,\infty) \times \Omega, \\ \dfrac{\partial v}{\partial t} - d_2 \Delta v = \lambda \frac{\alpha uv}{1+kv} - \sigma v & \text{in } (0,\infty) \times \Omega, \\ u(0,x) = u_0(x), \ \ v(0,x) = v_0(x) & \text{on } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & \text{on } (0,\infty) \times \partial\Omega. \end{cases} \tag{4.37}$$

The imposed conditions (4.4) and (4.5) can be easily verified. It is evident that

$$\varphi(0) = 0 \text{ and } \varphi(v) > 0 \text{ for all } v > 0.$$

Also, the derivative of $\varphi(v)$ is given by

$$\begin{aligned} \varphi'(v) &= \left(\frac{\alpha v}{1+kv}\right)' = \frac{\alpha(1+kv) - k\alpha v}{(1+kv)^2} \\ &= \frac{\alpha}{(1+kv)^2} > 0 \text{ and } \varphi'(0) = \alpha. \end{aligned}$$

In addition, we have

$$v\varphi'(v) = v\frac{\alpha}{(1+kv)^2} \leq \frac{\alpha v}{1+kv} = \varphi(v).$$

The constant steady states of system (4.37) are

$$E_0 = \left(\frac{\Lambda}{\mu}, 0\right) \quad \text{and} \quad E^* = \left(\frac{\sigma v^*}{\lambda \varphi(v^*)}, v^*\right) = \left(\frac{\sigma v^*}{\lambda \frac{\alpha v^*}{1+kv^*}}, v^*\right) = \left(\frac{\sigma(1+kv^*)}{\lambda \alpha}, v^*\right),$$

Table 4.2: Simulation parameters for example 2: system (4.37).

	Set	Figure	u_0	v_0	d_1	d_2	λ	α	σ	Λ	μ	k	R_0
ODE case	Set 1	4.5(a)	0.2	4.3	–	–	$\frac{7}{12}$	$\frac{13}{4}$	$\frac{9}{4}$	$\frac{33}{4}$	$\frac{5}{4}$	$\frac{1}{2}$	5.5611
	Set 2	4.5(b)	0.2	4.3	–	–	2	$\frac{1}{3}$	1	5	4	7	0.8333
	Set 3	4.6(a)	8	10	–	–	$\frac{7}{12}$	$\frac{13}{4}$	$\frac{9}{4}$	$\frac{33}{4}$	$\frac{5}{4}$	$\frac{1}{2}$	5.5611
	Set 4	4.6(b)	8	10	–	–	2	$\frac{1}{3}$	1	5	4	7	0.8333
PDE case	Set 1	4.7	$0.2+\frac{\cos(x)}{10}$	$0.6+\frac{\sin(x)}{10}$	3	2	$\frac{7}{12}$	$\frac{13}{4}$	$\frac{9}{4}$	$\frac{33}{4}$	$\frac{5}{4}$	$\frac{1}{2}$	5.5611
	Set 2	4.8	$4+\frac{\cos(x)}{10}$	$5+\frac{\sin(x)}{10}$	3	2	$\frac{7}{12}$	$\frac{13}{4}$	$\frac{9}{4}$	$\frac{33}{4}$	$\frac{5}{4}$	3	5.5611
	Set 3	4.9	$0.2+\frac{\cos(x)}{10}$	$0.6+\frac{\sin(x)}{10}$	3	2	2	$\frac{1}{3}$	1	5	4	$\frac{2}{3}$	0.8333
	Set 4	4.10	$0.2+\frac{\cos(x)}{10}$	$0.6+\frac{\sin(x)}{10}$	$\frac{7}{2}$	$\frac{5}{4}$	2	$\frac{1}{3}$	1	5	4	7	0.8333

where $v^* = \mu\frac{(R_0-1)}{(\lambda\alpha+k\mu)}$.

Again, E^* exists and is globally asymptotically stable provided that the reproduction number $R_0 = \frac{\lambda\Lambda}{\mu\sigma}\varphi'(0) = \frac{\lambda\Lambda}{\mu\sigma}\alpha > 1$. On the other hand, E_0 is globally asymptotically stable in the ODE case with no conditions and in the PDE case if

$$\frac{\mu}{\Lambda}\left(\frac{\mu+\sigma}{\lambda\alpha} - \frac{\Lambda}{\sigma}\right) \geq \theta \geq \frac{(d_1+d_2)^2}{4d_1d_2},$$

when $d_1 \neq d_2$, and to $\theta = 1$ when $d_1 = d_2$ or in the ODE case. Table 4.2 shows the parameter sets used in the numerical simulations.

The following is a description of the results:

- Figures 4.5 and 4.6 depict the numerical solutions obtained using the four parameter sets in the ODE case with steady states $E^* = (2.5289, 2.2617)$, $E_0 = (1.25, 0)$, $E^* = (2.5289, 2.2617)$, and $E_0 = (1.25, 0)$, respectively. In all four scenarios, both the analytical and numerical methods agree that the equilibrium's are stable.

- Figure 4.7 shows the PDE solution obtained using parameter set 1 with $E^* = (2.5289, 2.2617)$. In this case, $R_0 = 5.5611 > 1$ and by Theorem 24, E^* is globally asymptotically stable.

- Figure 4.8 shows the PDE solution obtained using parameter set 2 with $E^* = (4.7823, 1.0098)$. Since $R_0 = 5.5611 > 1$, E^* is globally asymptotically stable.

- Figure 4.9 shows the PDE solution obtained using parameter set 3 with $E_0 = (1.25, 0)$. In this case, $R_0 = 0.8333 < 1$ and using Theorem 23 with $\theta \in \left[\frac{25}{24}, 2\right]$, E_0 is globally asymptotically stable.

- Figure 4.10 shows the PDE solution obtained using parameter set 4 with $E_0 = (1.25, 0)$. In this scenario, we again have $R_0 = 0.8333 < 1$ and with $\theta \in \left[\frac{361}{280}, 2\right]$, E_0 is globally asymptotically stable.

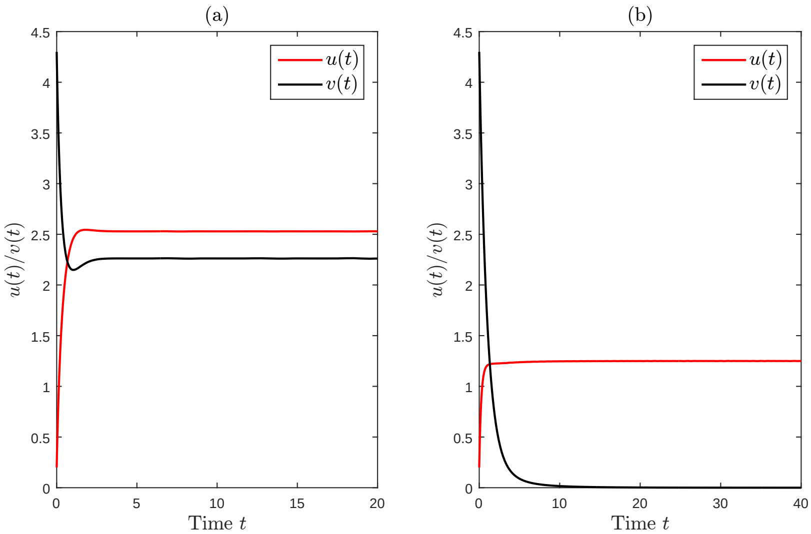


Figure 4.5: Numerical solutions of system (4.37) (ODE case) subject to the first and second sets of parameters.

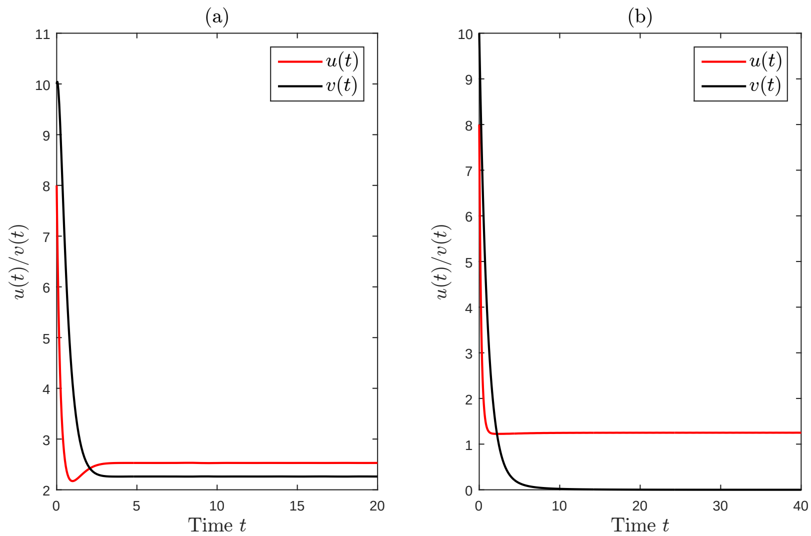


Figure 4.6: Numerical solutions of system (4.37) (ODE case) subject to the third and fourth sets of parameters.

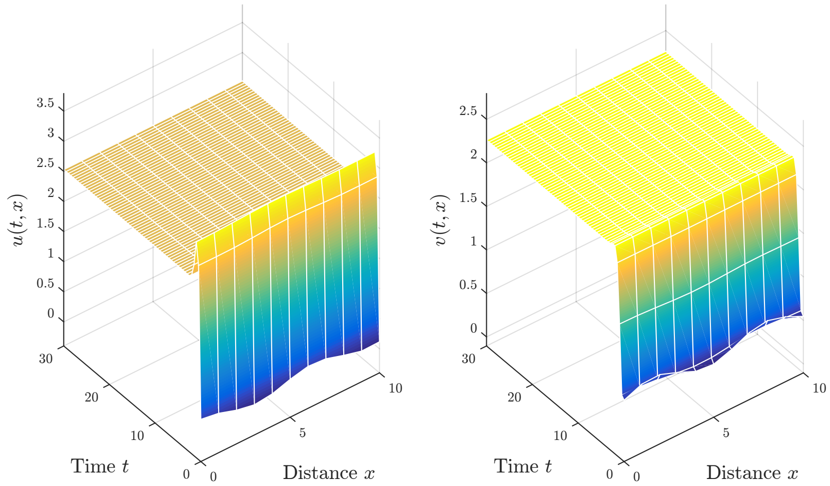


Figure 4.7: Numerical solutions of system (4.37) subject to the first set of parameters.

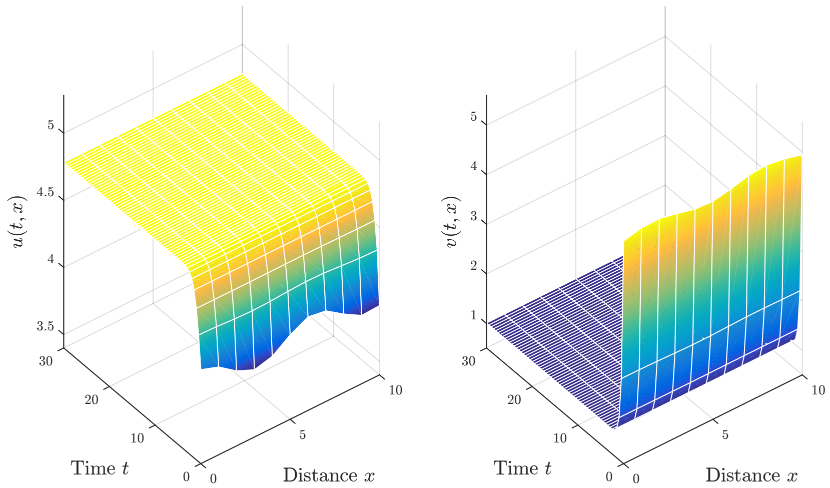


Figure 4.8: Numerical solutions of system (4.37) subject to the second set of parameters.

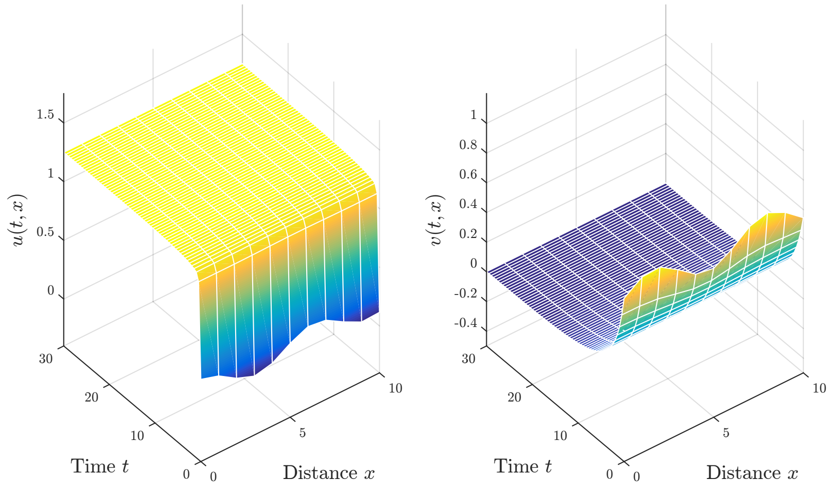


Figure 4.9: Numerical solutions of system (4.37) subject to the third set of parameters.

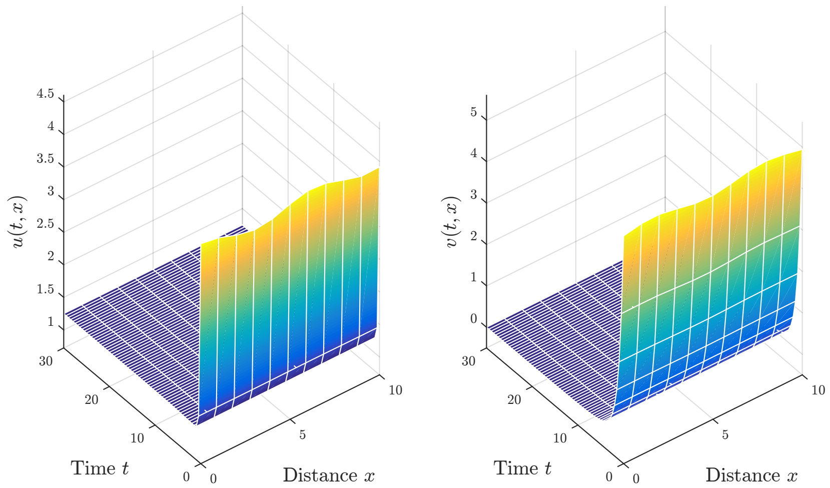


Figure 4.10: Numerical solutions of system (4.37) subject to the fourth set of parameters.

Table 4.3: Simulation parameters for example 3: system (4.37).

	Set	Figure	u_0	v_0	d_1	d_2	λ	α	σ	Λ	μ	k	R_0
ODE case	Set 1	4.11(a)	0.8	1.2	–	–	1	2	3	6	$\frac{1}{3}$	2	12
	Set 2	4.11(b)	0.4	6	–	–	$\frac{1}{2}$	1	2	$\frac{3}{4}$	$\frac{3}{7}$	$\frac{4}{3}$	0.5833
	Set 3	4.12(a)	0.2	4	–	–	1	2	3	8	$\frac{2}{3}$	2	8
	Set 4	4.12(b)	0.2	3	–	–	$\frac{1}{2}$	1	2	$\frac{3}{5}$	$\frac{3}{7}$	$\frac{6}{5}$	0.42
PDE case	Set 1	4.13	$4+\frac{\cos(x)}{10}$	$5+\frac{\sin(x)}{10}$	3	$\frac{5}{4}$	1	2	3	6	$\frac{1}{3}$	2	12
	Set 2	4.14	$0.6+\frac{\cos(x)}{7}$	$0.4+\frac{\sin(x)}{8}$	5	2	1	2	3	6	$\frac{1}{3}$	2	12
	Set 3	4.15	$2.6+\frac{\cos(x)}{7}$	$2.4+\frac{\sin(x)}{8}$	2	1	1	2	3	8	$\frac{2}{3}$	2	8
	Set 4	4.16	$4+\frac{\cos(x)}{10}$	$5+\frac{\sin(x)}{10}$	3	$\frac{5}{4}$	$\frac{1}{2}$	1	2	$\frac{3}{4}$	$\frac{3}{7}$	$\frac{4}{3}$	0.5833
	Set 5	4.17	$0.6+\frac{\cos(x)}{7}$	$0.4+\frac{\sin(x)}{8}$	$\frac{13}{4}$	2	$\frac{1}{2}$	1	2	$\frac{3}{5}$	$\frac{3}{7}$	$\frac{6}{5}$	0.42

4.6.3 Third Example

The last example is obtained by substituting $\varphi(v) = \frac{kv}{1+\left(\frac{v}{\alpha}\right)}$, which yields the same system studied in [14], [51] but with $d_1 = d_2 = 0$. The resulting system is given by

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1\Delta u = -\lambda k\frac{v}{1+\left(\frac{v}{\alpha}\right)}u + \Lambda - \mu u & \text{in } (0,\infty)\times\Omega, \\ \dfrac{\partial v}{\partial t} - d_2\Delta v = \lambda k\frac{v}{1+\left(\frac{v}{\alpha}\right)}u - \sigma v & \text{in } (0,\infty)\times\Omega, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x) & \text{on } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & \text{on } (0,\infty)\times\partial\Omega, \end{cases} \tag{4.38}$$

For $\alpha > 0$ and $k > 0$. The imposed conditions may be verified as

$$\begin{cases} \varphi(0) = 0, \text{ and } \varphi(v) > 0 \text{ for all } v > 0, \\ \varphi'(v) = \left(\frac{kv}{1+\left(\frac{v}{\alpha}\right)}\right)' = \frac{k}{\left(1+\left(\frac{v}{\alpha}\right)\right)^2} > 0, \quad \varphi'(0) = k, \\ v\varphi'(v) = v\frac{k}{\left(1+\left(\frac{v}{\alpha}\right)\right)^2} \leq \frac{kv}{1+\left(\frac{v}{\alpha}\right)} = \varphi(v). \end{cases}$$

The steady states of system (4.38) are given by $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$ and $E^* = \left(\frac{\sigma v^*}{\lambda\varphi(v^*)}, v^*\right) = \left(\frac{\sigma(\alpha+v^*)}{\lambda\alpha k}, v^*\right)$, where $v^* = \mu\alpha\frac{(R_0-1)}{(\lambda k\alpha+\mu)}$ with the reproduction number $R_0 = \frac{\lambda\Lambda}{\mu\sigma}\varphi'(0) = \frac{\lambda\Lambda}{\mu\sigma}k > 1$. Note that if E^* exists than it is globally asymptotically stable and that E_0 is globally asymptotically stable if

$$\frac{\mu}{\Lambda}\left(\frac{\mu+\sigma}{\lambda\alpha} - \frac{\Lambda}{\sigma}\right) \geq \theta \geq \frac{(d_1+d_2)^2}{4d_1d_2}$$

when $d_1 \neq d_2$, and if $\theta = 1$ when $d_1 = d_2$ or in the ODE case. Table 4.3 details the sets of parameters used in the numerical simulations.

The following is a description of the results:

- Figures 4.11 and 4.12 show the numerical solutions of system (4.38) resulting the four parameter sets in the ODE case with steady states $E^* = (2.7692, 1.6923)$, $E_0 = (1.75, 0)$, $E^* = (3, 2)$, and $E_0 = (1.4, 0)$, respectively. In all four scenarios, both the analytical and numerical methods agree that the equilibrium's are stable.

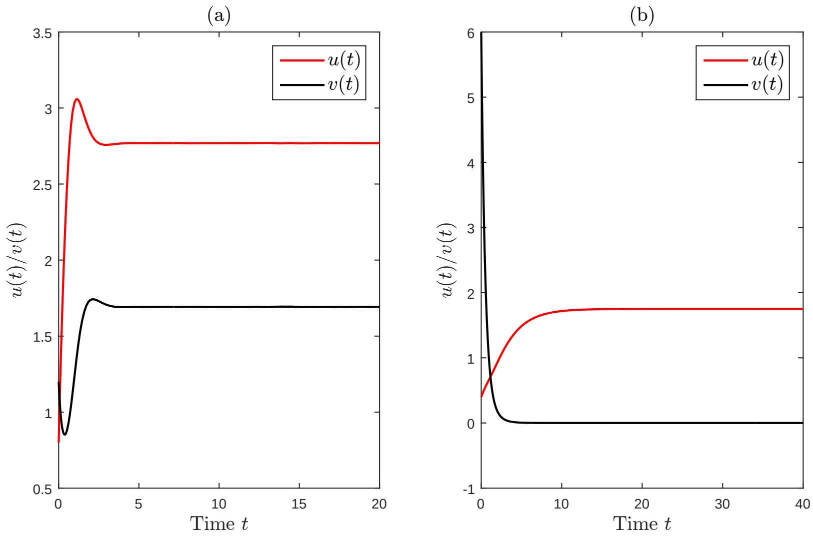


Figure 4.11: Numerical solutions of system (4.38) (ODE case) subject to the first and second sets of parameters.

- Figure 4.13 shows the PDE solution obtained using parameter set 1 with $E^* = (2.7692, 1.6923)$. In this case, $R_0 = 12 > 1$ and by Theorem 24, E^* is globally asymptotically stable.
- Figure 4.14 shows the PDE solution obtained using parameter set 2 with $E^* = (2.7692, 1.6923)$. In this case, $R_0 = 12 > 1$ and E^* is globally asymptotically stable.
- Figure 4.15 shows the PDE solution obtained using parameter set 3 with $E^* = (3, 2)$. In this case, $R_0 = 8 > 1$ and E^* is globally asymptotically stable.
- Figure 4.16 shows the PDE solution obtained using parameter set 4 with $E_0 = (1.75, 0)$. In this case, $R_0 = 0.5833 < 1$ and by Theorem 23 with $\theta \in \left[\frac{289}{240}, \frac{394}{211}\right]$, E_0 is globally asymptotically stable.
- Figure 4.17 shows the PDE solution obtained using parameter set 5 with $E_0 = (1.4, 0)$. In this case, $R_0 = 0.42 < 1$ and by Theorem 23 with $\theta \in \left[\frac{441}{416}, \frac{1831}{684}\right]$, E_0 is globally asymptotically stable.

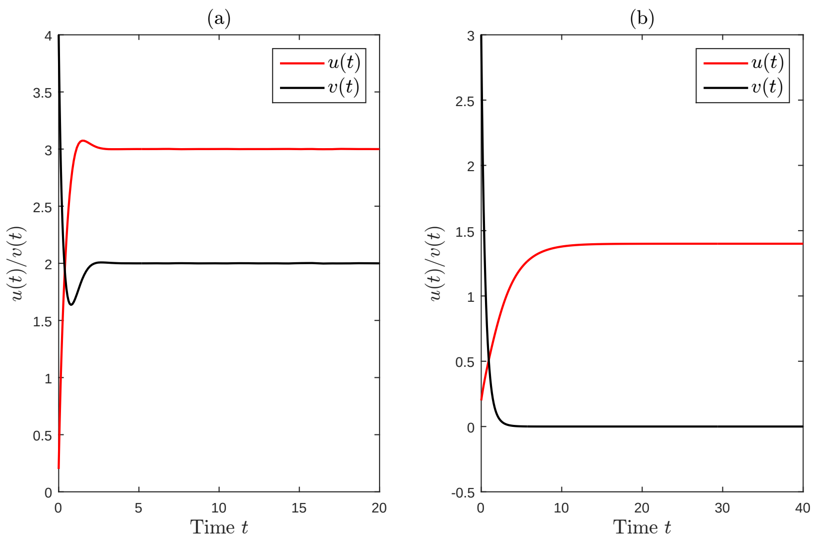


Figure 4.12: Numerical solutions of system (4.38) (ODE case) subject to the third and fourth sets of parameters.

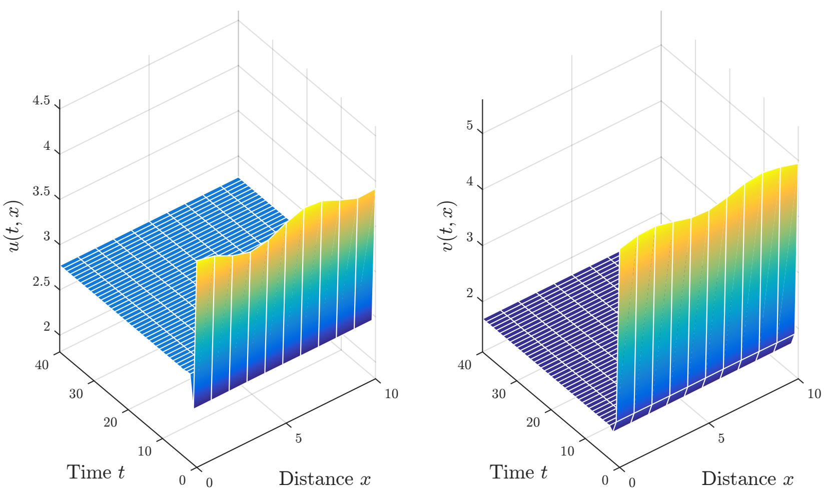


Figure 4.13: Numerical solutions of system (4.38) subject to the first set of parameters.

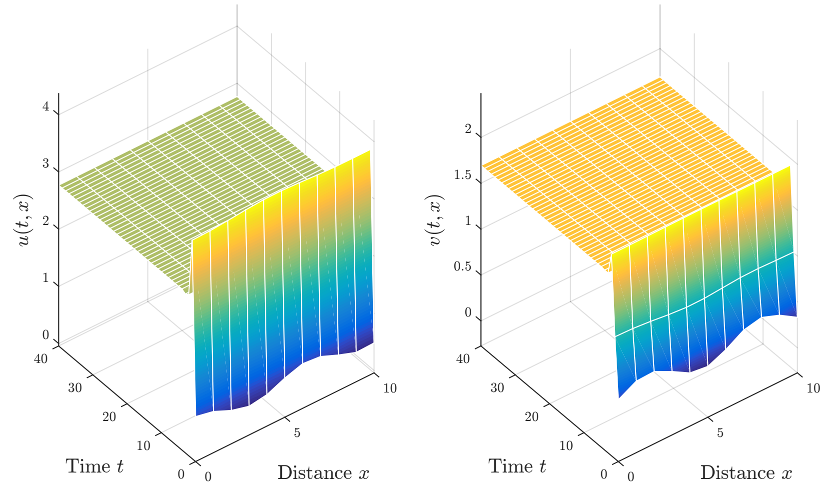


Figure 4.14: Numerical solutions of system (4.38) subject to the second set of parameters.

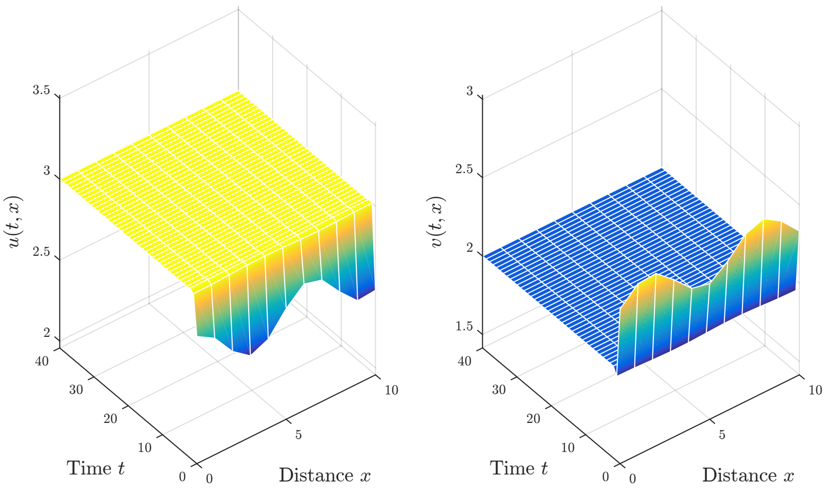


Figure 4.15: Numerical solutions of system (4.38) subject to the third set of parameters.

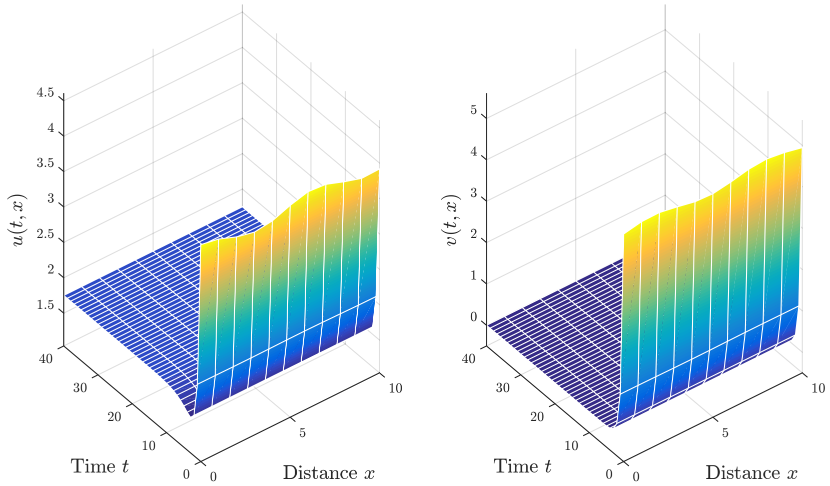


Figure 4.16: Numerical solutions of system (4.38) subject to the forth set of parameters.

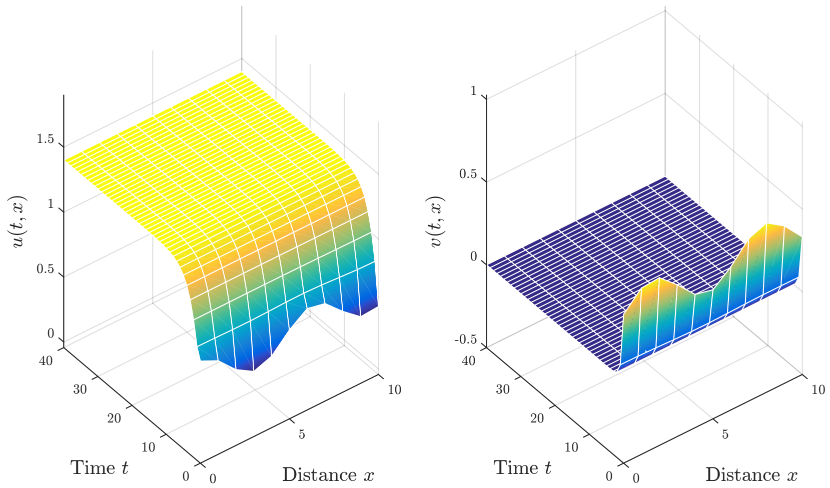


Figure 4.17: Numerical solutions of system (4.38) subject to the fifth set of parameters.

General Conclusion

This thesis is devoted to study a general HIV/AIDS epidemic model with general nonlinear incidence rate, which describes the diffusion of an epidemic in a population or chemical situations. Rigorous mathematical results have been established for the model.

Our first contribution is devoted to the proof of the global existence and boundedness of solutions to a coupled reaction-diffusion systems by using the theory of Lyapunov function. We also give the result of asymptotic behaviour of solutions to a particular case of our reaction-diffusion systems.

Our second contribution in this work is to derive sufficient conditions for the local asymptotic stability of epidemic reaction-diffusion system by means of the stability theory in ODE and PDE cases with a general incidence function $u\varphi(v)$ that includes the bilinear case uv (studied by [41, 19]) and a family of saturated incidence rate of the form $\lambda\frac{\alpha uv}{1+kv}$ and $\lambda k\frac{v}{1+\left(\frac{v}{\alpha}\right)}u$ (studied by [49], [14], [51]), where if we assuming that the incidence function φ satisfies $\varphi(0) = 0$ and $0 < v\varphi'(v) \leq \varphi(v)$ for all $v > 0$, it is proved that the model always has one disease free equilibrium point E_0 locally asymptotically stable when $R_0 < 1$ and is unstable if $R_0 > 1$. Further, we poved that the model has an endemic equilibrium E^* when $R_0 > 1$ and is locally asymptotically stable.

The basic reproductive number R_0 is calculated by use of the method of the next generating matrix. Also by applying Lyapunov functional we derive conditions for the global asymptotic stability of the proposed model in ODE and PDE case, where the condition $0 < v\varphi'(v) \leq \varphi(v)$ holds, if $R_0 < 1$, then E_0 is a globally asymptotically stable disease-free equilibrium for the model under the condition $\varphi'(0) \leq \frac{\mu+\sigma}{\lambda\left(\theta\frac{\Delta}{\mu}+\frac{\Delta}{\sigma}\right)}$, $\theta \geq \frac{(d_1+d_2)^2}{4d_1d_2}$. If $R_0 > 1$ assuming that u_0, $v_0 > 0$ and (4.5) holds, then E^* is a globally asymptotically stable endemic steady-state. It is important to note that, the model in the ODE case if $R_0 < 1$, the disease-free equilibrium E_0 is globally asymptotically stable. Finaly we present some numerical examples.

Open problems

In this section we discuss about some open and interesting problems.

- It would be interesting to study the behavior of system at $R_0 = 1$ is, in order to complete the analysis that we have made.We could not obtain any results about the local and global asymptotic stability of the solutions to the problem studied in the fourth chapter in the case where $R_0 = 1$. It could be an area of our future research.
- Another interesting topic is to ask about the global existence and stability of solutions for the Time-Fractional Epidemic Reaction-Diffusion System.

- It would be interesting to consider a family of incidence rates more generally, changing $u\varphi(v)$ by $f(u,v)$ and trying to obtain results of stability similar to the ones obtained in this work.

- It would be interesting to study global existence and stability of solutions for an epidemic model with three or more equations.

- We propose use of treatment control by using of the optimal control technique.

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