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Numerical solutions of the heat equation via Crank-Nicolson difference method

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Abstract

Numerical partial differential equations is the branch of numerical analysis that studies the numerical solution of partial differential equations (PDEs). In this work we describe the Finite Difference Methods and some of its applications. We focus on three methods and apply to the heat equation:(Explicit Forward difference method, Implicit Backward difference method and Crank-Nicolson method).

Résumé

Les équations aux dérivées partielles numériques sont la branche de l'analyse numérique qui étudie la solution numérique des équations aux dérivées partielles (EDP). Dans ce travail, nous décrivons la méthode des différences finies et certaines de ses applications. Nous nous concentrons sur trois méthodes et appliquons à l'équation de la chaleur:(méthode de différence en avant explicite, méthode de différence en arrière implicite et méthode de Crank-Nicolson).

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At the end of this work, I thank Allah who gave me the strength and the patience to accomplish this modest work.

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Finally, thank you to the people who participated directly or indirectly in the realization of this work.

Dedication

I dedicate this modest work:

To my dear parents.To my sisters and my brother. Nothing in the world is worth the effort put in day and night for our education and well-being. This work is the result of the sacrifices you have made for our. Thank you.

Contents

Introduction

An ordinary differential equation (ODE) is an equation that involves an unknown function (the dependent variable) and some of its derivatives with respect to a single independent variable. An nth-order equation has the highest order derivative of order n:

$$f(x, y, y', y'', \dots, y^{(n)}) = 0, a \leq x \leq b$$

where $y = y(x)$, and $y(n)$ denotes the nth derivative with respect to x. An nth-order ODE requires the specification of n conditions to assure uniqueness of the solution. If all conditions are imposed at $x = a$, the conditions are called initial conditions (I.C.), and the problem is an initial value problem (IVP). If the conditions are imposed at both $x = a$ and $x = b$,the conditions are called boundary conditions (B.C.), and the problem is a boundary value problem (BVP).

For example, consider the initial value problem

$$\begin{cases} y' = ky, \\ y(0) = y_0 \in \mathbb{R}. \end{cases}$$

A partial differential equation (PDE) is an equation that involves an unknown function (the dependent variable) and some of its partial derivatives with respect to two or more independent variables. An nth-order equation has the highest order derivative of order n. For example

$$u_{xx} + u_{yy} = f(x, y),$$

which is of second order. Numerical techniques for solving Partial differential equations are several and various, we site as examples: Finite difference method, Finite element method, Method of lines Gradient discretization Method, Finite volume method, Spectral method, Meshfree method, Domain decomposition methods, Multigrid methods and Comparison of methods. We shall focus on the first method.

Our work is organized as follows: In the first chapter, we give some definitions of the Partial differential equation and some of theirs related topics, we shall delineate theirs classification as three types of field equations, namely hyperbolic, parabolic and elliptic and consolidate by some examples, we also know Hibert space and semigroups, as well as Hille Yosida's theorem. In the second chapter, we present two analytical techniques for solving Partial differential equations: the separation of variables and the Fourier transformation and we shall focus on the heat equation. In the last chapter, we shall describe one of the simplest and of the oldest methods to solve ordinary and partial differential equation. It is known under the name " finite difference method" and based on approximations for derivatives. We describe three methods: the Explicit Forward, the Implicit Backward and the Crank-Nicolson methods which we apply all to the heat equation and prove the stability condition of the first one, as the above methods lead, generally to algebraic linear systems with Tridiagonal principal matrices, some techniques are presented to solve such systems.Finally,we shall focus in the Crank-Nicolson method by giving more examples.

Chapter 1

Preliminaries

1.1 Partial differential equation

If we well talk about partial differential equation then we must start this section with some definitions about (Partial differential equations).

Definition 1.1.1 *A partial differential equation (PDE) is an equation containing partial derivatives of the dependent variable with two or more independent variables.*

Example 1.1.1 *The simple three-dimensional heat equation is in the form*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{c\rho}{k}\frac{\partial u}{t}$$

Definition 1.1.2 *The order of a partial differential equation is the order of the highest order derivative in the equation.*

Example 1.1.2 *The equation*

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

is of first order, and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f$$

is of second order.

Definition 1.1.3 *A partial differential equation is linear if it is linear in the unknown function and all its derivatives with coefficient depending only on the independent variables.*

Example 1.1.3 *The following PDE is linear*

$$\alpha(x,y)\tfrac{\partial^2 u}{\partial x^2} + 2\tfrac{\partial^2 u}{\partial x \partial y} + 3x^2\tfrac{\partial^2 u}{\partial y^2} = 4e^x$$

Definition 1.1.4 *A partial differential equation is called homogeneous if the equation does not contain a term independent of the unknown function and its derivatives.*

Example 1.1.4 *In general form the next PDE is homogeneous ⇔ G(x,y)=0 otherwise the PDE is called non homogeneous*

$$A(x,y)\tfrac{\partial^2 u}{\partial x^2} + B(x,y)\tfrac{\partial^2 u}{\partial x \partial y} + C(x,y)\tfrac{\partial^2 u}{\partial y^2} + D(x,y)\tfrac{\partial u}{\partial x} + E(x,y)\tfrac{\partial u}{\partial y} + F(x,y)u = G(x,y).$$

1.2 Classification of the second order linear partial differential equation in two variables

Consider the general, second-order, linear partial differential equation in two variables

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \tag{1.1}$$

where the coefficient are functions of the independent variables x and y (i.e., A = A(x,y), B = B(x,y), etc.), and we have used subscripts to denote partial derivatives, e.g.,

$$\tfrac{\partial^2 u}{\partial x^2} = u_{xx}$$

The quantity $\Delta = B^2 - 4AC$ *is referred to be as the discriminant of the equation.The behavior of the solution of equation (1.1) depends on the sign of the discriminant such that:*

- *The equation is said* ***elliptic*** $\Leftrightarrow \Delta < 0$

- *The equation is said* ***parabolic*** $\Leftrightarrow \Delta = 0$

- *The equation is said* ***hyperbolic*** $\Leftrightarrow \Delta > 0$

1. *As example of the elleptic case, whene we take*

$$A = C = 1 \text{ and } B = 0;$$

which give

$$\Delta = -4 < 0,$$

we find the poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g,$$

which when g = 0, it is called Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

2. *For*

$$A = -\alpha^2 \text{ and } B = C = 0,$$

We have

$$\Delta = 0$$

we find the heat equation

$$\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

which is parabolic.

3. *When we take*

$$A = 1,\ C = -\frac{1}{c^2} \text{ and } B = 0;$$

we have

$$\Delta = \frac{4}{c^2} > 0,$$

we find the following equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial y^2} = 0$$

which it is hyperbolic.

1.3 Hilbert space

Definition 1.3.1 *(Inner product): An inner product on a real linear space X is a map*

$$(.,.) : X \times X \to \mathbb{R}$$

Such that, for all $x, y, z \in X$ *and* $\lambda, \mu, \in \mathbb{R}$

a) *(x, λy + μz)= λ(x,y)+ μ(x,z) (linear in the second argument);*

b) *(x,y)=(y,x)*

c) *(x,x)=0 if and only if x =0 (positive definitel);*

d) *(x,x) $\geqslant$ 0 (non negative);*

We call a linear space with an inner product an inner product space or pre-Hilbert space.

Definition 1.3.2 *A Hilbert space is a complete inner product space. Complete is all convergent sequence is a Cauchy sequence.*

1.4 Semigroups

Definition 1.4.1 *Let X be a Banach space. A one parameter family T(t)* $0 \leqslant t < \infty$*, of bounded linear operators from X into X is a semigroups bounded linear operator on X if*

$T(0) = I$*, (I is the identity operator on X).*

$T(s+t) = T(s)T(t)$ *for every* $t, s \leq 0$ *(the semigroup property).*

A semigroup of bounded linear operator, T(t) is uniformly continuous $\lim\limits_{t\to 0} \|T(t) - I\| = 0$

The linear operator A defined by $D(A) = \{x \in X : \lim\limits_{t\to 0} \dfrac{T(t)x - x}{t} exists\}$

and $Ax = \lim\limits_{t\to 0} \dfrac{T(t)x - x}{t}$ *for* $x \in D(A)$*[2]*

Definition 1.4.2 *Let* $A : D(A) \subset H \to H$ *(H is a Hilbert) linear. A is monotone if* $\forall x \in D(A) :< Au, u >\geqslant 0$.

Ais maximal if $\exists \mu_0 \geqslant 0, \forall f \in H, \exists \mu : (\mu_0 + A)\mu = f$

Théorème 1.4.1 *(Hille Yosida- Hilbert space case-)*

Let A be a monotone maximal operator in H (H is a Hilbert).

$\Rightarrow \forall u_0 \in D(A), \exists! u \in C^1(0, +\infty, H) \cap C(0, +\infty, D(A))$ *is solution cauchy problem (P):*

$$(P) = \begin{cases} u_t + Au = 0 \\ u(0,x) = u_0(x) \end{cases}$$

And we have $|u(t,x)| \leqslant \|u_0(x)\|, forall\ t \geqslant 0$

$|u_t| \leqslant |Au_0|$.

Chapter 2

Analytical solutions to the heat equation

Before addressing the solutions to the heat equation, we will first prove the existence and uniqueness of the solution.

Let be the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u(t,x) = 0, (t,x) \in Q \\ u(t,x) = 0 on \Gamma = \partial Q \\ u(0,x) = u_0(x) \end{cases}$$

that $Q =]0, +\infty[\times \Omega$

We pose $A = -\Delta$ *and* $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$ *and* $H = L^2(\Omega)$ *muni the norm* $\|f\|^2_{L^2(\Omega)} = \int_\Omega |f|^2$

1- We show that A is monotone:

A is monotone if $\forall u \in D(A), < Au, u >_H \geqslant 0$ *Let* $u \in D(A)$

$$< Au, u >_{L^2(\Omega)} = < -\Delta u, u >_{L^2(\Omega)} = -\int_\Omega \Delta u u$$

We use Green's formula

$$-\int_\Omega \Delta u u = \int_\Omega \nabla u \nabla u - \int_{\partial\Omega} \nabla u \vec{\eta}$$

we have $u = 0$ *on* $\partial\Omega$

$$\Rightarrow < Au, u >_{L^2(\Omega)} = \int_\Omega \nabla u \nabla u = \int_\Omega \nabla u^2 = \|\nabla u\|^2_{L^2(\Omega)} \geqslant 0$$

$$\Rightarrow A \text{ is monotone}$$

2- we show that A is maximal:

A is maximal if $\forall\ f \in L^2(\Omega),\ \exists u \in D(A),\ Au{=}f$

$$-\Delta u\ = f \in L^2(\Omega) \tag{*}$$

We have $(*)$ *admits a unique solution* $u \in H_0^1(\Omega)$ *(according to the Lax-Milgram theorem).*

As

$$f \in L^2(\Omega) \Rightarrow -\Delta u \in L^2(\Omega) \Rightarrow u \in H^2(\Omega)$$

because

$$H^2(\Omega)\ = \{u \in L^2(\Omega), u' \in L^2(\Omega), u'' \in L^2(\Omega)\}$$

$$\Rightarrow u \in H_0^1(\Omega) \cap H^2(\Omega)\ = D(A)$$

$$\Rightarrow A \text{ is maximal.}$$

Conclusion:

A is maximal and monotone so according to Hille-Yosida

$$\Rightarrow \exists! u \in C^1(0, +\infty, H) \cap C(0, +\infty, D(A)$$

2.1 Solving heat equation using separation variables

In mathematics, separation of variables (also known as the Fourier method) is any of several methods for solving ordinary and partial differential equations, in which algebra allows one to rewrite an equation so that each of two variables occurs on a different side of the equation. The method of separation of variables is also used to solve a wide range of linear partial differantial equations with boundary and initial conditions, such as the heat equation, wave equation, Laplace *equation, etc.*

The analytical method of separation of variables for solving partial differential equations has also been generalized into a computationel method of decomposition in invariant structures that can be used to solve systems of partial differential equations.

Example 2.1.1 *Consider the one-dimensional heat equation .*

The equation is:

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \qquad \forall x \in [0, L], t >, 0 \tag{2.1}$$

and the boundary condition is homogeneous, that is :

$$u(0, t) = u(L, t) = 0 \quad for\ all\ t > 0 \tag{2.2}$$

with initial condition: $u(x, 0) = \varphi 0 \leqslant x \leqslant L$ *we find the solution on the following form :*

$$u(x, t) = g(x)f(t) \tag{2.3}$$

substituting u *back into equation (2.1)*

$$g(x)f'(t) = \alpha g''(x)f(t)$$

we divided by $g(x)f(t)$ *we get*

$$\frac{f'(t)}{\alpha f(t)} = \frac{g''(x)}{g(x)} \tag{2.4}$$

Since the right hand side depends only on x *and the left hand side only on* t*, both sides are equal to some constant value* $(+\lambda)$ *thus:*

$$f'(t) = +\lambda \alpha f(t) \tag{2.5}$$

and

$$g''(x) = +\lambda g(x) \tag{2.6}$$

$(+\lambda)$ here is the eigenvalue for both differential operators, and $f(t)$ and $g(x)$ are corresponding eigen functions.

Integrating equation (2.5), we get

$$f(t) = Ce^{+\lambda\alpha t} \qquad , C = cte$$

we will now show that solutions for $g(x)$ for values of λ

- *Suppose that $\lambda > 0$ and we pose $\lambda = \omega^2 > 0$ so the equation (2.6) becomes*

$$g''(x) - \omega^2 g(x) = 0 \tag{2.7}$$

The homogeneous second order equation with constant coefficients :

The characteristics equation :

We pose: $g''(x) = m^2$

so

$$m^2 - \omega^2 = 0$$

so

$$m = \omega \ , \ m = -\omega$$

$$g(x) = C_1' e^{\omega x} + C_2' e^{-\omega x}$$

or

$$g(x) = C_2 ch(\omega x) + C_3 sh(\omega x)$$

so

$$u(x,t) = g(x)f(t)$$

$$u(x,t) = (C_2ch(\omega x) + C_3sh(wx))Ce^{\omega^2\alpha t}$$

$$u(x,t) = e^{\omega^2\alpha t}[Ach(\omega x) + Bsh(\omega x)]$$

we get

$$u(x,t) = 0$$

$$e^{\omega^2\alpha t}[Ach(0) + Bsh(0)] = 0$$

and

$$e^{\omega^2\alpha t} > 0$$

so

$$Ach(0) = 0 \Rightarrow A = 0 \text{ because } ch(0) = 1$$

and $u(L,t) = 0$

$$e^{\omega^2 xt}[Ach(\omega L) + Bsh(\omega L)] = 0$$

so $Bsh(\omega L) = 0$

let get

$$B = 0 \text{ or } sh(\omega L) = 0$$

$$sh(\omega L) \neq 0 \text{ because } L \neq 0 \text{ and } \omega \neq 0 \text{ so } B = 0$$

$$u(x,t) = 0$$

- *suppose that* $\lambda < 0 \Rightarrow \lambda = -\omega^2$

  so

$$u(x,t) = e^{-\omega^2 \alpha t}[Acos(\omega x) + Bsin(\omega x)]$$

$$m^2 + \omega^2 = 0 \Rightarrow m^2 = -\omega^2$$

so

$$m = i\omega \, , m = -i\omega \in \mathbb{C}$$

we get

$$u(0,t) = 0$$

$$e^{-\omega^2 \alpha t}[A\cos(0) + B\sin(0)] = 0 \Rightarrow A = 0,$$

and we get

$$u(L,t) = 0$$

$$e^{-\omega^2 \alpha t}[A\cos(\omega L) + B\sin(\omega L)] = 0 \Rightarrow B\sin(\omega L) = 0$$

$$\textit{because } A = 0 \textit{ and } e^{-\omega^2 \alpha t} > 0.$$

If $B = 0$ $u(x,t)$

if $\sin(\omega L) = 0 \Rightarrow \omega L = n\pi \Rightarrow \omega = \dfrac{n\pi}{L},$

so the solution is $u(x,t) = Be^{(\frac{n\pi}{L})^2 \alpha t} \sin(\frac{n\pi}{L}x)$

so $u_n(x,t) = C_n e^{(\frac{n\pi}{L})^2 \alpha t} \sin(\frac{n\pi}{L}x)$

Recal that the principle of superposition admits that n linear combination of the fanction $u_n(x,t)$ *also satisfies the given equation and boundary condition .*

Therfore,using this principle gives the general solution by

$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{(\frac{n\pi}{L})^2 \alpha t} \sin(\frac{n\pi}{L}x) \tag{2.8}$$

where the arbitrary constants C_n,$n \geq 1$ *are still undetrmined .*

To determine B_n,$n \geq 1$,*we substitue* $t = 0$ *...(1.8) and using the ... intial we find*

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{L}x) = \varphi(x) \tag{2.9}$$

a Fourier series of the function (sin) *and* C_n *... Fourier coefficient. Functions propres* $\phi(x) = \sin(\frac{n\pi}{L}x)$ *are orthogonal in* $[0, L]$*:*

$$(\phi_n, \phi_m) = \int_0^L \sin(\tfrac{n\pi}{L}x)\sin(\tfrac{m\pi}{L}x)dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n, \end{cases}$$

so $C_n = \frac{2}{L}\int_0^L C(x)\sin(\frac{n\pi}{L}x)dx$.

The function defined by (2.8) satisfies (2.1).

2.2 Solving heat equation using Fourier transformation

The Fourier transform is one of the most powerful and fundamental tools in linear analysis, converting constant-coefficient linear differential operators into multiplication by polynomials.

Definition 2.2.1 *(Fourier transform) Let* $f \in L^1(\mathbb{R})$, *then we define the Fourier transform* $\mathcal{F}f$ *of* f

$$(\mathcal{F}f) = \widehat{f}(\omega) = \int_{-\infty}^{+\infty} f(x)e^{-i\omega x}dx,\ \forall\omega \in \mathbb{R}.$$

The operator $\mathcal{F}$ *is called the Fourier transform.*

This integral is defined since $|f(x)e^{-2\pi i\omega x}| = |f(x)|$ *and* $f \in L^1(\mathbb{R})$

Proposition 2.2.1 *Let* $f \in L^1(\mathbb{R})$,*and derivable and* $f' \in L^1(\mathbb{R})$,

$$\mathcal{F}(f'(x)) = \widehat{f}'(\omega) = i\omega\widehat{f}(\omega)$$

If $f \in C^\infty$ *and* $f^{(n)} \in L^1(\mathbb{R}$, *so*

$$\mathcal{F}(f^{(n)}(x)) = \widehat{f^{(n)}}(\omega) = (i\omega)^n\widehat{f}(\omega)$$

Proposition 2.2.2 *Let* $f,\ g \in L^1(\mathbb{R})$, *so* $f * g \in L^1(\mathbb{R})$, *and*

$$\mathcal{F}((f * g)(x))(\omega) = \widehat{f}(\omega).\widehat{g}(\omega)$$

Let us apply the proposition to the heat equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \alpha\frac{\partial^2 u(x,t)}{\partial x^2} \\ \\ u(x,0) = \varphi(x) \end{cases}$$

The Fourier transform

$$\mathcal{F}(\tfrac{\partial u(x,t)}{\partial t})(\omega) = \mathcal{F}(\alpha\tfrac{\partial^2 u(x,t)}{\partial x^2})(\omega)$$

And the Fourier transform is linear, so

$$\tfrac{\partial}{\partial t}\mathcal{F}(u(x,t))(\omega) = \alpha\mathcal{F}(\tfrac{\partial^2 u(x,t)}{\partial x^2})(\omega)$$

By application of the proposition 2.2.1

$$\mathcal{F}(\tfrac{\partial^2 u(x,t)}{\partial x^2})(\omega) = (2\pi i\omega)^2\mathcal{F}(u(x,t))(\omega)$$

We find

$$\Rightarrow \widehat{u}_t^{\,'}(\omega,t) = -4\pi^2\omega^2\alpha\widehat{u}(\omega,t)$$

$$\Rightarrow \widehat{u_t}^{\,'}(\omega,t) + -4\pi^2\omega^2\alpha\widehat{u}(\omega,t) = 0$$

Is a homogeneous equation, so

$$\Rightarrow \frac{\partial \widehat{u}(\omega,t)}{\widehat{u}(\omega,t)} = -4\pi^2\omega^2\alpha\partial t$$

$$\Rightarrow \ln|\widehat{u}(\omega,t)| = -4\pi^2\omega^2\alpha t + c$$

$$\Rightarrow \widehat{u}(\omega,t) = e^{-4\pi^2\omega^2\alpha t}.e^c$$

We pose $k = e^c$

$$\Rightarrow \widehat{u}(\omega,t) = ke^{-4\pi^2\omega^2\alpha t}$$

We have: $\widehat{u}(\omega,0) = ke^{-4\pi^2\omega^2\alpha 0} = \widehat{\varphi}(x)$

$$\Rightarrow k = \widehat{\varphi}(\omega)$$

$$\widehat{u}(\omega,t) = \widehat{\varphi}(\omega)e^{-4\pi^2\omega^2\alpha t}$$

$\mathcal{F}^{-1}(\widehat{u}(\omega,t)) = \mathcal{F}^{-1}(\widehat{\varphi}(\omega)) * \mathcal{F}^{-1}(e^{-4\pi^2\omega^2\alpha t})$

We calculate $\mathcal{F}^{-1}(e^{-4\pi^2\omega^2\alpha t})$

$$\mathcal{F}^{-1}(e^{-4\pi^2\omega^2\alpha t}) = \int_{\mathbb{R}} e^{-4\pi^2\omega^2\alpha t}.e^{2\pi i\omega x}d\omega$$

$$= \int_{\mathbb{R}} e^{2\pi(\omega ix - 2\pi\omega^2\alpha t)}d\omega$$

The typical shape of $(\omega ix - 2\pi\omega^2\alpha t)$ *where* $a = -2\pi\alpha t$, $b = ix$, $c = 0$, *is*

$-2\pi\alpha t[(\omega + \frac{ix}{-4\pi\alpha t})^2 + \frac{x^2}{16\pi^2\alpha^2t^2}]$

$$\Rightarrow \int_{\mathbb{R}} e^{-4\pi^2\alpha t[(\omega+\frac{ix}{-4\pi\alpha t})^2+\frac{x^2}{16\pi^2\alpha^2t^2}]}d\omega$$

$$= \int_{\mathbb{R}} e^{-4\pi^2\alpha t(\omega+\frac{ix}{-4\pi\alpha t})^2}.e^{-4\pi^2\alpha t(\frac{x^2}{16\pi^2\alpha^2t^2})}d\omega$$

$$= \int_{\mathbb{R}} e^{-4\pi^2\alpha t(\omega+\frac{ix}{-4\pi\alpha t})^2}.e^{\frac{x^2}{-4\alpha t}}d\omega$$

$$= e^{-\frac{x^2}{4\alpha t}}\sqrt{\frac{\pi}{4\pi^2\alpha t}} = \sqrt{\frac{1}{4\pi\alpha t}}e^{-\frac{x^2}{4\alpha t}}$$

$$\Rightarrow u(x,t) = \varphi(x) * \sqrt{\frac{1}{4\pi\alpha t}}e^{-\frac{x^2}{4\alpha t}}$$

So

$$u(x,t) = \sqrt{\frac{1}{4\pi\alpha t}}\int_{\mathbb{R}} \varphi(x)e^{-\frac{(x-y)^2}{4\alpha t}}dy$$

Chapter 3

Numerical solutions to the heat equation via the finite difference method

Suppose that k, c, ρ are functions of x, y and z and represent, respectively, the thermal conductivity, specific heat, and density of an isotropic the body at the point (x,y,z). Then the temperature, $u \equiv u(x, y, z, t)$*, in a body can be found by solving the partial differential equation*

$$\frac{\partial}{\partial x}(k\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial u}{\partial y}) + \frac{\partial}{\partial z}(k\frac{\partial u}{\partial z}) = c\rho\frac{\partial U}{t} \tag{3.1}$$

When k, c, ρ are constants, this equation is known as the simple three-dimensional heat equation and is expressed as

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{c\rho}{k}\frac{\partial U}{t} \tag{3.2}$$

If the boundary of the body is relatively simple, the solution to this equation can be found using Fourier series. In most situations where k, c, ρ are not constant or when the boundary is irregular, the solution to the partial differential equation must be obtained by approximation techniques. it means that the numerical solution is the study of approximation techniques for solving more *difficult problems that is hard to solve analytically taking into account the extent of possible errors, in these work we will consider the truncation error and compeer the approximate solution to the exact solution.[1]*

Definition 3.0.1 *The truncation error is the amount by which the solution of the differential equation fails to satisfy the approximate equation. [1]*

We present in this section the simplest of all approximation methods, the finite difference method. The idea is simply to approximate any derivative by a differential quotient using a Taylor expansion for t we have

$$u(x_i, t_j + k) = u(x_i, t_j) + ku_t(x_i, t_j) + \frac{k^2}{2} u_{tt}(x_i, \mu_j)$$

$$\Rightarrow u_t(x_i, t_j) = \frac{u(x_i,t_j+k) - u(x_i,t_j)}{k} - \frac{k}{2} u_{tt}(x_i, \mu_j);$$

where $\mu_j \in (t_j, t_{j+1})$ *now for* x *we will need in our work the second derivative approximation and again we will use Taylor expansion*

$$u(x_i + h, t_j) = u(x_i, t_j) + hu_x(x_i, t_j) + \frac{h^2}{2} u_{xx}(x_i, t_j) + \frac{h^3}{6} u_{xxx}(x_i, t_j) + \frac{h^4}{24} u_{xxxx}(\epsilon_i^+, t_j)$$

$$u(x_i - h, t_j) = u(x_i, t_j) - hu_x(x_i, t_j) + \frac{h^2}{2} u_{xx}(x_i, t_j) - \frac{h^3}{6} u_{xxx}(x_i, t_j) + \frac{h^4}{24} u_{xxxx}(\epsilon_i^-, t_j)$$

adding the tow formulas above we thus obtain

$$u_{xx}(x_i, t_j) = \frac{u(x_i+h,t_j) - 2u(x_i,t_j) + u(x_i-h,t_j)}{h^2} - \frac{h^2}{12} u_{xxxx}(\epsilon_i, t_j)$$

where $\epsilon_i \in (x_{i-1}, x_{i+1})$ *.In this section we will consider the numerical solution to a problem involving the heat equation*

$$u_t(x,t) - \alpha^2 u_{xx}(x,t) = 0 \tag{3.3}$$

The physical problem considered here concerns the flow of heat along a rod of length L which has a uniform temperature within each cross-sectional element. This requires the rod to be perfectly insulated on its lateral surface. The constant α is assumed to be independent of the position in the rod. It is determined by the heat-conductive properties of the material of which the rod is composed. One of the typical sets of constraints for a heat- ow problem of this type is to specify the initial heat distribution in the rod

$$u(x, 0) = f(x)$$

and to describe the behavior at the ends of the rod. For example, if the ends are held at constant temperatures U_1 and U_2, the boundary conditions have the form

$$u(0, t) = U_1 \text{ and } u(l, t) = U_2$$

If, instead, the rod is insulated so that no heat flows through the ends, the boundary conditions are

$$u_x(0, t) = 0 \text{ and } u_x(L, t) = 0$$

Then no heat escapes from the rod and in the limiting case the temperature on the rod is constant. The parabolic partial differential equation is also of importance in the study of gas diffusion; in fact, it is known in some circles as the diffusion equation. now we will consider the heat equation

$$u_t(x, t) = \alpha^2 u_{xx}(x, t); 0 < x < L, t > 0 \tag{3.4}$$

subject to the conditions

$$u(0, t) = u(L, t) = 0 \; t > 0$$

$$u(x, 0) = f(x) : 0 \leq x \leq 1$$

The approach we use to approximate the solution to this problem involves finite differences, first select an integer $m > 0$ and define the x-axis step size $h = \frac{L}{m}$. Then select a time step size k.

The grid points for this situation are (x_i, t_j), where $x_i = ih$, for $i = 0, 1, ..., m$, and $t_j = jk$ for $j = 0, 1, ...$ [1]

3.1 Explicit Forward difference method

3.1.1 Presentation of the method

We obtain the difference method using the Taylor series in t to form the difference quotient

$$u_t(x_i, t_j) = \frac{u(x_i, t_j + k) - u(x_i, t_j)}{k} - \frac{k}{2} u_{tt}(x_i, \mu_j) \tag{3.5}$$

for some $\mu_j \in (t_j, t_{j+1})$, and the Taylor series in x to form the difference quotient

$$u_{xx}(x_i, t_j) = \frac{u(x_i + h, t_j) - 2u(x_i, t_j) + u(x_i - h, t_j)}{h^2} - \frac{h^2}{12} u_{xxxx}(\epsilon_i, t_j) \tag{3.6}$$

where $\epsilon_i \in (x_{i-1}, x_{i+1})$. The parabolic partial differential equation (3.4) implies that at interior grid points $(x_i; t_j)$, for each $i = 1, 2, ..., m$ and $j = 1, 2, ...$ we have

$$u_{tt}(x_i, t_j) - \alpha^2 u_{xx}(x_i, t_j) = 0 \tag{3.7}$$

so the difference method using the difference quotients Equation (3.5) and Equation (3.6) is

$$\frac{\omega_{i,j+1} - \omega_{i,j}}{k} - \alpha^2 \frac{\omega_{i+1,j} - 2\omega_{i,j} + \omega_{i-1,j}}{h^2} = 0 \tag{3.8}$$

where $w_{i,j}$ approximates $u(x_i, t_j)$. The local truncation error for this difference equation is

$$\tau_{ij} = \frac{k}{2}u_{tt}(x_i, \mu_j) - \alpha^2\frac{h^2}{12}u_{xxxx}(\epsilon_i, t_j) \tag{3.9}$$

Solving Equation (3.8) for $w_{i,j+1}$ *gives*

$$\omega_{i,j+1} = (1 - \frac{2\alpha^2 k}{h^2})\omega_{ij} + \alpha^2\frac{k}{h^2}(\omega_{i+1,j}\omega_{i-1,j} \tag{3.10}$$

for each $i = 1, 2, ...m + 1$ *and* $j = 1, 2, ...$ *So we have*

$w_{0,0} = f(x_0)$, $w_{1,0} = f(x_1)$,.... $w_{m,0} = f(x_m)$.

Then we generate the next t-row by

$$\omega_{0,1} = u(0, t_1) = 0$$

$$\omega_{1,1} = (1 - \tfrac{2\alpha^2 k}{h^2})\omega_{1,0} + \alpha^2\tfrac{k}{h^2}(\omega_{2,0}\omega_{0,0}$$

$$\omega_{2,1} = (1 - \tfrac{2\alpha^2 k}{h^2})\omega_{2,0} + \alpha^2\tfrac{k}{h^2}(\omega_{3,0}\omega_{1,0}$$

$$\vdots$$

$$\omega_{m-1,1} = (1 - \tfrac{2\alpha^2 k}{h^2})\omega_{m-1,0} + \alpha^2\tfrac{k}{h^2}(\omega_{m,0}\omega_{m-2,0}$$

$$\omega_{m,1} = u(m, t_1) = 0$$

Now we can use the $w_{i,1}$ *values to generate all the* $w_{i,2}$ *values and so on. The explicit nature of the difference method implies that the* $(m-1)(m-1)$ *matrix associated with this system can* *be written in the Tridiagonal form*

$$A = \begin{pmatrix} (1-2\lambda) & \lambda & 0 & \dots & \dots & \dots & \dots & 0 \\ \lambda & (1-2\lambda) & \lambda & \ddots & & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & \lambda \\ 0 & \dots & \dots & \dots & \dots & 0 & \lambda & (1-2\lambda) \end{pmatrix} \tag{3.11}$$

where $\lambda = \alpha^2(\frac{k}{h^2})$ *if we let*

$$W^{(0)} = (f(x_1), f(x_2), \dots f(x_{m-1}))^t$$

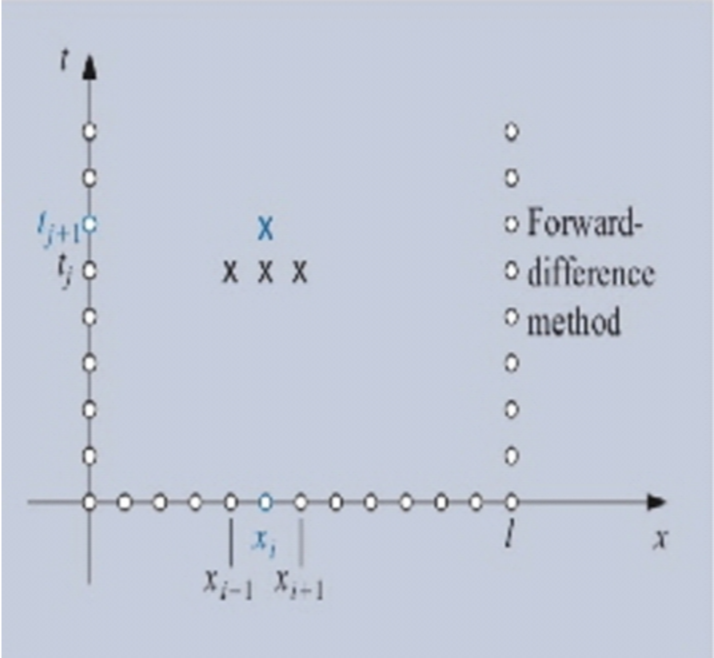
and

$$W^{(j)} = (\omega_{1j}, \omega_{2j}, \dots\dots, \omega_{m-1,j})^t \textit{ for each } j = 1, 2, \dots$$

then the approximate solution is given by

$$W^{(j)} = AW^{(j-1)} \textit{for each } j = 1, 2, \dots$$

so $W^{(j)}$ *is obtained from* $W^{(j-1)}$ *by a simple matrix multiplication. This is known as the Forward-Difference method, and the approximation at the cyan point shown in the next Figure [1]*



3.1.2 Stability of the forward difference method

We say that the forward difference method is stable if it gives an accurate solutions to the problem, that happens when

$$\lambda = \alpha^2 \tfrac{k}{h^2} \leqslant \tfrac{1}{2}$$

so this inequality requires that h and k be chosen so that $\lambda \leq \frac{1}{2}$

Proof 3.1.1 *Suppose that an error* $e^{(0)} = (e_1^{(0)}, e_2^{(0)}, \ldots e_{m-1}^{(0)})^t$ *is made in representing the initial data* $\omega^{(0)} = (f(x_1), f(x_2), \ldots f(x_{m-1}))^t$ *(or in any particular step). An error of* $Ae^{(0)}$ *propagates in* $\omega^{(1)}$*, because* $\omega^{(1)} = A(\omega^{(0)} + e^{(0)}) = A\omega^{(0)} + Ae^{(0)}$ *this process continues. At the nth time step, the error in* $\omega^{(n)}$ *due to* $e^{(0)}$ *is* $A^n e^{(0)}$*. The method is consequently stable precisely when these errors do not grow as n increases. But this is true if and only if for any initial error* $e^{(0)}$*, we have* $\|A^n e^{(0)}\| \leqslant \|e^{(0)}\|$ *for all n. Hence,we must have* $\|A^n\| \leq 1$ *[1]*

Definition 3.1.1 *The spectral radius* $\rho(A)$ *of a matrix A is defined by* $\rho(A) = max|\lambda|$*, where is an eigenvalue of A. (For complex* $\lambda = \alpha + \beta i$*, we define* $|\lambda| = (\alpha^2 + \beta^2)^{\frac{1}{2}}$*.)*

Théorème 3.1.1 *If A is an $n \times n$ matrix, then $\rho(A) \leq \|A\|$, for any natural norm $\|.\|$.*

We can see the condition $\|A^n\| \leq 1$ requires that $\rho(A^n) = (\rho(A))^n \leqslant 1$. therefore the Forward-Difference method is stable only if $\rho(A) \leqslant 1$. From the fact that $AV^{(i)} = \mu_i V^{(i)}$ we can find that The eigenvalues of A are

$$\mu_i = 1 - 4\lambda(\sin \tfrac{i\pi}{2m})^2 \text{ for each } i = 1, \ldots, m-1,$$

with corresponding eigenvectors $V^{(i)}$, where $v_j^{(i)} = \sin(\frac{ij\pi}{m})$ so the condition for stability consequently reduces to determining whether

$$\rho(A) = \max_{1 \leqslant i \leqslant m-1} |1 - 4\lambda(\sin \frac{i\pi}{2m})^2| \leqslant 1$$

and this can be simplified to

$$0 \leqslant (\sin \tfrac{i\pi}{2m})^2 \leqslant \tfrac{1}{2} \text{ for each } i = 1, \ldots, m-1$$

Stability requires that this inequality condition hold as $h \to 0$, or, equivalently, as $m \to \infty$. using the limit chain role we find that

$$\lim_{m \to \infty} [\sin(\frac{(m-1)\pi}{2m})]^2 = 1$$

it means that stability will occur only if $0 \leqslant \lambda \leq \frac{1}{2}$ and we have

$$\lambda = \alpha^2 \tfrac{k}{h^2} \leqslant \tfrac{1}{2}$$

this inequality requires that h and k be chosen so that $\lambda \leq \frac{1}{2}$ ∎ [1]

Example 3.1.1 *We will use the Forward-difference method to approximate the solution to the following parabolic partial differential equation*

$$\begin{cases} u_t - u_{xx} = 0; 0 < x < 2, t > 0 \\ u(0,t) = u(2,t) = 0; t > 0 \\ u(,0) = \sin(2\pi x); 0 \leq x \leq \pi \end{cases}$$

for $h = 0.4$ and $k = 0.1$ and we will compare the results at $T = 0.5$ to the actual solution

$$u(x,t) = e^{-4\pi^2 t}\sin(2\pi x)$$

First we should find m to know the size of the matrix A

$$h = \tfrac{L}{m} \Rightarrow 0.4 = \tfrac{2}{m} \Rightarrow m = 5 \Rightarrow A$$

$$k = \tfrac{T}{N} \Rightarrow 0.1 = \tfrac{0.5}{N} \Rightarrow N = 5$$

have the size 4×4; second we most calculate λ in order to find the elements the matrix A,

$$\lambda = 1 \times \tfrac{0.1}{0.4^2} = 0.625$$

using (3.11)

$$A = \begin{pmatrix} -0.25 & 0.625 & 0 & 0 \\ 0.625 & -0.25 & 0.625 & 0 \\ 0 & 0.625 & -0.25 & 0.625 \\ 0 & 0 & 0.625 & -0.25 \end{pmatrix}$$

now we calculate $w^{(0)}$ using

$$W^{(0)} = (f(x_1), \dots, f(x_{m-1}))^t$$

$$x_1 = 0.4,\ x_2 = 0.8,\ x_3 = 1.2,\ x_4 = 1.6$$

$$f(0.4) = \sin 2\pi(0.4) = 0.5878$$

$$f(0.8) = -0.9511$$

$$f(1.2) = 0.9511$$

$$f(1.6) = -0.5877$$

$$w^{(0)} = \begin{pmatrix} 0.5878 \\ -0.9511 \\ 0.9511 \\ -0.5878 \end{pmatrix}$$

now we use $w^{(j)} = Aw^{(j-1)}$ *for each* $j = 1, 2, 3, 4, 5$

$$w^{(1)} = A \times w^{(0)} = \begin{pmatrix} -0.7414 \\ 1.1996 \\ -1.1996 \\ 0.7441 \end{pmatrix}$$

$$w^{(2)} = A \times w^{(1)} = \begin{pmatrix} 0.9351 \\ -1.5130 \\ 1.5130 \\ -0.9351 \end{pmatrix}$$

$$w^{(3)} = A \times w^{(2)} = \begin{pmatrix} -1.1794 \\ 1.9083 \\ -1.9083 \\ 1.1794 \end{pmatrix}$$

$$w^{(4)} = A \times w^{(3)} = \begin{pmatrix} 1.4875 \\ -2.4069 \\ 2.4069 \\ -1.4875 \end{pmatrix}$$

$$w^{(5)} = A \times w^{(4)} = \begin{pmatrix} -1.8762 \\ 3.0357 \\ -3.0357 \\ 1.8762 \end{pmatrix}$$

When we compere the approximation result in the fthit eration with the actual solution which is after rounding 5 digits equals to 0, we find that it is not good approximation because $\lambda > \frac{1}{2}$ *which shows the instability of this case. [1]*

3.2 Implicit Backward difference method

To obtain a method that is unconditionally stable, we consider an implicit-difference method that results from using the backward-difference quotient for $u_t(x_i, t_j)$ in the form

$$u_t(x_i, t_j) = \frac{u(x_i, t_j) - u(x_i, t_{j-1})}{k} - \frac{k}{2} u_{tt}(x_i, \mu_j) \tag{3.12}$$

for some $\mu_j \in (t_j, t_{j+1})$, substituting this equation together with

$$u_{xx}(x_i, t_j) = \frac{u(x_{i+1},t_j)-2u(x_i,t_j)+u(x_{i-1},t_j)}{h^2} - \frac{h^2}{12} u_{xxxx}(\epsilon_i, t_j)$$

into the partial differential equation gives

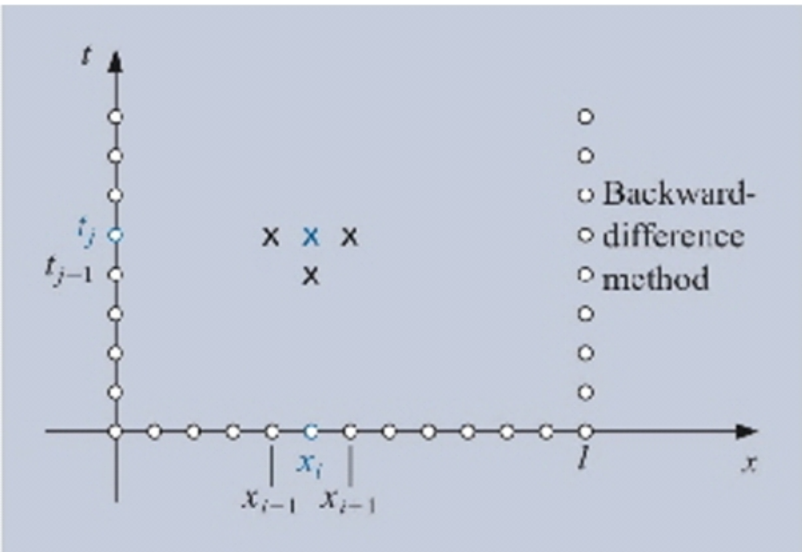
$$\frac{u(x_i,t_j)-u(x_i,t_{j-1})}{k} - \alpha^2 \frac{u(_xi+1,t_j)-2u(x_i,t_j)+u(x_{i-1},t_j)}{h^2}$$

$$= \frac{k}{2} u_{tt}(x_i, \mu_j) - \alpha \frac{h^2}{12} u_{xxxx}(\epsilon_i, t_j)$$

for some $\epsilon_i \in (x_{i-1}, x_{i+1})$. The Backward-Difference method that results is

$$\frac{\omega_{i,j}-\omega_{i,j-1}}{k} - \alpha^2 \frac{\omega_{i+1,j}-2\omega_{i,j}+\omega_{i-1,j}}{h^2} = 0$$

for each $i = 1, 2, \ldots, m-1$ and $j = 1, 2 \ldots$ The Backward-Difference method involves the mesh points (x_i, t_{j-1}), (x_{i-1}, t_j), and (x_{i+1}, t_j) to approximate the value at (x_i, t_j), as in the next Figure



If we again let $\lambda = \alpha^2 \frac{k}{h^2}$, the Backward-Difference method becomes

$$\omega_{i,j-1} = (1+2\lambda)\omega_{i,j} - \lambda\omega_{i+1,j} - \lambda\omega_{i-1,j}$$

for each $i = 1, 2, \ldots, m-1$ *and* $j = 1, 2, \ldots$. *Using the know ledge that* $w_{i,0} = f(xi)$, *for each* $i = 1, 2, \ldots m-1$ *and* $w_{m,j} = w_{0,j} = 0$, *for each* $j = 1, 2, \ldots,$ *this difference method has the matrix representation:*

$$\begin{pmatrix} (1+2\lambda) & -\lambda & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -\lambda & (1+2\lambda) & -\lambda & \ddots & & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & -\lambda \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -\lambda & (1+2\lambda) \end{pmatrix} \begin{pmatrix} \omega_{1,j} \\ \omega_{2,j} \\ \vdots \\ \omega_{m-1,j} \end{pmatrix} = \begin{pmatrix} \omega_{1,j-1} \\ \vdots \\ \omega_{2,j-1} \\ \omega_{m-1,j-1} \end{pmatrix}$$

or

$$AW^{(j)} = W^{(j-1)}, \forall i = 1, 2, \ldots$$

Hence, we must now solve a linear system to obtain $W^{(j)}$ *from* $W^{(j-1)}$.*this method is stable, independent of the choice of* λ *we call the Backward-Difference method an unconditionally stable method and the local truncation error for the method is of order* $O(k+h^2)$,*[1]*

Example 3.2.1 *We will approximate the solution to the following partial differential equation using the Backward-Difference method.*

$$\begin{cases} u_t - u_{xx} = 0; 0 < x < 2, t > 0 \\ u(0,t) = u(2,t) = 0; t > 0 \\ u(x,0) = \sin(\frac{\pi}{2}x); 0 \le x \le 2 \end{cases}$$

for $m = 4$, $T = 0.1$ *and* $N = 2$ *and will compare the results to the actual solution*

$$u(x,t) = e^{-(\frac{\pi^2}{4})t} \sin \frac{\pi}{2} x$$

We have

$$\begin{cases} h = \frac{L}{m} = \frac{2}{4} = 0.5 and \\ k = \frac{0.1}{2} = 0,05, \\ \lambda = \alpha^2 \frac{k}{h^2} = \frac{0.05}{0.5^2} = 0.2. \end{cases}$$

The size of matrix A is $(m-1)(m-1) = 3 \times 3$, *then for* $j = 1$ *we have*

$$\begin{pmatrix} 1.4 & -0.2 & 0 \\ -0.2 & 1.4 & -0.2 \\ 0 & -0.2 & 1.4 \end{pmatrix} \begin{pmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{pmatrix} = \begin{pmatrix} \omega_{1,0} \\ \omega_{2,0} \\ \omega_{3,0} \end{pmatrix}$$

where

$$\omega_{1,0} = \sin \frac{\pi}{2} \times 0.5 = 0.707107$$

$$\omega_{2,0} = \sin \frac{\pi}{2} = 1,$$

$$\omega_{3,0} = \sin \frac{\pi}{2} \times 1.5 = 0.707107$$

then

$$\begin{pmatrix} 1.4 & -0.2 & 0 \\ -0.2 & 1.4 & -0.2 \\ 0 & -0.2 & 1.4 \end{pmatrix} \begin{pmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{pmatrix} = \begin{pmatrix} 0.707107 \\ 1 \\ 0.707107 \end{pmatrix}$$

Solving the linear system, yields

$$\begin{pmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{pmatrix} = \begin{pmatrix} 0.632952 \\ 0.895129 \\ 0.63952 \end{pmatrix}$$

Now the next iteration for $j = 2$

$$\begin{pmatrix} 1.4 & -0.2 & 0 \\ -0.2 & 1.4 & -0.2 \\ 0 & -0.2 & 1.4 \end{pmatrix} \begin{pmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{pmatrix} = \begin{pmatrix} 0.632952 \\ 0.895129 \\ 0.63952 \end{pmatrix}$$

Solving the new linear system, we get

$$\begin{pmatrix} \omega_{1,2} \\ \omega_{2,2} \\ \omega_{3,2} \end{pmatrix} = \begin{pmatrix} 0.566573 \\ 0.801256 \\ 0.566573 \end{pmatrix}$$

The next table shows the actual solution and the approximation at each step and the error between them [1]

i	j	x_i	t_j	$\omega_{i,j}$	$u(x_i, t_j)$	$\lvert\omega_{i,j} - u(x_i, t_j)\rvert$
1	1	0.5	0.05	0.632952	0.652037	0.019085
2	1	1	0.05	0.895129	0.883937	0.011192
3	1	1.5	0.05	0.632952	0.625037	7.915×10^{-3}
1	2	0.5	0.1	0.566573	0.552493	0.01408
2	2	1	0.1	0.801256	0.781344	0.019912
3	2	1.5	0.1	0.566573	0.552493	0.01408

3.3 Crank-Nicolson method

The weakness of the Backward-Difference method results from the fact that the local truncation error has one of order $O(h^2)$, and another of order $O(k)$. This requires that time intervals be made much smaller than the x-axis intervals A more rewarding method is derived by averaging the Forward-Difference method at the jth step in t,

$$\frac{\omega_{i,j+1}-\omega_{i,j}}{k} - \alpha^2 \frac{\omega_{i+1,j}-2\omega_{i,j}+\omega_{i-1,j}}{h^2} = 0$$

which has local truncation error of order $O(k, h^2)$ and the Backward-Difference method at the $(j+1)$ st step in t

$$\frac{\omega_{i,j+1}-\omega_{i,j}}{k} - \alpha^2 \frac{\omega_{i+1,j+1}-2\omega_{i,j+1}+\omega_{i-1,j+1}}{h^2} = 0$$

which has also local truncation error of order $O(k, h^2)$ *then the average difference method*

$$\frac{\omega_{i,j+1}-\omega_{i,j}}{k} - \frac{\alpha^2}{2}\left[\frac{\omega_{i+1,j}-2\omega_{i,j}+\omega_{i-1,j}}{h^2} + \frac{\omega_{i+1,j+1}-2\omega_{i,j+1}+\omega_{i-1,j+1}}{h^2}\right] = 0$$

which has local truncation error of order $O(k^2, h^2)$ *This is known as the Crank-Nicolson method and is represented in the matrix form*

$$AW^{(j+1)} = BW^{(j)},\ j = 0, 1, 2, \ldots$$

where

$$\lambda = \alpha^2 \frac{k}{h^2}$$

$$W^{(j)} = (\omega_{1,j}, \ldots, \omega_{m-1,j})^t$$

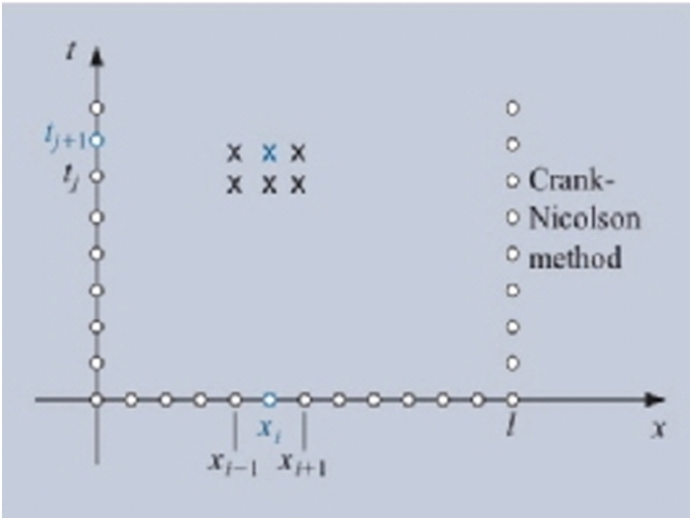
and the matrices A and B are given by:

$$A = \begin{pmatrix} (1+\lambda) & -\frac{\lambda}{2} & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ -\frac{\lambda}{2} & (1+\lambda) & -\frac{\lambda}{2} & \ddots & & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & -\frac{\lambda}{2} \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & -\frac{\lambda}{2} & (1+\lambda) \end{pmatrix}$$

and

$$B = \begin{pmatrix} (1-\lambda) & \frac{\lambda}{2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \frac{\lambda}{2} & (1-\lambda) & \frac{\lambda}{2} & \ddots & & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & \frac{\lambda}{2} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{\lambda}{2} & (1-\lambda) \end{pmatrix}$$

A diagram showing the interaction of the nodes for determining an approximation at (x_i, t_j) is shown in next Figure



Remark 3.3.1 *Crank-Nicolson method is much better than the forward method and the backward method as we described before it is more rewarding method because it doesnt have a stability condition as the forward method and it has local truncation error of order $O(k^2, h^2)$, while the backward method and the forward method have local truncation error of order $O(k, h^2)$,this means that the Crank-Nicolson method is faster and have less error than the backward method and the forward method.[1]*

Example 3.3.1 *We will approximate the solution to the same partial differential equation in the previous example using the Crank-Nicolson method.*

$$\begin{cases} u_t - u_{xx} = 0; 0 < x < 2, t > 0 \\ u(0,t) = u(2,t) = 0; t > 0 \\ u(x,0) = \sin(\frac{\pi}{2}x); 0 \leq x \leq 2 \end{cases}$$

for $m = 4$, $T = 0.1$ *and* $N = 2$ *and will compare the results to the actual solution*

$$u(x,t) \ = e^{-(\frac{\pi^2}{4})t} \sin \frac{\pi}{2} x$$

We have

$$\begin{cases} h = \frac{L}{m} = \frac{2}{4} = 0.5 and \\ k = \frac{0.1}{2} = 0,05, \\ \lambda = \alpha^2 \frac{k}{h^2} = \frac{0.05}{0.5^2} = 0.2. \end{cases}$$

The size of matrix A is $(m-1)(m-1) \ = 3 \times 3$,

$$A = \begin{pmatrix} 1.2 & -0.1 & 0 \\ -0.1 & 1.2 & -0.1 \\ 0 & -0.1 & 1.2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0.8 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.1 & 0.8 \end{pmatrix}$$

Now we must solve the system

$$AW^{(j+1)} = BW^{(j)}.$$

for $j = 1$

$$\begin{pmatrix} 1.2 & -0.1 & 0 \\ -0.1 & 1.2 & -0.1 \\ 0 & -0.1 & 1.2 \end{pmatrix} \begin{pmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{pmatrix} = \begin{pmatrix} 0.8 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} \omega_{1,0} \\ \omega_{2,0} \\ \omega_{3,0} \end{pmatrix},$$

where

$$\omega_{1,0} = \sin\tfrac{\pi}{2} \times 0.5 = 0.707107,$$

$$\omega_{1,0} = \sin\tfrac{\pi}{2} = 1,$$

$$\omega_{1,0} = \sin\tfrac{\pi}{2} \times 1.5 = 0.707107,$$

then

$$\begin{pmatrix} 1.2 & -0.1 & 0 \\ -0.1 & 1.2 & -0.1 \\ 0 & -0.1 & 1.2 \end{pmatrix} \begin{pmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{pmatrix} = \begin{pmatrix} 0.8 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} 0.707107 \\ 1 \\ 0.707107 \end{pmatrix},$$

that is

$$\begin{pmatrix} 1.2 & -0.1 & 0 \\ -0.1 & 1.2 & -0.1 \\ 0 & -0.1 & 1.2 \end{pmatrix} \begin{pmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{pmatrix} = \begin{pmatrix} 0.6656 \\ 0.9414 \\ 0.6656 \end{pmatrix},$$

Solving the linear system gives

$$\begin{pmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{pmatrix} = \begin{pmatrix} 0.6288 \\ 0.8892 \\ 0.6288 \end{pmatrix}.$$

Now the next iteration for $j = 2$

$$\begin{pmatrix} 1.2 & -0.1 & 0 \\ -0.1 & 1.2 & -0.1 \\ 0 & -0.1 & 1.2 \end{pmatrix} \begin{pmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{pmatrix} = \begin{pmatrix} 0.8 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} 0.6288 \\ 0.8892 \\ 0.6288 \end{pmatrix},$$

then

$$\begin{pmatrix} 1.2 & -0.1 & 0 \\ -0.1 & 1.2 & -0.1 \\ 0 & -0.1 & 1.2 \end{pmatrix} \begin{pmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{pmatrix} = \begin{pmatrix} 0.5919 \\ 0.8371 \\ 0.5919 \end{pmatrix},$$

Solving the linear system gives

$$\begin{pmatrix} \omega_{1,2} \\ \omega_{2,2} \\ \omega_{3,2} \end{pmatrix} = \begin{pmatrix} 0.5591 \\ 0.7908 \\ 0.5591 \end{pmatrix}.$$

The next table shows the actual solution and the approximation at each step. and the error between them

i	j	x_i	t_j	$\omega_{i,j}$	$u(x_i,t_j)$	$\lvert\omega_{i,j} - u(x_i,t_j)\rvert$
1	1	0.5	0.05	0.6288	0.6520	0.0232
2	1	1	0.05	0.8892	0.8839	5.3×10^{-3}
3	1	1.5	0.05	0.6288	0.6250	3.8×10^{-3}
1	2	0.5	0.1	0.5591	0.5525	6.6×10^{-3}
2	2	1	0.1	0.7908	0.7813	9.5×10^{-3}
3	2	1.5	0.1	0.5591	0.5525	6.6×10^{-3}

3.3.1 Examples for C-N method using Matlab code

For some examples we can't solve the iterations by hand when the matrix is too big or the iterations are 20 or 30 or more ,then we need some help from matlab

Example 3.3.2 *Useing the Crank-Nicolson method with $h = 0.1$ and $k = 0.01$ we will approximate the solution to the problem*

$$\begin{cases} u_t - u_{xx} = 0; 0 < x < 1, t > 0, \\ u(0,t) = u(1.t) = 0; t > 0, \\ u(x,0) = \sin(\pi x); 0 \le x \le 1, \end{cases}$$

$$\begin{cases} h = 0.1 and \\ k = 0.01 \\ gives m = 10, N = 50 and T = 0.5 \end{cases}$$

After running a code Matlab and comper the approximation with the exact solution we find the solution in the 50 iteration as the following table

x_i	$\omega_{i,50}$	$u(x_i, 0.5)$	$\lvert\omega_{i,50} - u(x_i, 0.5)\rvert$
0.1	0.00230512	0.0022241	8.271×10^{-5}
0.2	0.00438461	0.00422728	1.573×10^{-4}
0.3	0.00603489	0.00581836	2.165×10^{-4}
0.4	0.00709444	0.00683989	2.546×10^{-4}
0.5	0.00745954	0.00719188	2.677×10^{-4}
0.6	0.00709444	0.00683989	2.546×10^{-4}
0.7	0.00603489	0.00581836	2.165×10^{-4}
0.8	0.00438461	0.00422728	1.573×10^{-4}
0.9	0.00230512	0.0022241	8.271×10^{-5}

Example 3.3.3 *In this example we will solve the following PDE*

$$\begin{cases} u_t - u_{xx} = 0; 0 < x < 2, t > 0, \\ u(0,t) = u(2.t) = 0; t > 0, \\ u(x,0) = \sin(2\pi x); 0 \leq x \leq \pi, \end{cases}$$

using C-N method and the matlab code . we did solve this example before using the forward method example but the approximation wasn't satisfying ,for $h = 0.4$ *and* $k = 0.1$ *and we will compare the results at* $T = 0.5$ *to the actual solution*

$$u(x,t) = e^{-4\pi^2 t} \sin(2\pi x).$$

$$h = \frac{L}{m} \Rightarrow 0.4 = \frac{2}{m} \Rightarrow m = 5 \Rightarrow A$$

$$k = \frac{T}{N} \Rightarrow 0.1 = \frac{0.5}{N} \Rightarrow N = 5$$

$$L = 2$$

By replacing the values of n,m,h,k T,L and the initial condition,in the same matlab code as the previous example ,after rounding 5 digits we find the next table

i	j	x_i	t_j	$\omega_{i,j}$	$u(x_i, t_j)$
1	5	0.4	0.5	0	0
2	5	0.8	0.5	0	0
3	5	1.2	0.5	0	0
4	5	1.6	0.5	0	0

Conclusion

In this thesis, we used analytical and numerical solutions to solve heat equation. In this we observed that analytical solutions are not enough, because it takes the less computing time, but gives less accuracy solution. Thus we rely on numerical solutions to obtain more knowledge on the inherent problems. The numerical method is very convenient for solving boundary value problems, since the boundary condition are taken in to account automatically. Also numerical approach enables solution of a complex problem with a great number (but) of very simple operations and it is very simple in implementation. Thus, we need to undertake more research on heat equation to further our knowledge so that we can effectively utilize our limited resources for the betterment of people.

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