



الجمهورية الجزائرية الديمقراطية الشعبية
People's Democratic Republic of Algeria

وزارة التعليم العالي والبحث العلمي

Ministry of Higher Education and Scientific Research

جامعة الشهيد عباس لغرور خنشلة

Abbes Laghrou- Khenchela University



Faculty of Sciences and Technology

Department of Mathematics

Serial number:.....

كلية العلوم والتكنولوجيا

قسم الرياضيات

Final Master's Thesis

Submitted in the Fulfilment of Requirements for the Master's Degree in Mathematics

Major :Mathematics

Specialization :Applied Mathematics

Titled by:

SOLVING NONLINEAR TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS USING THE RESIDUAL POWER SERIES METHOD

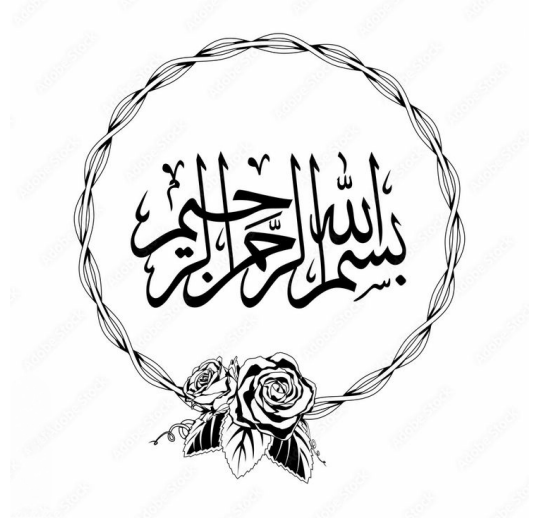
Presented by: Bouhafer Mebarka

Defended on: 24/05/2026

In the presence of the Examination Committee:

Saoudi Khaled	Pr	Abbes Laghrou - Khenchela university	Chair
Bekhouche Fares	MCB	Abbes Laghrou - Khenchela university	Supervisor
Derdoukh Asma	MCB	Abbes Laghrou - Khenchela university	Examiner

Academic year 2025/2026



Contents

Acknowledgements	4
Dedication	i
Notation	v
List of Figures	vi
List of Tables	vii
Introduction	1
Preliminaries	2
1 Special Functions	2
1.1 The Gamma Function	2
1.2 The Beta Function	5
1.3 The Mittag-Leffler Function	6
2 Fractional Integration	8
2.1 Riemann-Liouville Fractional integration	8
3 Fractional Derivative	11
3.1 Riemann-Liouville Fractional Derivative	11
3.2 Caputo Fractional Derivative	13
3.3 Relationship Between the Caputo and Riemann–Liouville	16
3.4 Comparison Between the Caputo and Riemann–Liouville Fractional Derivatives	16
4 Fractional Power Series Expansions	17
The Residual Power Series Method (RPSM)	19
1 Methodology	19
2 Convergence Analysis	24
Numerical Implementation	27
1 General Application	27
2 Examples And Case Studies	32
2.1 Application 1	32
2.2 Application 2	37
2.3 Application 3	43
Conclusion	49
References	50

Acknowledgements

In the Name of Allah the Most Gracious, the Most Merciful

Praise be to Allah, by Whose grace good deeds are completed. I praise Him for His guidance and support in bringing this thesis to completion.

I extend my sincere gratitude and appreciation to my esteemed professor, **Dr.Bekhouche Fares**, who generously shared with me his knowledge and time.

I also extend my special thanks to my distinguished examiners, **Pr.Saoudi Khaled** and **Dr.Derdoukh Asma**, members of the examining committee, who honored this work with their discussion and enriched it with their valuable observations.

To every professor who has left a lasting mark on my academic journey, I say: *thank you from the heart.*

But the gratitude that words cannot fully capture is reserved for my beloved parents, the fountain of love and affection. May Allah prolong their lives and fill their hearts with every joy.

And to my dear sisters, **Loubna** and **Manar**, and my beloved brother, **Nour Elyamine** — you are among the most beautiful blessings Allah has bestowed upon me, my unwavering support.

Nor can I forget every hand extended in assistance — a kind word, a sincere prayer, or a word of wisdom. To you all, I offer my heartfelt prayers and deepest gratitude.

In closing, I ask Allah to make this work sincerely dedicated to Him alone, to make it beneficial, and to grant it acceptance. *Whatever is correct in it is from Allah alone, and whatever shortcomings remain are my own.*

And may peace and blessings be upon our Master Muhammad, his family, and his companions.

Dedication

To my beloved parents,
the fountain of love and affection.

To Loubna, Manar, and Nour Elyamine,
my unwavering support and most beautiful blessings.

With all my love and gratitude.

Abstract

This memoir deals with the solution of nonlinear fractional-time partial differential equations using the Residual Power Series Method (RPSM). The method is applied to the generalized Kuramoto–Sivashinsky (KS) equation, adopting the Caputo time-fractional derivative of order α . The memoir is divided into three chapters: the first chapter covers the fundamentals of fractional calculus and special functions, the second reviews the RPSM methodology and its convergence theory, while the third presents a numerical application to three test cases with different parameters and initial conditions. The resulting approximate solutions are compared with exact solutions and the results of other numerical methods through error tables and two- and three-dimensional graphical plots. The results demonstrate high accuracy, numerical stability, and low computational cost. They also highlight the memory effect induced by the fractional order α on wave behavior and temporal convergence, confirming the efficiency of the method in modeling complex physical phenomena such as flame fronts, thin liquid films, and plasma.

Keywords: Fractional partial differential equations -- Caputo derivative -- Residual Power Series Method (RPSM) -- Approximate solutions -- Kuramoto–Sivashinsky equation.

Résumé

Ce mémoire traite de la résolution des équations aux dérivées partielles non linéaires à temps fractionnaire en utilisant la méthode des séries de puissances résiduelles (RPSM). La méthode est appliquée à l'équation de Kuramoto–Sivashinsky (KS) généralisée, en adoptant la dérivée fractionnaire temporelle de Caputo d'ordre α . Le mémoire est divisé en trois chapitres : le premier chapitre couvre les fondamentaux du calcul fractionnaire et les fonctions spéciales, le deuxième expose la méthodologie de la RPSM et sa théorie de convergence, tandis que le troisième présente une application numérique à trois cas tests avec différents paramètres et conditions initiales. Les solutions approchées obtenues sont comparées aux solutions exactes et aux résultats d'autres méthodes numériques à travers des tableaux d'erreurs et des représentations graphiques bidimensionnelles et tridimensionnelles. Les résultats démontrent une haute précision, une stabilité numérique et un faible coût de calcul. Ils mettent également en évidence l'effet de mémoire introduit par l'ordre fractionnaire α sur le comportement des ondes et la convergence temporelle, confirmant l'efficacité de la méthode pour la modélisation de phénomènes physiques complexes tels que les fronts de flamme, les films liquides minces et les plasmas.

Mots-clés : Équations aux dérivées partielles fractionnaires -- Dérivée de Caputo -- Méthode des séries de puissances résiduelles (RPSM) -- Solutions approchées -- Équation de Kuramoto–Sivashinsky.

ملخص

تتناول هذه المذكرة حلّ المعادلات التفاضلية الجزئية غير الخطية ذات الزمن الكسري باستخدام طريقة متسلسلة القوى المتبقية (RPSM). تُطبّق الطريقة على معادلة كوراموتو-سيفاشينسكي (KS) المعمّمة، مع اعتماد مشتقة كابوتو الكسرية الزمنية من الرتبة α . تنقسم المذكرة إلى ثلاثة فصول: يغطي الفصل الأول أساسيات الحساب الكسري والدوال الخاصة، ويستعرض الثاني منهجية طريقة RPSM ونظرية تقاربها، في حين يقدم الفصل الثالث تطبيقاً عددياً على ثلاث حالات اختبارية بوسائط وشروط ابتدائية مختلفة. تُجرى مقارنة الحلول التقريبية الناتجة مع الحلول المضبوطة ونتائج الطرق العددية الأخرى، من خلال جداول الخطأ ورسوم بيانية ثنائية وثلاثية الأبعاد. تُظهر النتائج دقة عالية واستقراراً عددياً وبتكلفة حسابية منخفضة، كما تبرز أثر الذاكرة الذي تُحدثه الرتبة الكسرية α على سلوك الموجات والتقارب الزمني، مما يؤكد كفاءة الطريقة في نمذجة ظواهر فيزيائية معقدة، مثل جبهات اللهب والأغشية السائلة الرقيقة والبلازما.

الكلمات المفتاحية: المعادلات التفاضلية الجزئية الكسرية -- مشتقة كابوتو -- طريقة متسلسلة القوى المتبقية (RPSM) -- الحلول التقريبية -- معادلة كوراموتو-سيفاشينسكي.

Notation

Γ	: The Gamma function.
β	: The Beta function.
E_α	: The Mittag-Leffler function (one-parameter).
$E_{\alpha,\beta}$	: The Mittag-Leffler function (two-parameter).
$\Re$	: Real part of a complex number.
$\mathbb{N}$	: Set of positive integers.
$\mathbb{R}$	: Set of real numbers.
$\mathbb{C}$	: Set of complex numbers.
I^α	: Riemann-Liouville fractional integral.
${}^{RL}D^\alpha$	: Riemann-Liouville fractional derivative.
${}^CD^\alpha$	: Caputo fractional derivative.
RPSM	: Residual Power Series Method.
KS	: Kuramoto–Sivashinsky.
NTDM	: Natural Transform Decomposition Method.
q-HATM	: q -Homotopy Analysis Transform Method.
FPS	: Fractional Power Series.

List of Figures

1.1	The Gamma Function	5
1.2	The Beta function	6
1.3	The Metag-Leffler function	7
3.1	2D comparison of the exact solution and the RPSM approximation for various fractional orders α ($\alpha = 1, \alpha = 0.8, \alpha = 0.6, \alpha = 0.4$) for Application 1.	35
3.2	3D surface plots of the exact solution and the RPSM approximation for various fractional orders α ($\alpha = 1, \alpha = 0.8, \alpha = 0.6, \alpha = 0.4$) for Application 1.	36
3.3	2D comparison of the exact solution and the RPSM approximation for various fractional orders α ($\alpha = 1, \alpha = 0.8, \alpha = 0.6, \alpha = 0.4$) for Application 2.	41
3.4	3D surface plots of the exact solution and the RPSM approximation for various fractional orders α ($\alpha = 1, \alpha = 0.8, \alpha = 0.6, \alpha = 0.4$) for Application 2.	42
3.5	2D comparison of the exact solution and the RPSM approximation for various fractional orders α ($\alpha = 1, \alpha = 0.9, \alpha = 0.75, \alpha = 0.6$) for Application 3.	47
3.6	3D surface plots of the exact solution and the RPSM approximation for various fractional orders α ($\alpha = 1, \alpha = 0.9, \alpha = 0.75, \alpha = 0.6$) for Application 3.	48

List of Tables

1.1	Comparison between the Riemann–Liouville and Caputo fractional derivatives	16
3.1	Numerical comparison between the error ratio of RPSM and NTDM [2] at different fractional orders of α ($\alpha = 0.4, \alpha = 0.6, \alpha = 0.8$ and $\alpha = 1$) for Application 1. .	34
3.2	Numerical comparison of approximate solutions of RPSM and NTDM [2] at different fractional orders of α ($\alpha = 0.4, \alpha = 0.6, \alpha = 0.8$ and $\alpha = 1$) for Application 2.	40
3.3	Numerical comparison of approximate solutions of RPSM and q-HATM [22] at different fractional orders of α ($\alpha = 0.6, \alpha = 0.75, \alpha = 0.9$ and $\alpha = 1$) for Application 3.	46

Introduction

Fractional calculus is a branch of mathematical analysis that aims to generalise the notions of differentiation and integration to non-integer (fractional or real) orders, thereby enabling mathematical models to describe memory effects and accumulated history that classical integer-order models cannot capture.

The roots of this field date back to 1695, when the Marquis de l'Hôpital asked about the meaning of a derivative of order $\frac{1}{2}$. Leibniz replied that such a concept could one day lead to useful results. Later, Euler paved the way by extending the factorial concept through the Gamma function in 1730, while Fourier introduced in 1822 an integral representation for derivatives of arbitrary order. The first practical application was given by Abel in 1823, when he solved the tautochrone problem using a derivative of order $\frac{1}{2}$ [18],[21].

Later on, Liouville and Riemann established the theoretical foundations of this field through the introduction of the Riemann–Liouville [19],[15] integral, followed by the Grünwald–Letnikov and Caputo definitions [19], which opened the way to modern applications in viscoelastic materials, anomalous diffusion, and control theory.

The theory of fractional differential equations is considered a natural extension of the theory of ordinary differential equations, in which integer-order derivatives are replaced by fractional ones. This theory relies on fractional differential and integral operators, among which the Riemann–Liouville and Caputo operators are the most prominent. The Caputo operator is generally preferred whenever the initial conditions possess a clear physical interpretation.

A fractional derivative is a nonlocal operator in time or space, which gives the solutions a memory or accumulated-effect property absent in classical models. Over the past decades, fractional calculus has evolved into a unified mathematical framework due to its wide range of applications in engineering, physics, chemistry, biology, and many other scientific fields [5],[23],[11],[1],[8]

Due to the complexity of nonlinear fractional partial differential equations and the difficulty of obtaining exact analytical solutions in most cases, many numerical and semi-analytical methods have been developed to approximate their solutions, such as the Laplace Adomian Decomposition Method (LADM) [7],[6], the Variational Iteration Method (VIM) [13], the Reproducing Kernel Method (RKM) [14], the Optimised Decomposition Method (ODM) [12], and the Homotopy Perturbation Method (HPM) [16].

In this context, the present memoir aims to highlight a promising and relatively simple semi-analytical technique, namely the Residual Power Series Method (RPSM) [24], [20], which was introduced in 2013 by Omar Abu Arqub [3]. This method is based on constructing rapidly convergent power series solutions of differential equations without requiring linearisation, decomposition of nonlinear terms, or the use of complicated integral transforms. Moreover, it does not rely on domain discretisation or perturbation techniques, which explains its rapid spread in applications of fractional calculus. In this work, we apply the Residual Power Series Method (RPSM) to the time-fractional Kuramoto–Sivashinsky equation. The Kuramoto–Sivashinsky (KS) equation [2],[22], is one of the important nonlinear partial differential equations used in the mathematical modelling of several physical phenomena. It is a fourth-order equation that first appeared in modelling flame fronts, thin-film instabilities, and plasma physics. The equation also combines nonlinearity, dispersion, and damping, which leads to rich dynamics encompassing bifurcations, chaos, and the formation of complex patterns. In this memoir, we study the time-fractional generalisation of this equation, which is given as follows:

$$\begin{cases} D_t^\alpha u + u \frac{\partial u}{\partial x} + a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^3 u}{\partial x^3} + c \frac{\partial^4 u}{\partial x^4} = 0, \\ 0 < \alpha \leq 1, \quad x \in \mathbb{R}, \quad t \in [0, T], \\ u(x, 0) = \varphi(x), \end{cases}$$

where a , b , and c are constants, while α denotes the fractional order that introduces memory and hereditary properties into the model, leading to more complex dynamical behaviour compared to the classical integer-order case.

This memoir is organised into three chapters:

Chapter 1 presents the analytical foundations of fractional calculus through the introduction of the Gamma function, the Beta function, and the Mittag–Leffler function. It also introduces the two principal definitions of fractional integration and differentiation (Riemann–Liouville and Caputo), together with their properties and a comparison between them. The chapter concludes with the concept of fractional power series, their convergence conditions, and the formulas of their coefficients.

Chapter 2 is devoted to the Residual Power Series Method (RPSM) for solving nonlinear time-fractional partial differential equations in the Caputo sense. The method is presented through clear algorithmic steps, together with a nonlinear example including convergence analysis and error estimation.

Chapter 3 applies the RPSM to the fractional Kuramoto–Sivashinsky equation for three different test cases with varying coefficients and initial conditions. Approximate solutions up to the third-order term (U_3) were obtained using *Mathematica*. The obtained results are compared with exact solutions and with published results achieved by the NTDM [2] and q-HATM [22] methods. The results are illustrated through error tables and two- and three-dimensional graphics, highlighting the high accuracy and numerical stability of the method for various values of the fractional order α .

Preliminaries



This chapter covers the basic concepts of fractional calculus that will be needed later in the thesis. It starts by introducing three important special functions: the Gamma function, the Beta function, and the Mittag-Leffler function. These functions form the foundation of fractional analysis. Next, the chapter explains the main definitions and properties of fractional integrals and derivatives, focusing specifically on the Riemann–Liouville and Caputo approaches. Finally, it introduces fractional power series expansions and shows how they can be used as a practical tool for solving fractional differential equations.

1. Special Functions

This section introduces a group of mathematical functions known as special functions, including notable examples like the Gamma, Beta, and Mittag-Leffler functions. These functions are termed "special" due to their recurring role in tackling complex mathematical problems, particularly those involving differential equations. Their distinctive properties and wide-ranging applications make them valuable tools in both theoretical and applied mathematics.

1.1 The Gamma Function

The Gamma function, originally developed by Euler during the eighteenth century, serves as an extension of factorial notation to encompass non-integer values, including real and complex numbers. This mathematical tool proves indispensable across numerous disciplines, from probability theory and complex analysis to engineering applications and fractional calculus.

Definition 1.1.1: [19] Let $z \in \mathbb{C}$ with $\Re(z) > 0$, the Gamma function $\Gamma(z)$ is defined by the integral:

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \quad (1.1)$$

Key properties include:

$$\Gamma(1) = 1, \Gamma(0^+) = +\infty,$$

and $\Gamma(z)$ as a strictly decreasing, monotonic for $0 < z < 1$.

Properties 1.1.1: [19] The Gamma function satisfies several fundamental properties that make it indispensable in analysis. Among the most important are :

1. $\Gamma(z+1) = z\Gamma(z) \quad \Re(z) > 0$
2. $\Gamma(n+1) = n! \quad \forall n \in \mathbb{N}$

Proof:

1. We begin with the definition of the Gamma function :

$$\Gamma(z+1) = \int_0^{+\infty} t^z e^{-t} dt$$

we apply integration by parts, Let us set :

$$\begin{aligned} u &= t^z & u' &= z t^{z-1} \\ v' &= e^{-t} & v &= -e^{-t}. \end{aligned}$$

Hence,

$$\begin{aligned} \Gamma(z+1) &= \int_0^{+\infty} t^z e^{-t} dt \\ &= [-t^z e^{-t}]_0^{+\infty} + \int_0^{+\infty} z t^{z-1} e^{-t} dt \\ &= 0 + z \int_0^{+\infty} t^{z-1} e^{-t} dt \\ &= z\Gamma(z). \end{aligned}$$

Thus,

$$\Gamma(z+1) = z\Gamma(z) \quad \Re(z) > 0.$$

2. By repeating property 1 each time, we obtain:

$$\Gamma(2) = 1\Gamma(1) = 1!$$

$$\Gamma(3) = 2\Gamma(2) = 2 \cdot 1! = 2!$$

$$\Gamma(4) = 3\Gamma(3) = 3 \cdot 2! = 3!$$

$$\vdots$$

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!$$

Remark 1.1.1:

$$\Gamma(1) = \int_0^{+\infty} t^{1-1} e^{-t} dt = \int_0^{+\infty} e^{-t} dt = 1.$$

Example 1.1.1: Calculating the gamma function when : $z = \frac{1}{2}$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-1/2} e^{-t} dt.$$

By changing the variable :

$$t = y^2 \quad \Rightarrow \quad dt = 2y dy,$$

We obtain the following integration:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} (y^2)^{-1/2} e^{-y^2} \cdot 2y dy = 2 \int_0^{+\infty} e^{-y^2} dy.$$

We have the well-known Gaussian integral, and its value is:

$$\int_0^{+\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2},$$

therefore,

$$\Gamma\left(\frac{1}{2}\right) = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

we have,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

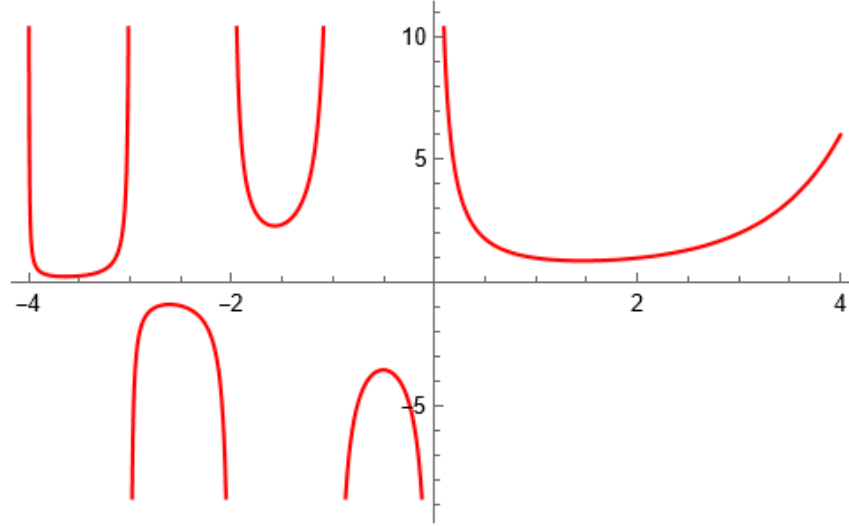


Figure 1.1: The Gamma Function

1.2 The Beta Function

In mathematics, the Beta function stands as a fundamental special function with broad utility in evaluating definite integrals. Its importance extends across multiple disciplines, including mathematical analysis, probability, and statistics. The function is particularly noteworthy for its intimate connection to the Gamma function, a relationship that proves invaluable in simplifying complex integral expressions.

Definition 1.1.2: [19] The Beta function $\beta(z, w)$ is a fundamental special function defined for two complex z and w . Its standard integral form is given by :

$$\beta(z, w) = \int_0^1 t^{z-1} (1-t)^{w-1} dt, \quad \Re(z) > 0, \quad \Re(w) > 0. \quad (1.2)$$

Remark 1.1.2: The relationship between the two functions Beta and Gamma can be written as follows:

$$\beta(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad \Re(z) > 0, \quad \Re(w) > 0. \quad (1.3)$$

From (1.3), we deduce the symmetry of the beta function:

$$\beta(z, w) = \beta(w, z), \quad \Re(z) > 0, \quad \Re(w) > 0.$$

Example 1.1.2: Calculating the $\beta(1, 1)$: $\beta(1, 1) = \frac{\Gamma(1)\Gamma(1)}{\Gamma(2)} = 1.$

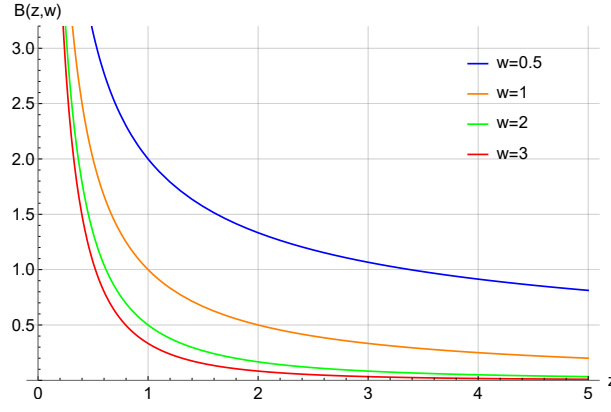


Figure 1.2: The Beta function

1.3 The Mittag-Leffler Function

Originally introduced by Gösta Magnus Mittag-Leffler (1846–1927), the function that bears his name extends the exponential function to a broader class of arguments. It is indispensable in fractional calculus, where it provides solutions to fractional-order differential equations. These equations arise naturally when describing systems with memory or hereditary properties, making the Mittag-Leffler function essential in fields such as anomalous diffusion, signal processing, and biomechanics.

Definition 1.1.3: [19] Let $z \in \mathbb{C}$, the Mittag-Leffler function of order α is defined by the following infinite series:

$$E_{\alpha}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0. \quad (1.4)$$

A more general formula, often referred to as the Wright function, exists for parameters α and β , and is defined by the following formula:

$$E_{\alpha, \beta}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \beta > 0. \quad (1.5)$$

Properties 1.1.2: The Mittag-Leffler function has several properties that simplify its form by varying the values of α and β each time, as follows:

1. $\beta = 1$:

In this case, the Mittag-Leffler function becomes as follows:

$$E_{\alpha, 1}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\alpha n + 1)} = E_{\alpha}(z).$$

2. $\alpha = 1$ and $\beta = 1$:

In this case, we obtain the well-known exponential function, because:

$$E_{1,1}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(n+1)} = \sum_{n=0}^{+\infty} \frac{z^n}{n!} = e^z.$$

3. $\alpha = 1$ and $\beta = 2$:

The Metag-Leffler function becomes more simplified:

$$E_{1,2}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(n+2)} = \sum_{n=0}^{+\infty} \frac{z^n}{(n+1)!} = \frac{1}{z} \sum_{n=0}^{+\infty} \frac{z^{n+1}}{(n+1)!}, \quad z \neq 0.$$

By changing the variable:

$$k = n + 1$$

$$\begin{cases} n = 0 \Rightarrow k = 1, \\ n \rightarrow +\infty \Rightarrow k \rightarrow +\infty. \end{cases}$$

We have :

$$\sum_{n=0}^{+\infty} \frac{z^{n+1}}{(n+1)!} = \sum_{k=1}^{+\infty} \frac{z^k}{k!} = e^z - 1$$

consequently :

$$E_{1,2}(z) = \frac{e^z - 1}{z}, \quad z \neq 0.$$

4. For $\alpha = 1$ and $\beta = m(m \geq 1)$:

$$E_{1,m}(z) = \frac{1}{z^{m-1}} \left(e^z - \sum_{n=0}^{m-2} \frac{z^n}{n!} \right), \quad m \in \mathbb{N}, m \geq 1. \quad (1.6)$$

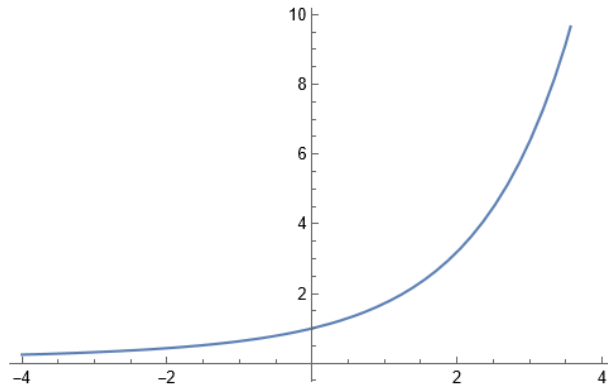


Figure 1.3: The Metag-Leffler function

2. Fractional Integration

Fractional integration lies at the heart of fractional calculus. While conventional integration is restricted to positive integer orders, fractional calculus generalizes this operation to accommodate arbitrary real or complex orders. This continuous range of integration orders offers a richer mathematical framework, enabling more accurate descriptions of physical and engineering systems where memory, heredity, or non-local effects play a significant role.

2.1 Riemann-Liouville Fractional integration

The Riemann–Liouville fractional integral is one of the earliest and most fundamental definitions in fractional calculus. It extends the concept of repeated integration to non-integer orders by means of a convolution integral involving the Gamma function. To understand its origin, consider a continuous function f defined on the interval $[0, T]$ with $T > 0$.

The first primitive of f is simply the standard integral:

$$I^1 f(t) = \int_0^t f(\tau) d\tau$$

For a second primitive, using Fubini's theorem :

For : $0 \leq \tau \leq u \leq t$.

we obtain:

$$\begin{aligned} I^2 f(t) &= \int_0^t I^1 f(u) du \\ &= \int_0^t \left(\int_0^u f(\tau) d\tau \right) du \\ &= \int_0^t \left(\int_\tau^t f(\tau) du \right) d\tau \\ &= \int_0^t \left(\int_\tau^t du \right) f(\tau) d\tau \\ &= \int_0^t (t - \tau) f(\tau) d\tau. \end{aligned}$$

By repeating this process n times, we generalize formula n of the function f to the following formula:

$$I^n f(t) = \frac{1}{(n-1)!} \int_0^t (t - \tau)^{n-1} f(\tau) d\tau, \quad t > 0, n \in \mathbb{N}^*. \quad (1.7)$$

Because the factorial is generalized using the gamma function, the following definition is given by:

Definition 1.2.1: [19],[15] The Riemann-Liouville fractional integral of the function f of order $\alpha \in \mathbb{C}$, $(\Re(\alpha) > 0)$, denoted by I^α , Let $f \in L^1([0, T])$, $T > 0$ is defined by:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0, \alpha > 0. \quad (1.8)$$

where : $\Gamma(\alpha)$ is the Gamma function .

Properties 1.2.1: Let α and β be two real numbers, and let f be an integrable function on the interval $[a, b)$

The following properties hold:

1. $I^\alpha (I^\beta f(t)) = I^{\alpha+\beta} f(t), \quad \alpha > 0, \beta > 0.$
2. $I^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}, \quad \alpha > 0.$

Proof:

1. By definition of the Riemann–Liouville fractional integral:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0, \alpha > 0.$$

Applying I^α to $I^\beta f(t)$, we get:

$$I^\alpha (I^\beta f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left[\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau-\xi)^{\beta-1} f(\xi) d\xi \right] d\tau.$$

Substitute $\tau = \xi + (t-\xi)s$, $d\tau = (t-\xi)ds$, we obtain:

$$\int_\xi^t (t-\tau)^{\alpha-1} (\tau-\xi)^{\beta-1} d\tau = (t-\xi)^{\alpha+\beta-1} \int_0^1 (1-s)^{\alpha-1} s^{\beta-1} ds,$$

where:

$$\int_0^1 (1-s)^{\alpha-1} s^{\beta-1} ds = B(\beta, \alpha) = \frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\alpha+\beta)}, \quad \alpha, \beta > 0.$$

So,

$$I^\alpha (I^\beta f)(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\xi)^{\alpha+\beta-1} f(\xi) d\xi = I^{\alpha+\beta} f(t), \quad \alpha > 0, \beta > 0.$$

2. By definition of the Riemann–Liouville fractional integral:

$$I^\alpha t^\beta = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^\beta d\tau, \quad t > 0, \alpha > 0, \beta > -1.$$

Let $\tau = ts$, then $d\tau = tds$, thus the integral becomes:

$$\begin{aligned} I^\alpha t^\beta &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^\beta d\tau \\ &= \frac{t^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^\beta ds \\ &= \frac{t^{\alpha+\beta}}{\Gamma(\alpha)} B(\beta+1, \alpha) \\ &= \frac{t^{\alpha+\beta}}{\Gamma(\alpha)} \cdot \frac{\Gamma(\beta+1)\Gamma(\alpha)}{\Gamma(\alpha+\beta+1)} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}. \end{aligned} \tag{1.9}$$

Example 1.2.1: Calculating the Riemann-Liouville fractional integral of the function $f(x) = x^3$, if $\alpha = 1 > 0, \beta = 3 > -1$, the integral is given Thus,

$$\begin{aligned} I^\alpha x^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta} \\ I^\alpha x^3 &= \frac{\Gamma(3+1)}{\Gamma(1+3+1)} x^{1+3} = \frac{\Gamma(4)}{\Gamma(5)} x^4 = \frac{6}{24} x^4 = \frac{1}{4} x^4 \end{aligned}$$

Remark 1.2.1: Relation (1.9) shows that the Riemann-Liouville fractional integral of order α of a constant is given by:

$$I^\alpha C = \frac{C}{\Gamma(\alpha+1)} x^\alpha, \quad x > 0, \alpha > 0.$$

Proof: Let C be a constant function. The Riemann–Liouville fractional integral of order $\alpha > 0$ is defined by:

$$I^\alpha C = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} C dt, \quad x > 0.$$

Thus:

$$I^\alpha C = \frac{C}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} dt.$$

We make the substitution $t = xu$, so $dt = xdu$. When $t = 0, u = 0$; and when $t = x, u = 1$. This gives:

$$I^\alpha C = \frac{C}{\Gamma(\alpha)} \int_0^1 (x-xu)^{\alpha-1} x du = \frac{Cx^\alpha}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} du.$$

The integral $\int_0^1 (1-u)^{\alpha-1} du$ is the Beta function $B(1, \alpha)$, which satisfies:

$$B(1, \alpha) = \frac{\Gamma(1)\Gamma(\alpha)}{\Gamma(1+\alpha)} = \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \quad (\text{since } \Gamma(1) = 1).$$

Substituting this result, we obtain:

$$I^\alpha C = \frac{Cx^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} = \frac{C}{\Gamma(\alpha+1)} x^\alpha.$$

Hence, for any constant function C , the Riemann–Liouville fractional integral of order α is given by:

$$I^\alpha C = \frac{C}{\Gamma(\alpha+1)} x^\alpha, \quad x > 0, \alpha > 0.$$

3. Fractional Derivative

We now introduce fractional derivatives, which generalize the familiar concept of differentiation to non-integer orders. Unlike classical derivatives that are limited to positive integers, fractional derivatives provide a continuous spectrum of differentiation orders. This generalization is not merely a mathematical curiosity; it has proven essential for capturing the complex dynamics observed in many physical, biological, and engineering systems where integer-order models fall short.

3.1 Riemann-Liouville Fractional Derivative

Originally developed by Riemann and Liouville in the 1800s, the Riemann–Liouville fractional derivative stands among the most frequently employed definitions in fractional calculus. Its construction—combining fractional integration with conventional differentiation—extends the notion of derivative to arbitrary non-integer orders. This formulation plays a central role in modeling complex systems and remains fundamental to the study and application of fractional differential equations.

Definition 1.3.1: [19],[15] Let $f \in L^1([0, T])$, with $T > 0$, and let $\alpha \in \mathbb{C}$ such that $\Re(\alpha) > 0$. The Riemann-Liouville fractional derivative of the function f of order α , denoted by ${}^{RL}D^\alpha f(t)$, is defined as:

$${}^{RL}D^\alpha f(t) = \begin{cases} \frac{d^n}{dt^n} [I^{n-\alpha} f(t)] = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, & n-1 \leq \Re(\alpha) < n, \\ \frac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N}. \end{cases} \quad (1.10)$$

Proposition 1.3.1: [19],[15] Let $n - 1 < \alpha \leq n$ and $m - 1 < \beta \leq m$, where $n, m \in \mathbb{N}$, and let f and g be functions for which ${}^{RL}D^\alpha f(t)$ and ${}^{RL}D^\beta g(t)$ exist.

Then we have the following properties:

1. **Linearity:** The Riemann-Liouville fractional derivative is a linear operator:

$${}^{RL}D^\alpha(\lambda f + g)(t) = \lambda ({}^{RL}D^\alpha f)(t) + ({}^{RL}D^\alpha g)(t), \quad \lambda \in \mathbb{R}.$$

2. **Reduction to classical derivative:** In the limit of integer order:

$$\lim_{\alpha \rightarrow n} {}^{RL}D^\alpha f(t) = f^{(n)}(t), \quad \lim_{\alpha \rightarrow n-1} {}^{RL}D^\alpha f(t) = f^{(n-1)}(t).$$

3. **Non-commutativity:** In general, Riemann-Liouville fractional derivatives do not commute:

$${}^{RL}D^\alpha ({}^{RL}D^\beta f(t)) \neq {}^{RL}D^\beta ({}^{RL}D^\alpha f(t)).$$

Properties 1.3.1: We proceed to compute the Riemann-Liouville fractional derivative of $f(t) = t^\beta$. Let $\alpha > 0$ such that $n - 1 < \alpha < n$ with $\beta \geq -1$ and $n \in \mathbb{N}$. According to equations (1.11) and (1.10), we have:

$${}^{RL}D^\alpha t^\beta = {}^{RL}D^n \left[I^{n-\alpha} t^\beta \right] = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + n - \alpha + 1)} {}^{RL}D^n t^{\beta+n-\alpha}. \quad (1.11)$$

Considering the identity:

$$\begin{aligned} {}^{RL}D^n t^{\beta+n-\alpha} &= (\beta + n - \alpha)(\beta + n - \alpha - 1) \cdots (\beta - \alpha + 1) t^{\beta-\alpha} \\ &= \frac{\Gamma(\beta + n - \alpha + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta-\alpha}. \end{aligned} \quad (1.12)$$

Substituting the result from (1.13) into formula (1.12), we obtain:

$$\begin{aligned} {}^{RL}D^\alpha t^\beta &= \frac{\Gamma(\beta + 1)}{\Gamma(\beta + n - \alpha + 1)} {}^{RL}D^n t^{\beta+n-\alpha} \\ &= \frac{\Gamma(\beta + 1)}{\Gamma(\beta + n - \alpha + 1)} \cdot \frac{\Gamma(\beta + n - \alpha + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta-\alpha} \\ &= \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta-\alpha}. \end{aligned}$$

Thus, the Riemann-Liouville fractional derivative of the function $f(t) = t^\beta$ is given by:

$${}^{RL}D^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}. \quad (1.13)$$

Example 1.3.1: The Riemann-Liouville fractional derivative of the function

$f(x) = (x-1)^2$ and $\alpha = 1, \beta = 2$ is given by:

$${}^{RL}D^\alpha (x-1)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x-1)^{\beta-\alpha} = \frac{\Gamma(3)}{\Gamma(2)} (x-1) = 2x-2$$

$${}^{RL}D^1 (x-1)^2 = 2x-2$$

Remark 1.3.1: In particular, if $\beta = 0$ and $\alpha > 0$, the Riemann-Liouville fractional derivative of a constant function, $f(t) = C$, is nonzero, its value is:

$${}^{RL}D^\alpha C = \frac{C}{\Gamma(1-\alpha)} t^{-\alpha},$$

3.2 Caputo Fractional Derivative

In 1967, Michele Caputo introduced the fractional Caputo derivative, a modification of the Riemann-Liouville derivative that generalizes classical derivatives to include non-integer orders.

This model is of great importance in the context of physical systems, as it allows the use of initial conditions as derivatives of integer orders, making it more suitable for modeling real-world problems.

Definition 1.3.2: [19] Let $f \in C^n([0, T])$, with $T > 0$, and let $\alpha \in \mathbb{C}$ such that $\Re(\alpha) > 0$. The Caputo fractional derivative of the function f of order α , denoted by ${}^CD^\alpha f(t)$, is defined as:

$${}^CD^\alpha f(t) = \begin{cases} I^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 \leq \Re(\alpha) < n, \\ \frac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N}. \end{cases} \quad (1.14)$$

Proposition 1.3.2: [19],[15] Let $n-1 < \alpha \leq n$ and $m-1 < \beta \leq m$, where $n, m \in \mathbb{N}$, and let f and g be functions for which ${}^CD^\alpha f(t)$ and ${}^CD^\alpha g(t)$ exist, Then we have the following properties:

1. **Linearity:** The Caputo fractional derivative is a linear operator:

$${}^CD^\alpha (\lambda f + g)(t) = \lambda ({}^CD^\alpha f)(t) + ({}^CD^\alpha g)(t), \quad \lambda \in \mathbb{R}.$$

2. **Reduction to classical derivative:** In the limit of integer order:

$$\lim_{\alpha \rightarrow n} {}^C D^\alpha f(t) = f^{(n)}(t), \quad \lim_{\alpha \rightarrow n-1} {}^C D^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0).$$

3. **Non-commutativity:** In general, Caputo fractional derivatives do not commute:

$${}^C D^\alpha ({}^C D^\beta f(t)) \neq {}^C D^\beta ({}^C D^\alpha f(t)).$$

Proof:

1. **Linearity:** Consider the linear combination $\lambda f(t) + g(t)$ and apply the definition of the Caputo derivative:

$$\begin{aligned} {}^C D^\alpha (\lambda f + g)(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t [\lambda f^{(n)}(\tau) + g^{(n)}(\tau)] (t-\tau)^{n-\alpha-1} d\tau \\ &= \frac{\lambda}{\Gamma(n-\alpha)} \int_0^t f^{(n)}(\tau) (t-\tau)^{n-\alpha-1} d\tau + \frac{1}{\Gamma(n-\alpha)} \int_0^t g^{(n)}(\tau) (t-\tau)^{n-\alpha-1} d\tau \\ &= \lambda {}^C D^\alpha f(t) + {}^C D^\alpha g(t). \end{aligned}$$

2. **Reduction to classical derivative:** Using integration by parts, set

$$u = f^{(n)}(\tau), \quad dv = (t-\tau)^{n-\alpha-1} d\tau \Rightarrow du = f^{(n+1)}(\tau) d\tau, \quad v = -\frac{(t-\tau)^{n-\alpha}}{n-\alpha}.$$

Then

$$\begin{aligned} {}^C D^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \left[-\frac{f^{(n)}(\tau)(t-\tau)^{n-\alpha}}{n-\alpha} \Big|_0^t + \int_0^t \frac{f^{(n+1)}(\tau)(t-\tau)^{n-\alpha}}{n-\alpha} d\tau \right] \\ &= \frac{f^{(n)}(0)t^{n-\alpha}}{\Gamma(n-\alpha+1)} + \frac{1}{\Gamma(n-\alpha+1)} \int_0^t f^{(n+1)}(\tau)(t-\tau)^{n-\alpha} d\tau. \end{aligned}$$

Considering the limits:

$$\begin{aligned} \lim_{\alpha \rightarrow n} {}^C D^\alpha f(t) &= f^{(n)}(t), \\ \lim_{\alpha \rightarrow n-1} {}^C D^\alpha f(t) &= f^{(n-1)}(t) - f^{(n-1)}(0). \end{aligned}$$

Example 1.3.2: We seek to determine the Caputo fractional derivative of the function $f(t) = t^\beta$.

Let $n \in \mathbb{N}$ be such that $n - 1 < \alpha < n$ with $\beta > n - 1$.

The n -th derivative of $f(\tau) = \tau^\beta$ is given by:

$$f^{(n)}(\tau) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - n + 1)} \tau^{\beta - n}. \quad (1.15)$$

Using the definition of the Caputo fractional derivative, we have:

$${}^C D^\alpha t^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(n - \alpha)\Gamma(\beta - n + 1)} \int_0^t (t - \tau)^{n - \alpha - 1} \tau^{\beta - n} d\tau. \quad (1.16)$$

By making the change of variable $\tau = ts$, with $s = 0$ when $\tau = 0$ and $s = 1$ when $\tau = t$, so that $d\tau = t ds$, equation (1.16) becomes:

$$\begin{aligned} {}^C D^\alpha t^\beta &= \frac{\Gamma(\beta + 1)}{\Gamma(n - \alpha)\Gamma(\beta - n + 1)} \int_0^1 (t - ts)^{n - \alpha - 1} (ts)^{\beta - n} t ds \\ &= \frac{\Gamma(\beta + 1)}{\Gamma(n - \alpha)\Gamma(\beta - n + 1)} t^{\beta - \alpha} \int_0^1 (1 - s)^{n - \alpha - 1} s^{\beta - n} ds. \end{aligned}$$

Using the definition of the Beta function $B(x, y) = \int_0^1 s^{x-1} (1 - s)^{y-1} ds$ and the relation $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we obtain:

$$\begin{aligned} {}^C D^\alpha t^\beta &= \frac{\Gamma(\beta + 1)B(n - \alpha, \beta - n + 1)}{\Gamma(n - \alpha)\Gamma(\beta - n + 1)} t^{\beta - \alpha} \\ &= \frac{\Gamma(\beta + 1)\Gamma(n - \alpha)\Gamma(\beta - n + 1)}{\Gamma(n - \alpha)\Gamma(\beta - n + 1)\Gamma(\beta - \alpha + 1)} t^{\beta - \alpha} \\ &= \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta - \alpha}. \end{aligned}$$

Thus, the Caputo fractional derivative of $f(t) = t^\beta$ is:

$${}^C D^\alpha t^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta - \alpha}. \quad (1.17)$$

Remark 1.3.2: For a constant function C defined on $[0, t]$ and $\alpha > 0$, the Caputo fractional derivative is zero:

$${}^C D^\alpha C = 0.$$

3.3 Relationship Between the Caputo and Riemann–Liouville

Understanding the relationship between the Caputo and Riemann–Liouville definitions is essential for several reasons. First, it allows us to translate results obtained with one formulation into the language of the other, facilitating comparison and cross-validation. Second, it clarifies the circumstances under which each definition is most appropriate, guiding the selection of the right tool for a given problem. Finally, exploring this connection deepens our theoretical insight into the underlying structure of fractional calculus itself.

Proposition 1.3.3: [19],[15] Let $\alpha \geq 0$ with $n - 1 < \alpha \leq n$, and let $f \in C^n([a, b))$. Then the following relation holds:

$${}^C D^\alpha f(t) = {}^{RL} D^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^{k-\alpha}. \quad (1.18)$$

Remark 1.3.3: The Caputo fractional derivative coincides with the Riemann–Liouville derivative, i.e.,

$${}^C D^\alpha f(t) = {}^{RL} D^\alpha f(t), \quad (1.19)$$

if

$$f^{(k)}(0) = 0, \quad k = 0, 1, 2, \dots, n-1.$$

3.4 Comparison Between the Caputo and Riemann–Liouville Fractional Derivatives

The main advantage of the Caputo approach is that the initial conditions of fractional differential equations involving Caputo derivatives have the same form as those of classical integer-order differential equations.

Property	Riemann–Liouville	Caputo
Representation	${}^{RL} D^\alpha f(t) = \frac{d^n}{dt^n} (I^{n-\alpha} f(t))$	${}^C D^\alpha f(t) = I^{n-\alpha} \left(\frac{d^n}{dt^n} f(t) \right)$
Reduction $\alpha \rightarrow n$	$\lim_{\alpha \rightarrow n} {}^{RL} D^\alpha f(t) = f^{(n)}(t)$	$\lim_{\alpha \rightarrow n} {}^C D^\alpha f(t) = f^{(n)}(t)$
Reduction $\alpha \rightarrow n-1$	$\lim_{\alpha \rightarrow n-1} {}^{RL} D^\alpha f(t) = f^{(n-1)}(t)$	$\lim_{\alpha \rightarrow n-1} {}^C D^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0)$
Linearity	${}^{RL} D^\alpha (\lambda f + g) = \lambda {}^{RL} D^\alpha f + {}^{RL} D^\alpha g$	${}^C D^\alpha (\lambda f + g) = \lambda {}^C D^\alpha f + {}^C D^\alpha g$
Non-commutativity	$\frac{d^n}{dt^n} ({}^{RL} D^\alpha f) = {}^{RL} D^{\alpha+n} f \neq {}^{RL} D^\alpha \left(\frac{d^n}{dt^n} f \right)$	${}^C D^\alpha \left(\frac{d^n}{dt^n} f \right) = {}^C D^{\alpha+n} f \neq \frac{d^n}{dt^n} ({}^C D^\alpha f)$
Constant $f(t) = c$	${}^{RL} D^\alpha C = \frac{C}{\Gamma(1-\alpha)} t^{-\alpha}$	${}^C D^\alpha C = 0$

Table 1.1: Comparison between the Riemann–Liouville and Caputo fractional derivatives

4. Fractional Power Series Expansions

Fractional power series (FPS) expansions are the focus of this section. These expansions provide an effective analytical approach for tackling fractional differential equations, especially in situations where classical techniques break down. Their ability to capture solution behavior near singularities or initial values makes them a versatile tool, bridging theoretical developments and real-world applications in fractional calculus.

Definition 1.4.1: [10] A fractional power series centred at t_0 is given by :

$$\sum_{n=0}^{+\infty} c_n (t - t_0)^{n\alpha} = c_0 + c_1 (t - t_0)^\alpha + c_2 (t - t_0)^{2\alpha} + \dots, \quad (1.20)$$

where $0 \leq m - 1 < \alpha < m$, $m \in \mathbb{N}$, $t \geq t_0$, and c_n are constants called the coefficients of the series.

Theorem 1.4.1: [10] Let $\sum_{n=0}^{+\infty} c_n t^{n\alpha}$, $t \geq 0$, be a fractional power series.

1. If the series converges at $t = b > 0$, then it converges for all $0 \leq t < b$.
2. If the series diverges at $t = d > 0$, then it diverges for all $t > d$.

Theorem 1.4.2: [10] For the fractional power series $\sum_{n=0}^{\infty} c_n t^{n\alpha}$, $t \geq 0$, there are only three possibilities:

1. The series converges only at $t = 0$.
2. The series converges for all $t \geq 0$.
3. There exists a positive real number R such that the series converges for $0 \leq t < R$ and diverges for $t \geq R$.

Remark 1.4.1: The number R in Theorem (1.4.2) is called the radius of convergence of the FPS. In case 1, $R = 0$, and in case 2, $R = +\infty$.

Definition 1.4.2 [4] For $0 \leq m - 1 < \alpha \leq m$, a series of the form

$$\sum_{n=0}^{+\infty} a_n(x) (t - t_0)^{n\alpha} = a_0(x) + a_1(x) (t - t_0)^\alpha + a_2(x) (t - t_0)^{2\alpha} + \dots, \quad (1.21)$$

where $x \in I$ and $t \geq t_0$, is called a *multiple fractional power series* about $t = t_0$, with $a_n(x)$ being functions of x called the coefficients.

Theorem 1.4.3: [4] Suppose $u(x, t)$ has a multiple FPS representation at $t = t_0$ of the form

$$u(x, t) = \sum_{n=0}^{+\infty} a_n(x) (t - t_0)^{n\alpha}, \quad 0 \leq m - 1 < \alpha, \quad x \in I, \quad t_0 \leq t < t_0 + R. \quad (1.22)$$

If $D_t^{n\alpha}u(x, t)$ are continuous on $I \times (t_0, t_0 + R)$ for $n = 0, 1, 2, \dots$, then the coefficients are given by:

$$a_n(x) = \frac{D_t^{n\alpha}u(x, t_0)}{\Gamma(n\alpha + 1)}, \quad n = 0, 1, 2, \dots, \quad (1.23)$$

where:

$$D_t^{n\alpha} = \underbrace{\frac{\partial^\alpha}{\partial t^\alpha} \cdot \frac{\partial^\alpha}{\partial t^\alpha} \cdots \frac{\partial^\alpha}{\partial t^\alpha}}_{n\text{-times}}, \quad R = \min_{c \in I} R_c,$$

and R_c is the radius of convergence of the FPS $\sum_{n=0}^{+\infty} a_n(c)(t - t_0)^{n\alpha}$.

Proof: Assume $u(x, t)$ can be represented by the series (1.22). Substituting $t = t_0$ into (1.22), all terms after the first vanish, giving

$$a_0(x) = u(x, t_0).$$

Applying D^α once to (1.22) gives

$$D_t^\alpha u(x, t) = \Gamma(\alpha + 1)a_1(x) + \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)}a_2(x)(t - t_0)^\alpha + \frac{\Gamma(3\alpha + 1)}{\Gamma(2\alpha + 1)}a_3(x)(t - t_0)^{2\alpha} + \dots \quad (1.24)$$

Substituting $t = t_0$ in (1.24) determines

$$a_1(x) = \frac{D_t^\alpha u(x, t_0)}{\Gamma(\alpha + 1)}. \quad (1.25)$$

Similarly, applying D_t^α twice gives

$$D_t^{2\alpha} u(x, t) = \Gamma(2\alpha + 1)a_2(x) + \frac{\Gamma(3\alpha + 1)}{\Gamma(\alpha + 1)}a_3(x)(t - t_0)^\alpha + \frac{\Gamma(4\alpha + 1)}{\Gamma(2\alpha + 1)}a_4(x)(t - t_0)^{2\alpha} + \dots \quad (1.26)$$

Substituting $t = t_0$ in (1.26) gives

$$a_2(x) = \frac{D_t^{2\alpha} u(x, t_0)}{\Gamma(2\alpha + 1)}. \quad (1.27)$$

By induction, applying D_t^α recursively n times and substituting $t = t_0$ gives the general formula

$$a_n(x) = \frac{D_t^{n\alpha} u(x, t_0)}{\Gamma(n\alpha + 1)}, \quad n = 0, 1, 2, \dots \quad (1.28)$$

Hence, the proof is complete.

The Residual Power Series Method(RPSM)



This chapter presents the Residual Power Series Method (RPSM), an analytical technique for solving nonlinear fractional partial differential equations involving Caputo derivatives. The approach follows a clear algorithmic structure. Its application is illustrated through a practical example that highlights both accuracy and ease of implementation. A theoretical convergence analysis is then provided, grounded in the framework of the fractional Taylor series.

1. Methodology

In this section, we describe the step-by-step application of the Residual Power Series Method (RPSM) to solve a nonlinear time-fractional partial differential equation involving Caputo derivatives. The equation under consideration takes the following general form:

$${}^C D_t^\alpha u(x,t) + \mathcal{L}[u(x,t)] + \mathcal{N}[u(x,t)] = f(x,t), \quad (2.1)$$

$$t > 0, x \in \mathbb{R}, \quad 0 < \alpha \leq 1.$$

subject to the following initial condition:

$$u(x,0) = \varphi(x) = \varphi_0(x) \quad (2.2)$$

In this formulation, the symbol ${}^C D_t^\alpha$ denotes the Caputo fractional derivative of order α , while $\mathcal{L}$ and $\mathcal{N}$ are linear and nonlinear operators, respectively, and $f(x,t)$ is a continuous function.

The RPSM procedure for obtaining an approximate solution to the initial value problem (2.1)–(2.2) consists of the following steps:

Step 1: Fractional Power Series Expansion

Assume that the solution $u(x, t)$ can be expressed in the fractional power series form

$$u(x, t) = \sum_{n=0}^{+\infty} \varphi_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \quad 0 < \alpha \leq 1, \quad x \in I, \quad 0 \leq t < R \quad (2.3)$$

From the initial condition, we obtain

$$\varphi_0(x) = \varphi(x).$$

Step 2: Truncated Series Solution

Define the k -th truncated series approximation as

$$u_k(x, t) = \sum_{n=0}^k \varphi_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \quad 0 < \alpha \leq 1, \quad x \in I, \quad 0 \leq t < R, \quad k = 1, 2, 3, \dots \quad (2.4)$$

Step 3: Define the Residual Functions

The residual function measures how well a candidate solution satisfies the original equation:

$$\text{Res}(x, t) = D_t^\alpha u(x, t) + \mathcal{L}[u(x, t)] + \mathcal{N}[u(x, t)] - f(x, t). \quad (2.5)$$

For the truncated series, we define the k -th residual function as

$$\text{Res}_k(x, t) = D_t^\alpha u_k(x, t) + \mathcal{L}[u_k(x, t)] + \mathcal{N}[u_k(x, t)] - f(x, t). \quad (2.6)$$

Ideally, $\text{Res}(x, t) = 0$ for the exact solution and $\lim_{k \rightarrow \infty} \text{Res}_k(x, t) = \text{Res}(x, t)$.

Step 4: Determination of the Unknown Coefficients

The key idea of the Residual Power Series Method (RPSM) is to enforce that the fractional derivatives of the residual function vanish at $t = 0$. Because the Caputo fractional derivative of a constant is zero, we impose the condition

$${}^C D_t^{(k-1)\alpha} \text{Res}_k(x, t) \Big|_{t=0} = 0, \quad k = 1, 2, 3, \dots \quad (2.7)$$

These conditions yield algebraic equations for the unknown coefficients $\varphi_k(x)$. Starting with $k = 1$:

- For $k = 1$, we compute

$$D_t^0 \text{Res}_1(x, 0) = \text{Res}_1(x, 0) = 0,$$

which allows us to determine $\varphi_1(x)$.

- For $k = 2$, we compute

$$D_t^\alpha \text{Res}_2(x, 0) = 0,$$

to determine $\varphi_2(x)$, using the previously obtained functions $\varphi_0(x)$ and $\varphi_1(x)$.

This procedure continues recursively. At each step k , we solve

$$D_t^{(k-1)\alpha} \text{Res}_k(x, 0) = 0$$

to obtain the coefficient function $\varphi_k(x)$.

Step 5: Construction of the Approximate Solution

After determining the coefficients, the approximate solution of order k is obtained as

$$u_k(x, t) = \varphi_0(x) + \frac{\varphi_1(x)}{\Gamma(1 + \alpha)} t^\alpha + \frac{\varphi_2(x)}{\Gamma(1 + 2\alpha)} t^{2\alpha} + \dots + \frac{\varphi_k(x)}{\Gamma(1 + k\alpha)} t^{k\alpha}.$$

$$u_k(x, t) = \varphi(x) + \sum_{n=1}^k \varphi_n(x) \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)}. \quad (2.8)$$

Example 2.1.1:

Consider the nonlinear time-fractional partial differential equation [17]:

$$\begin{cases} {}^C D_t^\alpha u(x, t) = -u(x, t) \frac{\partial u}{\partial x}(x, t), & 0 < \alpha \leq 1, \\ u(x, 0) = -x \end{cases} \quad (2.9)$$

Using the Residual Power Series Method (RPSM), we determine the RPS solution for $n = 4$.

Solution:

To determine the first unknown coefficient $\varphi_1(x)$ in the series expansion given by Eq. (2.6), we begin by substituting the first-order truncated series $u_1(x, t)$ into the first residual function $\text{Res}_1(x, t)$. This gives

$$\text{Res}_1(x, t) = {}^C D_t^\alpha u_1(x, t) + u_1(x, t) \frac{\partial u_1}{\partial x}(x, t). \quad (2.10)$$

Recall that the first truncated series is

$$u_1(x, t) = \varphi_0(x) + \varphi_1(x) \frac{t^\alpha}{\Gamma(1 + \alpha)}.$$

Substituting this into Eq. (2.10) gives

$$\text{Res}_1(x, t) = \varphi_1(x) + \left[\varphi_0(x) + \frac{t^\alpha}{\Gamma(1+\alpha)} \varphi_1(x) \right] \left[\varphi_0'(x) + \frac{t^\alpha}{\Gamma(1+\alpha)} \varphi_1'(x) \right] \quad (2.11)$$

Now, evaluating this expression at $t = 0$ simplifies it considerably, as all terms containing powers of t vanish. We obtain

$$\text{Res}_1(x, 0) = \varphi_1(x) + \varphi_0(x) \varphi_0'(x) \quad (2.12)$$

Imposing the condition that the residual function must be zero at the initial time, i.e., $\text{Res}_1(x, 0) = 0$, leads to an algebraic equation for $\varphi_1(x)$:

$$\varphi_1(x) = -\varphi_0(x) \varphi_0'(x) \quad (2.13)$$

By compensation we find :

$$\varphi_1(x) = -x$$

With $\varphi_0(x)$ known from the initial condition, this relation directly provides the first coefficient. Subsequently, the first-order RPS approximate solution can be constructed as

$$u_1(x, t) = x \left(-1 - \frac{t^\alpha}{\Gamma(1+\alpha)} \right) \quad (2.14)$$

This process is then repeated for higher-order approximations. For the second coefficient $\varphi_2(x)$, we substitute the second truncated series

$$u_2(x, t) = \varphi_0(x) + \varphi_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + \varphi_2(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

into the second residual function

$$\text{Res}_2(x, t) = D_t^\alpha u_2(x, t) + u_2(x, t) \frac{\partial u_2}{\partial x}(x, t).$$

This substitution yields a more involved expression:

$$\begin{aligned}
\text{Res}_2(x, t) = & \varphi_1(x) + \varphi_2(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + \varphi_0(x) \varphi_0'(x) + \frac{t^\alpha}{\Gamma(1+\alpha)} \varphi_1(x) \varphi_0'(x) \\
& + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \varphi_2(x) \varphi_0'(x) + \frac{t^\alpha}{\Gamma(1+\alpha)} \varphi_0(x) \varphi_1'(x) + \frac{t^{2\alpha}}{\Gamma^2(1+\alpha)} \varphi_1(x) \varphi_1'(x) \\
& + \frac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} \varphi_2(x) \varphi_1'(x) + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \varphi_0(x) \varphi_2'(x) \\
& + \frac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} \varphi_1(x) \varphi_2'(x) + \frac{t^{4\alpha}}{\Gamma^2(1+2\alpha)} \varphi_2(x) \varphi_2'(x)
\end{aligned} \tag{2.15}$$

Applying the fractional derivative operator D_t^α to both sides of Eq. (2.15) and then setting $t = 0$ (so that the condition $D_t^\alpha \text{Res}_2(x, 0) = 0$ holds) gives an equation for $\varphi_2(x)$:

$$\varphi_2(x) = -\varphi_0(x) \varphi_1'(x) - \varphi_1(x) \varphi_0'(x) \tag{2.16}$$

By compensation we find :

$$\varphi_2(x) = -2x$$

Thus, using previously determined coefficients, we find $\varphi_2(x)$. The second-order approximate solution then becomes

$$u_2(x, t) = \varphi_0(x) + \varphi_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + \varphi_2(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \tag{2.17}$$

$$u_2(x, t) = x \left(-1 - \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)} \right)$$

Continuing in the same fashion, we can compute higher-order coefficients $\varphi_3(x), \varphi_4(x), \dots$ by imposing successive conditions $D_t^{(k-1)\alpha} \text{Res}_k(x, 0) = 0$. The resulting expressions become increasingly lengthy but are straightforward to derive with the aid of symbolic computation. For instance, the third and fourth coefficients take the form

$$\begin{aligned}
\varphi_3(x) &= x \left[-4 - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} \right], \\
u_3(x, t) &= x \left[-1 - \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{4t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{t^{3\alpha} \Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2 \Gamma(1+3\alpha)} \right], \\
\varphi_4(x) &= 2x \left[-4 - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} - \frac{2\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} \right],
\end{aligned}$$

$$u_4(x, t) = x \left[-1 - \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha} \left(-4 - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} \right)}{\Gamma(1+3\alpha)} + \frac{2t^{4\alpha} \left(-4 - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} - \frac{2\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} \right)}{\Gamma(1+4\alpha)} \right]$$

2. Convergence Analysis

In this section, we introduce key foundational results related to the convergence and error bounds of approximate solutions obtained using the residual power series method (RPSM) for fractional differential equations. We begin by providing an estimate for the limit of the residual in the context of a fractional Taylor series. Next, we present a sufficient condition to ensure that the approximate RPSM solutions converge to the exact solution of the problem.

Theorem 2.2.1:

[9] Let $(X, \|\cdot\|)$ be a Banach space. we have a sequence $\{u_n(x, t)\}_{n \geq 0} \subset X$ satisfying

$$\|u_{n+1}\| \leq K \|u_n\| \quad \forall n \in \mathbb{N}, \quad 0 < K < 1.$$

for some constant K and for every fixed $0 < t < R < 1$ and x in the parameter domain. Then the sequence $\{u_n(x, t)\}_{n \geq 0}$ converges (geometrically) in X to a limit $u(x, t) \in X$. and the convergence is uniform in the parameters on any domain where the above inequality holds.

Moreover, for the N -th partial sum $S_N = \sum_{n=0}^N u_n$, the remainder satisfies

$$\left\| u - \sum_{n=0}^N u_n \right\| \leq \frac{K^{N+1}}{1-K} \|u_0\|.$$

Proof: We prove by induction that

$$\|u_n\| \leq K^n \|u_0\| \quad \forall n \geq 0. \quad (*)$$

For $n = 0$ the statement is trivial. Assuming it holds for some n , we have

$$\|u_{n+1}\| \leq K \|u_n\| \leq K \cdot K^n \|u_0\| = K^{n+1} \|u_0\|,$$

which completes the induction.

Next, we estimate the difference between successive terms using the triangle inequality and the inductive bound (*):

$$\|u_{n+1} - u_n\| \leq \|u_{n+1}\| + \|u_n\| \leq (K^{n+1} + K^n)\|u_0\| = K^n(1 + K)\|u_0\|. \quad (**)$$

For any integers $n > m \geq 0$, write the difference as a telescoping sum:

$$u_n - u_m = \sum_{j=m+1}^n (u_j - u_{j-1}).$$

Applying the triangle inequality and the estimate (**), we obtain

$$\|u_n - u_m\| \leq \sum_{j=m+1}^n \|u_j - u_{j-1}\| \leq \sum_{j=m+1}^n K^{j-1}(1 + K)\|u_0\|.$$

The right-hand side involves a finite geometric series. Indeed,

$$\sum_{j=m+1}^n K^{j-1} = K^m + K^{m+1} + \dots + K^{n-1}.$$

This is a geometric series with first term K^m , common ratio K , and $n - m$ terms. Using the standard formula for the sum of a finite geometric series,

$$\sum_{j=m+1}^n K^{j-1} = K^m \frac{1 - K^{n-m}}{1 - K}, \quad (K \neq 1).$$

Substituting this into the estimate yields

$$\|u_n - u_m\| \leq (1 + K)\|u_0\| \cdot K^m \frac{1 - K^{n-m}}{1 - K}.$$

Since $1 - K^{n-m} \leq 1$, we obtain the simpler bound

$$\|u_n - u_m\| \leq \frac{(1 + K)K^m}{1 - K} \|u_0\|. \quad (***)$$

Because $0 < K < 1$, we have $K^m \rightarrow 0$ as $m \rightarrow \infty$. Hence $\|u_n - u_m\| \rightarrow 0$ as $n, m \rightarrow \infty$. Therefore, $\{u_n\}$ is a Cauchy sequence.

Since X is a Banach space (i.e., a complete normed vector space), every Cauchy sequence converges in X .

Consequently, there exists an element $u \in X$ such that

$$u_n \rightarrow u \quad \text{in } X.$$

Now, let $S_N = \sum_{n=0}^N u_n$ denote the N -th partial sum of the series $\sum_{n=0}^{\infty} u_n$. Then the remainder is given by

$$u - S_N = \sum_{n=N+1}^{\infty} u_n.$$

Using the triangle inequality and the inductive bound (*),

$$\|u - S_N\| \leq \sum_{n=N+1}^{\infty} \|u_n\| \leq \sum_{n=N+1}^{\infty} K^n \|u_0\|.$$

The infinite series on the right-hand side is geometric with ratio K . Its sum is

$$\sum_{n=N+1}^{\infty} K^n = K^{N+1} (1 + K + K^2 + \cdots) = \frac{K^{N+1}}{1 - K}.$$

Thus, we obtain the remainder estimate

$$\|u - S_N\| \leq \frac{K^{N+1}}{1 - K} \|u_0\|. \quad (****)$$

In particular, since $K^{N+1} \rightarrow 0$ as $N \rightarrow \infty$, this confirms the convergence of the series $\sum_{n=0}^{\infty} u_n$ to u . Finally, we address the uniformity of convergence with respect to the parameters x, t . Assume that the constant K is independent of the parameters and that $\|u_0(x, t)\|$ is uniformly bounded; i.e., there exists a constant $M < \infty$ such that

$$M = \sup_{x, t} \|u_0(x, t)\| < \infty.$$

Then from estimate (**), we have

$$\sup_{x, t} \left\| u(x, t) - \sum_{n=0}^N u_n(x, t) \right\| \leq \frac{K^{N+1}}{1 - K} M.$$

The right-hand side tends to zero as $N \rightarrow \infty$ and does not depend on x, t . Hence, the convergence is uniform in the parameters.

Numerical Implementation



This chapter focuses on the practical implementation of the Residual Power Series Method (RPSM) for solving nonlinear partial differential equations involving fractional derivatives in time. To illustrate its capabilities, the chapter applies the method to several test cases derived from the fractional-time nonlinear Kuramoto–Sivashinsky(KS)Equation. It first details how the RPSM is adapted to the general form of this equation, then moves on to explore specific instances. The accuracy of the obtained solutions is evaluated by comparing them with both exact analytical results and those produced by alternative numerical approaches. Tables and various graphical illustrations, including two- and three-dimensional plots, are included to clarify the findings and support the discussion.

1. General Application

This section outlines the general steps for using the Residual Power Series Method (RPSM) to solve the nonlinear fractional-time Kuramoto Sivashinski (KS) equation. It begins with the general formula of the equation and then reviews the key components of the process: how to formulate the problem and how to construct the power series solution. The discussion focuses on the initial value problem formulated using Caputo's derivative.

The equation is defined as follows:

$$\begin{cases} D_t^\alpha u(x,t) + u(x,t) \frac{\partial u}{\partial x}(x,t) + a \frac{\partial^2 u}{\partial x^2}(x,t) + b \frac{\partial^3 u}{\partial x^3}(x,t) + c \frac{\partial^4 u}{\partial x^4}(x,t) = 0, \\ x \in \mathbb{R}, t \in [0, T], 0 < \alpha \leq 1, \\ u(x, 0) = \varphi(x). \end{cases} \quad (3.1)$$

where a , b , and c are constants,

Given the nonlinear fractional-time Kuramoto-Sivashinski (KS) equation in problem (3.1), and considering the initial condition, according to the Residual Power Series Method (RPSM), the solution is written as a fractional power series.

$$u(x, t) = \sum_{n=0}^{\infty} \varphi_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \quad (3.2)$$

Using the initial condition

$$\varphi_0(x) = \varphi(x). \quad (3.3)$$

The k -th truncated series approximation is given by :

$$u_k(x, t) = \varphi(x) + \sum_{n=1}^k \varphi_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \quad (3.4)$$

To obtain the value of coefficients $\varphi_n(x)$, $n = 1, 2, \dots, k$ in series expanded of Eq.(3.4), Residual function Res is given by

$$\text{Res}(x, t) = D_t^\alpha u(x, t) + u(x, t) \frac{\partial u}{\partial x}(x, t) + a \frac{\partial^2 u}{\partial x^2}(x, t) + b \frac{\partial^3 u}{\partial x^3}(x, t) + c \frac{\partial^4 u}{\partial x^4}(x, t). \quad (3.5)$$

and the k -th Residual functions of equation (3.1) is defined by:

$$\text{Res}_k(x, t) = D_t^\alpha u_k(x, t) + u_k(x, t) \frac{\partial u_k}{\partial x}(x, t) + a \frac{\partial^2 u_k}{\partial x^2}(x, t) + b \frac{\partial^3 u_k}{\partial x^3}(x, t) + c \frac{\partial^4 u_k}{\partial x^4}(x, t). \quad (3.6)$$

to find the needed coefficients $\varphi_n(x)$ for $n = 1, 2, \dots, k$ in Eq.(3.4).

To obtain $\varphi_1(x)$, we consider $k = 1$ in Eq. (3.6),

$$\begin{aligned} \text{Res}_1(x, t) = & \varphi_1(x) + \left[\varphi(x) + \frac{t^\alpha \varphi_1(x)}{\Gamma(1+\alpha)} \right] \left[\varphi'(x) + \frac{t^\alpha \varphi_1'(x)}{\Gamma(1+\alpha)} \right] + a \left[\varphi''(x) + \frac{t^\alpha \varphi_1''(x)}{\Gamma(1+\alpha)} \right] \\ & + b \left[\varphi^{(3)}(x) + \frac{t^\alpha \varphi_1^{(3)}(x)}{\Gamma(1+\alpha)} \right] + c \left[\varphi^{(4)}(x) + \frac{t^\alpha \varphi_1^{(4)}(x)}{\Gamma(1+\alpha)} \right] \end{aligned}$$

Where, the first truncated solution is :

$$u_1(x, t) = \varphi(x) + \varphi_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)}.$$

Where, we know that $Res_1(x, t) = 0$ at $t = 0$ and thus,

$$\varphi_1(x) = -\varphi(x)\varphi'(x) - a\varphi''(x) - b\varphi^{(3)}(x) - c\varphi^{(4)}(x)$$

and

$$u_1(x, t) = \frac{\varphi(x) [\Gamma(1 + \alpha) - t^\alpha \varphi'(x)] - t^\alpha [a\varphi''(x) + b\varphi^{(3)}(x) + c\varphi^{(4)}(x)]}{\Gamma(1 + \alpha)}$$

Likewise, to obtain the form of the second unknown coefficient $\varphi_2(x)$, we write $k = 2$ in Eq.(3.6),

$$\begin{aligned} Res_2(x, t) = & -b\varphi^3(x) - \varphi(x)\varphi'(x) - a\varphi''(x) - c\varphi^{(4)}(x) + \frac{t^\alpha}{\Gamma(1 + \alpha)} \varphi_2(x) \\ & + \left[\varphi(x) + \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} \varphi_2(x) + \frac{t^\alpha}{\Gamma(1 + \alpha)} \left(-b\varphi^3(x) - \varphi(x)\varphi'(x) - a\varphi''(x) \right. \right. \\ & \left. \left. - c\varphi^{(4)}(x) \right) \right] \times \left[\varphi'(x) + \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} \varphi_2'(x) + \frac{t^\alpha}{\Gamma(1 + \alpha)} \left(-(\varphi'(x))^2 - b(\varphi^3)'(x) \right. \right. \\ & \left. \left. - \varphi(x)\varphi''(x) - a\varphi^{(3)}(x) - c\varphi^{(5)}(x) \right) \right] + a \left[\varphi''(x) + \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} \varphi_2''(x) \right. \\ & \left. + \frac{t^\alpha}{\Gamma(1 + \alpha)} \left(-3\varphi'(x)\varphi''(x) - b(\varphi^3)''(x) - \varphi(x)\varphi^{(3)}(x) - a\varphi^{(4)}(x) - c\varphi^{(6)}(x) \right) \right] \\ & + b \left[\varphi^{(3)}(x) + \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} \varphi_2^{(3)}(x) + \frac{t^\alpha}{\Gamma(1 + \alpha)} \left(-3(\varphi''(x))^2 - 4\varphi'(x)\varphi^{(3)}(x) \right. \right. \\ & \left. \left. - b(\varphi^3)^{(3)}(x) - \varphi(x)\varphi^{(4)}(x) - a\varphi^{(5)}(x) - c\varphi^{(7)}(x) \right) \right] + c \left[\varphi^{(4)}(x) + \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} \varphi_2^{(4)}(x) \right. \\ & \left. + \frac{t^\alpha}{\Gamma(1 + \alpha)} \left(-10\varphi''(x)\varphi^{(3)}(x) - 5\varphi'(x)\varphi^{(4)}(x) - b(\varphi^3)^{(4)}(x) - \varphi(x)\varphi^{(5)}(x) \right. \right. \\ & \left. \left. - a\varphi^{(6)}(x) - c\varphi^{(8)}(x) \right) \right]. \end{aligned}$$

where

$$u_2(x, t) = \varphi(x) + \varphi_1(x) \frac{t^\alpha}{\Gamma(1 + \alpha)} + \varphi_2(x) \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)}. \quad (3.7)$$

we know that $D_t^\alpha Res_2(x, t)|_{t=0} = 0$ and thus,

$$\begin{aligned}\varphi_2(x) = & b\varphi^3(x)\varphi'(x) + \varphi^2(x)\varphi''(x) + 4a\varphi'(x)\varphi''(x) + 3b(\varphi''(x))^2 + ab(\varphi^3)''(x) \\ & + 4b\varphi'(x)\varphi^{(3)}(x) + 10c\varphi''(x)\varphi^{(3)}(x) + b^2(\varphi^3)^{(3)}(x) + a^2\varphi^{(4)}(x) \\ & + 6c\varphi'(x)\varphi^{(4)}(x) + bc(\varphi^3)^{(4)}(x) + ab\varphi^{(5)}(x) + \varphi(x)\left[2(\varphi'(x))^2 + b(\varphi^3)'(x)\right. \\ & \left.+ 2a\varphi^{(3)}(x) + b\varphi^{(4)}(x) + 2c\varphi^{(5)}(x)\right] + 2ac\varphi^{(6)}(x) + bc\varphi^{(7)}(x) + c^2\varphi^{(8)}(x).\end{aligned}$$

and

$$\begin{aligned}u_2(x, t) = & \frac{1}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} \left[t^{2\alpha}\Gamma(1+\alpha)\varphi^2(x)\varphi''(x) + \varphi(x)\left(-t^\alpha\Gamma(1+2\alpha)\varphi'(x)\right. \right. \\ & + 2t^{2\alpha}\Gamma(1+\alpha)(\varphi'(x))^2 + \Gamma(1+\alpha)(\Gamma(1+2\alpha) + bt^{2\alpha}(\varphi^3)'(x) + 2at^{2\alpha}\varphi^{(3)}(x) \\ & + bt^{2\alpha}\varphi^{(4)}(x) + 2ct^{2\alpha}\varphi^{(5)}(x))\Big) + t^\alpha\left(-b\varphi^3(x)(\Gamma(1+2\alpha) - t^\alpha\Gamma(1+\alpha)\varphi'(x))\right. \\ & + 3bt^\alpha\Gamma(1+\alpha)(\varphi''(x))^2 + abt^\alpha\Gamma(1+\alpha)(\varphi^3)''(x) + 4bt^\alpha\Gamma(1+\alpha)\varphi'(x)\varphi^{(3)}(x) \\ & + \varphi''(x)(-a\Gamma(1+2\alpha) + 4at^\alpha\Gamma(1+\alpha)\varphi'(x) + 10ct^\alpha\Gamma(1+\alpha)\varphi^{(3)}(x)) \\ & + b^2t^\alpha\Gamma(1+\alpha)(\varphi^3)^{(3)}(x) + a^2t^\alpha\Gamma(1+\alpha)\varphi^{(4)}(x) - c\Gamma(1+2\alpha)\varphi^{(4)}(x) \\ & + 6ct^\alpha\Gamma(1+\alpha)\varphi'(x)\varphi^{(4)}(x) + bct^\alpha\Gamma(1+\alpha)(\varphi^3)^{(4)}(x) + abt^\alpha\Gamma(1+\alpha)\varphi^{(5)}(x) \\ & \left. \left. + 2act^\alpha\Gamma(1+\alpha)\varphi^{(6)}(x) + bct^\alpha\Gamma(1+\alpha)\varphi^{(7)}(x) + c^2t^\alpha\Gamma(1+\alpha)\varphi^{(8)}(x)\right)\right]\end{aligned}$$

Similarly to obtain $\varphi_3(x)$, we consider $k = 3$ in Eq. (3.6),

$$\begin{aligned}Res_3(x, t) = & -b(\varphi^3)[x] + \frac{t^{2\alpha}\varphi_3[x]}{\Gamma(1+2\alpha)} - \varphi[x](\varphi')[x] + \frac{bt^\alpha(\varphi^3)[x](\varphi')[x]}{\Gamma(1+\alpha)} + \frac{2t^\alpha\varphi[x](\varphi'[x])^2}{\Gamma(1+\alpha)} \\ & + \frac{bt^\alpha\varphi[x](\varphi^3)'[x]}{\Gamma(1+\alpha)} - a(\varphi'')[x] + \frac{t^\alpha\varphi[x]^2(\varphi'')[x]}{\Gamma(1+\alpha)} + \frac{4at^\alpha(\varphi')[x](\varphi'')[x]}{\Gamma(1+\alpha)} \\ & + \frac{3bt^\alpha(\varphi'')[x]^2}{\Gamma(1+\alpha)} + \frac{abt^\alpha((\varphi^3)'')[x]}{\Gamma(1+\alpha)} + \frac{2at^\alpha\varphi[x](\varphi^3)[x]}{\Gamma(1+\alpha)} + \frac{4bt^\alpha(\varphi')[x](\varphi^3)[x]}{\Gamma(1+\alpha)} \\ & + \frac{10ct^\alpha(\varphi'')[x](\varphi^3)[x]}{\Gamma(1+\alpha)} + \frac{b^2t^\alpha((\varphi^3)^3)[x]}{\Gamma(1+\alpha)} - c(\varphi^{(4)})[x] + \frac{a^2t^\alpha(\varphi^{(4)})[x]}{\Gamma(1+\alpha)} \\ & + \frac{bt^\alpha\varphi[x](\varphi^{(4)})[x]}{\Gamma(1+\alpha)} + \frac{6ct^\alpha(\varphi')[x](\varphi^{(4)})[x]}{\Gamma(1+\alpha)} + \frac{bct^\alpha((\varphi^3)^{(4)})[x]}{\Gamma(1+\alpha)} + \frac{abt^\alpha(\varphi^{(5)})[x]}{\Gamma(1+\alpha)} \\ & + \frac{2ct^\alpha\varphi[x](\varphi^{(5)})[x]}{\Gamma(1+\alpha)} + \frac{2act^\alpha(\varphi^{(6)})[x]}{\Gamma(1+\alpha)} + \frac{bct^\alpha(\varphi^{(7)})[x]}{\Gamma(1+\alpha)} + \frac{c^2t^\alpha(\varphi^{(8)})[x]}{\Gamma(1+\alpha)} \\ & + \left[\varphi[x] + \frac{t^{3\alpha}\varphi_3[x]}{\Gamma(1+3\alpha)} + \frac{t^\alpha\left(-b(\varphi^3)[x] - \varphi[x](\varphi')[x] - a(\varphi'')[x] - c(\varphi^{(4)})[x]\right)}{\Gamma(1+\alpha)}\right] \\ & + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\left(b(\varphi^3)[x](\varphi')[x] + \dots\right)\end{aligned}$$

where

$$u_3(x, t) = \varphi(x) + \varphi_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + \varphi_2(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \varphi_3(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}. \quad (3.8)$$

When $D_t^{2\alpha} \text{Res}_3(x, 0) = 0$ and thus,

$$\begin{aligned} \varphi_3(x) = & -16a(\varphi'[x])^2\varphi''[x] - \frac{a\Gamma[1+2\alpha](\varphi'[x])^2\varphi''[x]}{\Gamma[1+\alpha]^2} - 3ab((\varphi^3)'[x])\varphi''[x] \\ & - \frac{ab\Gamma[1+2\alpha]((\varphi^3)'[x])\varphi''[x]}{\Gamma[1+\alpha]^2} - 33b(\varphi'[x])(\varphi''[x])^2 - 30c(\varphi''[x])^3 - 4ab(\varphi'[x])(\varphi^3)''[x] \\ & - 6b^2(\varphi''[x])(\varphi^3)''[x] - \varphi[x]^3\varphi^{(3)}[x] - 24b(\varphi'[x])^2\varphi^{(3)}[x] - 4b^2((\varphi^3)'[x])\varphi^{(3)}[x] \\ & - 14a^2(\varphi''[x])\varphi^{(3)}[x] - \frac{a^2\Gamma[1+2\alpha](\varphi''[x])\varphi^{(3)}[x]}{\Gamma[1+\alpha]^2} - \dots \end{aligned}$$

and

$$\begin{aligned} u_3(x, t) = & \varphi[x] - \frac{t^\alpha(b(\varphi^3)[x] + \varphi[x]\varphi'[x] + a\varphi''[x] + c\varphi^{(4)}[x])}{\Gamma[1+\alpha]} \\ & + \frac{t^{2\alpha}}{\Gamma[1+2\alpha]} \left[b(\varphi^3)[x]\varphi'[x] + \varphi[x]^2\varphi''[x] + 4a\varphi'[x]\varphi''[x] + 3b(\varphi''[x])^2 \right. \\ & + ab(\varphi^3)''[x] + 4b\varphi'[x]\varphi^{(3)}[x] + 10c\varphi''[x]\varphi^{(3)}[x] + b^2(\varphi^3)^{(3)}[x] \\ & + a^2\varphi^{(4)}[x] + 6c\varphi'[x]\varphi^{(4)}[x] + bc(\varphi^3)^{(4)}[x] + ab\varphi^{(5)}[x] \\ & + \varphi[x](2(\varphi'[x])^2 + b(\varphi^3)'[x] + 2a\varphi^{(3)}[x] + b\varphi^{(4)}[x] + 2c\varphi^{(5)}[x]) \\ & \left. + 2ac\varphi^{(6)}[x] + bc\varphi^{(7)}[x] + c^2\varphi^{(8)}[x] \right] \\ & - \frac{t^{3\alpha}}{\Gamma[1+\alpha]^2\Gamma[1+3\alpha]} \left[16a\Gamma[1+\alpha]^2(\varphi'[x])^2\varphi''[x] + \dots \right] \end{aligned}$$

Repeating the above operation for, we find the unknown coefficients $\varphi_k(x)$. The solution of equation (3.1)

2. Examples And Case Studies

This section presents several test cases and numerical examples to demonstrate the efficiency and accuracy of the Residual Power Series Method (RPSM) in solving the time-fractional Kuramoto–Sivashinsky (KS) equation under different initial conditions and parameter values.

2.1 Application 1

By substituting $a = -1, b = 0, c = 1$. into the equation (3.1), we find :

$$\begin{cases} D_t^\alpha u(x, t) + u(x, t) \frac{\partial u}{\partial x}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) + \frac{\partial^4 u}{\partial x^4}(x, t) = 0, \\ x \in \mathbb{R}, t \in [0, T], 0 < \alpha \leq 1, \\ u(x, 0) = \mu + \frac{15 \tanh^3(\rho(x-\eta)) - 45 \tanh(\rho(x-\eta))}{19\sqrt{19}}. \end{cases} \quad (3.9)$$

where μ, ρ and η are real constants.

By taking $\alpha = 1$, we obtain the exact solution as [2]

$$u(x, t) = \mu + \frac{15 \tanh^3(\rho(x - \mu t - \eta)) - 45 \tanh(\rho(x - \mu t - \eta))}{19\sqrt{19}}$$

For: $\mu = \frac{3}{4}, \rho = \frac{1}{2\sqrt{19}}, \eta = -25$ Applying the RPSM method to solve equation (3.9), Suppose the solution is written as the following power series.

$$u(x, t) = \sum_{n=0}^{\infty} \varphi_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \quad (3.10)$$

We consider that $u_k(x, t)$ is k . truncated series of $u(x, t)$

$$u_k(x, t) = \varphi(x) + \sum_{n=1}^k \varphi_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \quad k = 1, 2, 3, \dots \quad (3.11)$$

where $u_0(x, t) = u(x, 0) = \varphi_0(x) = \varphi(x)$

To obtain the value of coefficients $\varphi_n(x), n = 1, 2, \dots, k$ in series expanded of Eq.(3.9), Residual function Res is given by

$$\text{Res}(x, t) = D_t^\alpha u(x, t) + u(x, t) \frac{\partial u}{\partial x}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) + \frac{\partial^4 u}{\partial x^4}(x, t). \quad (3.12)$$

and the k -th Residual functions of equation (3.9) is defined by:

$$\text{Res}_k(x, t) = D_t^\alpha u_k(x, t) + u_k(x, t) \frac{\partial u_k}{\partial x}(x, t) - \frac{\partial^2 u_k}{\partial x^2}(x, t) + \frac{\partial^4 u_k}{\partial x^4}(x, t). \quad (3.13)$$

to find the needed coefficients $\varphi_n(x)$ for $n = 1, 2, \dots, k$ in Eq.(3.11). To obtain $\varphi_1(x)$, we consider $k = 1$ in Eq. (3.13),

Where, we know that $\text{Res}_1(x, t) = 0$ at $t = 0$ and thus,

$$\varphi_1(x) = \frac{135}{2888} \text{sech}^4\left(\frac{25+x}{2\sqrt{19}}\right)$$

Likewise, to obtain the form of the second unknown coefficient $\varphi_2(x)$, we write $k = 2$ in Eq.(3.13), we know that $D_t^\alpha \text{Res}_2(x, t)|_{t=0} = 0$ and thus,

$$\varphi_2(x) = \frac{810 \text{csch}^5\left(\frac{25+x}{2\sqrt{19}}\right) \sinh^6\left(\frac{25+x}{2\sqrt{19}}\right)}{361\sqrt{19}}$$

Similarly to obtain $\varphi_3(x)$, we consider $k = 3$ in Eq. (3.13),

When $D_t^{2\alpha} \text{Res}_3(x, 0) = 0$ and thus,

$$\begin{aligned} \varphi_3(x) = & \frac{1215 \text{sech}^9\left(\frac{25+x}{2\sqrt{19}}\right)}{1267762688 \Gamma(1+\alpha)^2} \left[240\sqrt{19} \Gamma(1+2\alpha) \sinh\left(\frac{25+x}{2\sqrt{19}}\right) \right. \\ & + \Gamma(1+\alpha)^2 \left(-1805 \cosh\left(\frac{25+x}{2\sqrt{19}}\right) + 361 \cosh\left(\frac{5(25+x)}{2\sqrt{19}}\right) \right. \\ & \left. \left. - 480\sqrt{19} \sinh\left(\frac{25+x}{2\sqrt{19}}\right) \right) \right] \end{aligned}$$

and

$$\begin{aligned} u_3(x, t) = & \frac{3}{4} + \frac{135t^\alpha \text{sech}^4\left(\frac{25+x}{2\sqrt{19}}\right)}{2888 \Gamma(1+\alpha)} + \frac{810t^{2\alpha} \text{csch}^5\left(\frac{25+x}{2\sqrt{19}}\right) \sinh^6\left(\frac{25+x}{2\sqrt{19}}\right)}{361\sqrt{19} \Gamma(1+2\alpha)} \\ & + \frac{15 \tanh\left(\frac{25+x}{2\sqrt{19}}\right) \left(-3 + \tanh^2\left(\frac{25+x}{2\sqrt{19}}\right) \right)}{19\sqrt{19}} + \frac{1215t^{3\alpha} \text{sech}^9\left(\frac{25+x}{2\sqrt{19}}\right)}{1267762688 \Gamma(1+\alpha)^2 \Gamma(1+3\alpha)} \\ & \times \left[240\sqrt{19} \Gamma(1+2\alpha) \sinh\left(\frac{25+x}{2\sqrt{19}}\right) + \Gamma(1+\alpha)^2 \left(-1805 \cosh\left(\frac{25+x}{2\sqrt{19}}\right) \right. \right. \\ & \left. \left. + 361 \cosh\left(\frac{5(25+x)}{2\sqrt{19}}\right) - 480\sqrt{19} \sinh\left(\frac{25+x}{2\sqrt{19}}\right) \right) \right] \end{aligned}$$

Table 3.1: Numerical comparison between the error ratio of RPSM and NTDM [2] at different fractional orders of α ($\alpha = 0.4, \alpha = 0.6, \alpha = 0.8$ and $\alpha = 1$) for Application 1.

$\alpha = 0.4$				$\alpha = 0.6$				$\alpha = 0.8$				$\alpha = 1$					
t	x	RPSM	NTDM	RPSM	NTDM	RPSM	NTDM	RPSM	NTDM	RPSM	NTDM	RPSM	NTDM				
0.1	0.2	2.90	$\times 10^{-6}$	3.88	$\times 10^{-5}$	1.41	$\times 10^{-6}$	2.56	$\times 10^{-5}$	5.24	$\times 10^{-7}$	1.25	$\times 10^{-5}$	1.17	$\times 10^{-12}$	5.24	$\times 10^{-9}$
	0.4	2.64	$\times 10^{-6}$	3.54	$\times 10^{-5}$	1.28	$\times 10^{-6}$	2.34	$\times 10^{-5}$	4.79	$\times 10^{-7}$	1.14	$\times 10^{-5}$	1.07	$\times 10^{-12}$	5.19	$\times 10^{-9}$
	0.6	2.41	$\times 10^{-6}$	3.23	$\times 10^{-5}$	1.17	$\times 10^{-6}$	2.13	$\times 10^{-5}$	4.37	$\times 10^{-7}$	1.038	$\times 10^{-5}$	9.75	$\times 10^{-13}$	4.20	$\times 10^{-9}$
	0.8	2.20	$\times 10^{-6}$	2.95	$\times 10^{-5}$	1.07	$\times 10^{-6}$	1.95	$\times 10^{-5}$	3.99	$\times 10^{-7}$	9.47	$\times 10^{-6}$	8.91	$\times 10^{-13}$	4.26	$\times 10^{-9}$
0.2	1	2.01	$\times 10^{-6}$	2.70	$\times 10^{-5}$	9.77	$\times 10^{-7}$	1.78	$\times 10^{-5}$	3.64	$\times 10^{-7}$	8.64	$\times 10^{-6}$	8.14	$\times 10^{-13}$	3.76	$\times 10^{-9}$
	0.2	3.53	$\times 10^{-6}$	3.91	$\times 10^{-5}$	1.88	$\times 10^{-6}$	2.62	$\times 10^{-5}$	7.61	$\times 10^{-7}$	1.28	$\times 10^{-5}$	1.88	$\times 10^{-11}$	2.167	$\times 10^{-8}$
	0.4	3.22	$\times 10^{-6}$	3.56	$\times 10^{-5}$	1.72	$\times 10^{-6}$	2.39	$\times 10^{-5}$	6.95	$\times 10^{-7}$	1.17	$\times 10^{-5}$	1.72	$\times 10^{-11}$	2.00	$\times 10^{-8}$
	0.6	2.94	$\times 10^{-6}$	3.25	$\times 10^{-5}$	1.57	$\times 10^{-6}$	2.18	$\times 10^{-5}$	6.35	$\times 10^{-7}$	1.07	$\times 10^{-5}$	1.57	$\times 10^{-11}$	1.79	$\times 10^{-8}$
0.3	0.8	2.68	$\times 10^{-6}$	2.97	$\times 10^{-5}$	1.43	$\times 10^{-6}$	1.99	$\times 10^{-5}$	5.79	$\times 10^{-7}$	9.73	$\times 10^{-6}$	1.44	$\times 10^{-11}$	1.65	$\times 10^{-8}$
	1	2.45	$\times 10^{-6}$	2.71	$\times 10^{-5}$	1.31	$\times 10^{-6}$	1.82	$\times 10^{-5}$	5.29	$\times 10^{-7}$	8.88	$\times 10^{-6}$	1.31	$\times 10^{-11}$	1.50	$\times 10^{-8}$
	0.2	3.83	$\times 10^{-6}$	3.91	$\times 10^{-5}$	2.17	$\times 10^{-6}$	2.66	$\times 10^{-5}$	9.20	$\times 10^{-7}$	1.30	$\times 10^{-5}$	9.58	$\times 10^{-11}$	4.91	$\times 10^{-8}$
	0.4	3.50	$\times 10^{-6}$	3.57	$\times 10^{-5}$	1.98	$\times 10^{-6}$	2.42	$\times 10^{-5}$	8.40	$\times 10^{-7}$	1.19	$\times 10^{-5}$	8.76	$\times 10^{-11}$	4.51	$\times 10^{-8}$
0.4	0.6	3.20	$\times 10^{-6}$	3.26	$\times 10^{-5}$	1.81	$\times 10^{-6}$	2.21	$\times 10^{-5}$	7.67	$\times 10^{-7}$	1.08	$\times 10^{-5}$	8.00	$\times 10^{-11}$	4.10	$\times 10^{-8}$
	0.8	2.92	$\times 10^{-6}$	2.97	$\times 10^{-5}$	1.65	$\times 10^{-6}$	2.02	$\times 10^{-5}$	7.00	$\times 10^{-7}$	9.90	$\times 10^{-6}$	7.32	$\times 10^{-11}$	3.77	$\times 10^{-8}$
	1	2.66	$\times 10^{-6}$	2.71	$\times 10^{-5}$	1.51	$\times 10^{-6}$	1.84	$\times 10^{-5}$	6.39	$\times 10^{-7}$	9.04	$\times 10^{-6}$	6.68	$\times 10^{-11}$	3.43	$\times 10^{-8}$
	0.2	3.97	$\times 10^{-6}$	3.90	$\times 10^{-5}$	2.35	$\times 10^{-6}$	2.68	$\times 10^{-5}$	1.03	$\times 10^{-6}$	1.32	$\times 10^{-5}$	3.05	$\times 10^{-10}$	8.79	$\times 10^{-8}$
0.5	0.4	3.62	$\times 10^{-6}$	3.56	$\times 10^{-5}$	2.14	$\times 10^{-6}$	2.44	$\times 10^{-5}$	9.42	$\times 10^{-7}$	1.20	$\times 10^{-5}$	2.79	$\times 10^{-10}$	8.08	$\times 10^{-8}$
	0.6	3.31	$\times 10^{-6}$	3.25	$\times 10^{-5}$	1.96	$\times 10^{-6}$	2.23	$\times 10^{-5}$	8.60	$\times 10^{-7}$	1.098	$\times 10^{-5}$	2.55	$\times 10^{-10}$	7.32	$\times 10^{-8}$
	0.8	3.02	$\times 10^{-6}$	2.96	$\times 10^{-5}$	1.79	$\times 10^{-6}$	2.04	$\times 10^{-5}$	7.85	$\times 10^{-7}$	1.00	$\times 10^{-5}$	2.33	$\times 10^{-10}$	6.73	$\times 10^{-8}$
	1	2.75	$\times 10^{-6}$	2.71	$\times 10^{-5}$	1.63	$\times 10^{-6}$	1.86	$\times 10^{-5}$	7.16	$\times 10^{-7}$	9.15	$\times 10^{-6}$	2.13	$\times 10^{-10}$	6.12	$\times 10^{-7}$
0.6	0.2	3.98	$\times 10^{-6}$	3.88	$\times 10^{-5}$	2.45	$\times 10^{-6}$	2.69	$\times 10^{-5}$	1.11	$\times 10^{-6}$	1.33	$\times 10^{-5}$	7.50	$\times 10^{-10}$	1.39	$\times 10^{-7}$
	0.4	3.63	$\times 10^{-6}$	3.54	$\times 10^{-5}$	2.23	$\times 10^{-6}$	2.45	$\times 10^{-5}$	1.01	$\times 10^{-6}$	1.21	$\times 10^{-5}$	6.85	$\times 10^{-10}$	1.27	$\times 10^{-7}$
	0.6	3.32	$\times 10^{-6}$	3.23	$\times 10^{-5}$	2.04	$\times 10^{-6}$	2.24	$\times 10^{-5}$	9.24	$\times 10^{-7}$	1.12	$\times 10^{-5}$	6.26	$\times 10^{-10}$	1.16	$\times 10^{-7}$
	0.8	3.03	$\times 10^{-6}$	2.95	$\times 10^{-5}$	1.86	$\times 10^{-6}$	2.04	$\times 10^{-5}$	8.43	$\times 10^{-7}$	1.01	$\times 10^{-5}$	5.72	$\times 10^{-10}$	1.06	$\times 10^{-8}$
0.7	1	2.76	$\times 10^{-6}$	2.69	$\times 10^{-5}$	1.70	$\times 10^{-6}$	1.86	$\times 10^{-5}$	7.70	$\times 10^{-7}$	9.22	$\times 10^{-6}$	5.23	$\times 10^{-10}$	9.66	$\times 10^{-7}$

Table 3.1 shows that the RPSM achieves extremely small numerical errors (down to 10^{-13}) for all fractional orders α , with a clear superiority over NTDM [2], as the error of RPSM is several orders of magnitude smaller, reflecting its high efficiency. The parameter α represents the order of the fractional derivative and expresses the level of memory or cumulative effect of the system. When $\alpha < 1$, the system at a given time t does not depend only on its instantaneous state, but also on its past evolution over time, meaning that the solution exhibits a time-dependent memory effect. The results also indicate that the accuracy of the method improves as α approaches the classical case ($\alpha = 1$), where the memory effect gradually disappears and the system becomes local in time, depending only on the current state. Therefore, these findings confirm that α not only affects the form of the solution but also determines the temporal nature of the system, and that RPSM provides an accurate and stable numerical approximation over time for different values of α .

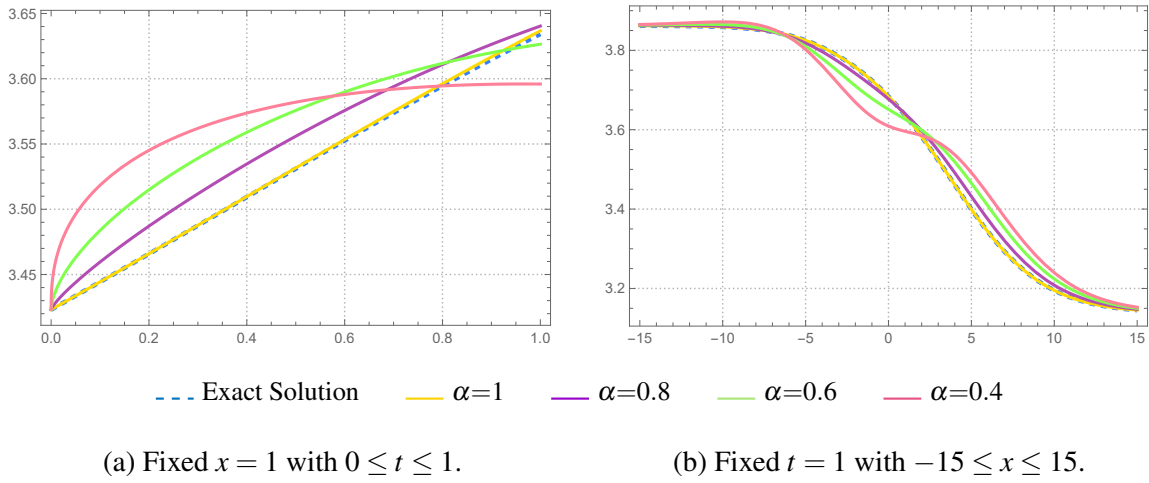


Figure 3.1: 2D comparison of the exact solution and the RPSM approximation for various fractional orders α ($\alpha = 1, \alpha = 0.8, \alpha = 0.6, \alpha = 0.4$) for Application 1.

Figure 3.1 presents a 2D comparison between the exact solution and the RPSM approximation for the fractional orders α ($\alpha = 1, \alpha = 0.8, \alpha = 0.6$ and $\alpha = 0.4$). Panel (a) illustrates the time evolution at the fixed spatial point $x = 1$ for $0 \leq t \leq 1$, while panel (b) depicts the spatial distribution at the fixed time $t = 1$ for $-15 \leq x \leq 15$. Both plots clearly show that the agreement between the approximate and exact solutions improves steadily as α approaches unity. A perfect match is observed at $\alpha = 1$, which is expected because the fractional derivative reduces to the classical integer-order derivative, allowing the RPSM representation to recover the exact solution fully. For smaller values, such as $\alpha = 0.4$, a slight discrepancy emerges due to the stronger memory effects inherent in fractional operators. Despite this, the RPSM approximation consistently exhibits excellent accuracy across the entire temporal and spatial domains, demonstrating the method's reliability for analyzing fractional differential models.

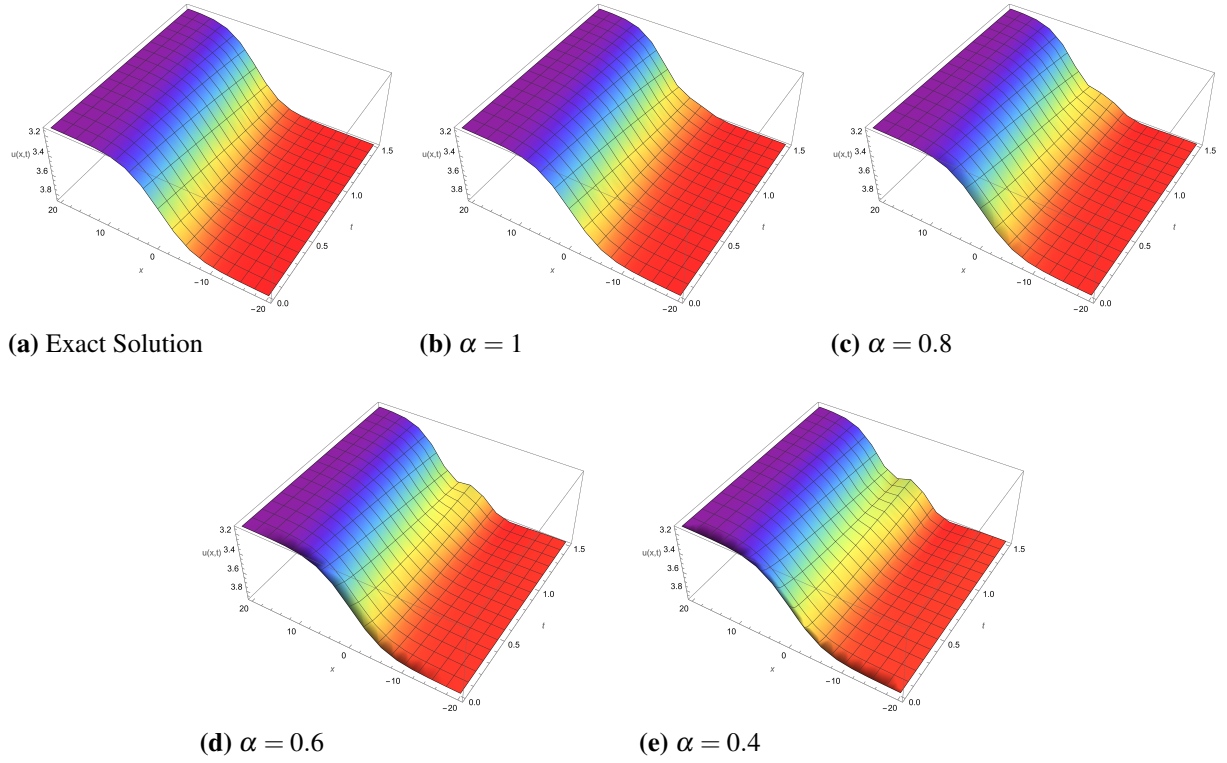


Figure 3.2: 3D surface plots of the exact solution and the RPSM approximation for various fractional orders α ($\alpha = 1, \alpha = 0.8, \alpha = 0.6, \alpha = 0.4$) for Application 1.

Figure 3.2 displays 3D surface plots comparing the exact solution and the RPSM approximation for fractional orders α ($\alpha = 1, \alpha = 0.8, \alpha = 0.6, \alpha = 0.4$). The horizontal axes correspond to the spatial variable $x \in [-20, 20]$ and the temporal variable $t \in [0, 1.5]$, while the vertical axis represents the solution $u(x, t)$. The surfaces clearly demonstrate that the approximate solution visually coincides with the exact solution across the entire domain when $\alpha = 1$, and the agreement remains excellent for values close to unity such as $\alpha = 0.8$. As α decreases towards 0.6 and 0.4, slight deviations become noticeable in certain regions, reflecting the stronger memory effects inherent in fractional operators. Nonetheless, the RPSM method consistently captures the overall structure of the solution with high fidelity, confirming its reliability for simulating fractional differential equations over the considered ranges.

2.2 Application 2

By substituting $a = 1, b = 0, c = 1$. into the equation (3.1), we find :

$$\begin{cases} D_t^\alpha u(x, t) + u(x, t) \frac{\partial u}{\partial x}(x, t) + \frac{\partial^2 u}{\partial x^2}(x, t) + \frac{\partial^4 u}{\partial x^4}(x, t) = 0, \\ x \in \mathbb{R}, t \in [0, T], 0 < \alpha \leq 1, \\ u(x, 0) = \mu + \frac{15}{19} \sqrt{\frac{11}{19}} (-9 \tanh(\rho(x - \eta)) + 11 \tanh^3(\rho(x - \eta))) \end{cases} \quad (3.14)$$

where μ, ρ and η are real constants.

By taking $\alpha = 1$, we obtain the exact solution as [2]

$$u(x, t) = \mu + \frac{15}{19} \sqrt{\frac{11}{19}} (-9 \tanh(\rho(x - \mu t - \eta)) + 11 \tanh^3(\rho(x - \mu t - \eta)))$$

For: $\mu = \frac{3}{4}, \rho = \frac{\sqrt{11}}{2\sqrt{19}}, \eta = -25$ Applying the RPSM method to solve equation (3.14), Suppose the solution is written as the following power series.

$$u(x, t) = \sum_{n=0}^{\infty} \varphi_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \quad (3.15)$$

We consider that $u_k(x, t)$ is k . truncated series of $u(x, t)$

$$u_k(x, t) = \varphi(x) + \sum_{n=1}^k \varphi_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \quad k = 1, 2, 3, \dots \quad (3.16)$$

where $u_0(x, t) = u(x, 0) = \varphi_0(x) = \varphi(x)$

To obtain the value of coefficients $\varphi_n(x), n = 1, 2, \dots, k$ in series expanded of Eq.(3.14), Residual function Res is given by

$$\text{Res}(x, t) = D_t^\alpha u(x, t) + u(x, t) \frac{\partial u}{\partial x}(x, t) + \frac{\partial^2 u}{\partial x^2}(x, t) + \frac{\partial^4 u}{\partial x^4}(x, t). \quad (3.17)$$

and the k -th Residual functions of equation (3.14) is defined by:

$$\text{Res}_k(x, t) = D_t^\alpha u_k(x, t) + u_k(x, t) \frac{\partial u_k}{\partial x}(x, t) + \frac{\partial^2 u_k}{\partial x^2}(x, t) + \frac{\partial^4 u_k}{\partial x^4}(x, t). \quad (3.18)$$

to find the needed coefficients $\varphi_n(x)$ for $n = 1, 2, \dots, k$ in Eq.(3.16). To obtain $\varphi_1(x)$, we consider $k = 1$ in Eq. (3.18),

Where, we know that $Res_1(x, t) = 0$ at $t = 0$ and thus,

$$\varphi_1(x) = -\frac{1485}{2888} \left[-7 + 4 \cosh \left(\sqrt{\frac{11}{19}} (25 + x) \right) \right] \operatorname{sech}^4 \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right)$$

Likewise, to obtain the form of the second unknown coefficient $\varphi_2(x)$, we write $k = 2$ in Eq.(3.18), we know that $D_t^\alpha Res_2(x, t)|_{t=0} = 0$ and thus,

$$\begin{aligned} \varphi_2(x) = -\frac{4455}{5776} \sqrt{\frac{11}{19}} \operatorname{sech}^5 \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right) & \left[-10 \sinh \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right. \\ & \left. + \sinh \left(\frac{3}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right] \end{aligned}$$

Similarly to obtain $\varphi_3(x)$, we consider $k = 3$ in Eq. (3.18),

When $D_t^{2\alpha} Res_3(x, 0) = 0$ and thus,

$$\begin{aligned} \varphi_3(x) = & -\frac{147015}{2535525376\Gamma(1+\alpha)^2} \operatorname{sech}^9 \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \\ & \times \left[-480\sqrt{209}\Gamma(1+2\alpha) \left[92 \sinh \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right) - 27 \sinh \left(\frac{3}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right. \right. \\ & \left. \left. + 2 \sinh \left(\frac{5}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right] + \Gamma(1+\alpha)^2 \left[30685 \cosh \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right. \right. \\ & \left. \left. - 3249 \cosh \left(\frac{3}{2} \sqrt{\frac{11}{19}} (25 + x) \right) - 7581 \cosh \left(\frac{5}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right. \right. \\ & \left. \left. + 361 \cosh \left(\frac{7}{2} \sqrt{\frac{11}{19}} (25 + x) \right) + 88320\sqrt{209} \sinh \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right. \right. \\ & \left. \left. - 25920\sqrt{209} \sinh \left(\frac{3}{2} \sqrt{\frac{11}{19}} (25 + x) \right) + 1920\sqrt{209} \sinh \left(\frac{5}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right] \right] \end{aligned}$$

and

$$\begin{aligned}
u_3(x, t) = & \frac{3}{4} - \frac{1485 t^\alpha \left[-7 + 4 \cosh \left(\sqrt{\frac{11}{19}} (25 + x) \right) \right] \operatorname{sech}^4 \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right)}{2888 \Gamma(1 + \alpha)} \\
& - \frac{4455 \sqrt{\frac{11}{19}} t^{2\alpha} \operatorname{sech}^5 \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right)}{361 \Gamma(1 + 2\alpha)} \left[-10 \sinh \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right. \\
& \left. + \sinh \left(\frac{3}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right] - \frac{147015 t^{3\alpha} \operatorname{sech}^9 \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right)}{2535525376 \Gamma(1 + \alpha)^2 \Gamma(1 + 3\alpha)} \\
& \times \left[-480 \sqrt{209} \Gamma(1 + 2\alpha) \left[92 \sinh \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right) - 27 \sinh \left(\frac{3}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right. \right. \\
& \left. \left. + 2 \sinh \left(\frac{5}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right] + \Gamma(1 + \alpha)^2 \left[30685 \cosh \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right. \right. \\
& \left. \left. - 3249 \cosh \left(\frac{3}{2} \sqrt{\frac{11}{19}} (25 + x) \right) - 7581 \cosh \left(\frac{5}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right. \right. \\
& \left. \left. + 361 \cosh \left(\frac{7}{2} \sqrt{\frac{11}{19}} (25 + x) \right) + 88320 \sqrt{209} \sinh \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right. \right. \\
& \left. \left. - 25920 \sqrt{209} \sinh \left(\frac{3}{2} \sqrt{\frac{11}{19}} (25 + x) \right) + 1920 \sqrt{209} \sinh \left(\frac{5}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right] \right] \\
& + \frac{15}{19} \sqrt{\frac{11}{19}} \tanh \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \left[-9 + 11 \tanh^2 \left(\frac{1}{2} \sqrt{\frac{11}{19}} (25 + x) \right) \right]
\end{aligned}$$

Table 3.2: Numerical comparison of approximate solutions of RPSM and NTDM [2] at different fractional orders of α ($\alpha = 0.4$, $\alpha = 0.6$, $\alpha = 0.8$ and $\alpha = 1$) for Application 2.

t	x	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$		$\alpha = 1$	
		RPSM	NTDM	RPSM	NTDM	RPSM	NTDM	RPSM	NTDM
0.1	0.2	3.57×10^{-8}	1.16×10^{-6}	1.66×10^{-8}	8.54×10^{-7}	6.02×10^{-9}	4.69×10^{-7}	4.46×10^{-14}	6.87×10^{-8}
	0.4	3.07×10^{-8}	1.00×10^{-6}	1.42×10^{-8}	7.39×10^{-7}	5.17×10^{-9}	4.09×10^{-7}	3.62×10^{-14}	6.45×10^{-8}
	0.6	2.64×10^{-8}	8.58×10^{-7}	1.22×10^{-8}	6.31×10^{-7}	4.44×10^{-9}	3.47×10^{-7}	6.13×10^{-14}	5.21×10^{-8}
	0.8	2.26×10^{-8}	7.38×10^{-7}	1.05×10^{-8}	5.43×10^{-7}	3.81×10^{-9}	3.00×10^{-7}	5.31×10^{-14}	4.60×10^{-8}
	1	1.94×10^{-8}	6.33×10^{-7}	9.02×10^{-9}	4.66×10^{-7}	3.27×10^{-9}	2.57×10^{-7}	3.18×10^{-14}	3.88×10^{-8}
0.2	0.2	4.59×10^{-8}	1.12×10^{-6}	2.32×10^{-8}	8.69×10^{-7}	9.07×10^{-9}	5.18×10^{-7}	9.67×10^{-13}	7.12×10^{-7}
	0.4	3.94×10^{-8}	9.58×10^{-7}	2.00×10^{-8}	7.46×10^{-7}	7.79×10^{-9}	4.45×10^{-7}	8.28×10^{-13}	1.02×10^{-7}
	0.6	3.38×10^{-8}	8.22×10^{-7}	1.71×10^{-8}	6.40×10^{-7}	6.69×10^{-9}	3.82×10^{-7}	7.39×10^{-13}	8.72×10^{-8}
	0.8	2.93×10^{-8}	7.07×10^{-7}	1.47×10^{-8}	5.51×10^{-7}	5.74×10^{-9}	3.29×10^{-7}	6.37×10^{-13}	7.59×10^{-8}
	1	2.49×10^{-8}	6.08×10^{-7}	1.26×10^{-8}	4.73×10^{-7}	4.93×10^{-9}	2.82×10^{-7}	5.32×10^{-13}	6.57×10^{-8}
0.3	0.2	5.21×10^{-8}	1.01×10^{-6}	2.79×10^{-8}	8.10×10^{-7}	1.14×10^{-8}	5.00×10^{-7}	5.01×10^{-12}	7.19×10^{-7}
	0.4	4.47×10^{-8}	8.69×10^{-7}	2.44×10^{-8}	7.00×10^{-7}	9.76×10^{-9}	4.34×10^{-7}	4.35×10^{-12}	1.07×10^{-7}
	0.6	3.84×10^{-8}	7.41×10^{-7}	2.06×10^{-8}	5.96×10^{-7}	8.38×10^{-9}	3.67×10^{-7}	3.72×10^{-12}	8.63×10^{-8}
	0.8	3.30×10^{-8}	6.36×10^{-7}	1.77×10^{-8}	5.11×10^{-7}	7.20×10^{-9}	3.15×10^{-7}	3.20×10^{-12}	7.39×10^{-8}
	1	2.83×10^{-8}	5.46×10^{-7}	1.52×10^{-8}	4.39×10^{-7}	6.18×10^{-9}	2.71×10^{-7}	2.74×10^{-12}	6.35×10^{-8}
0.4	0.2	5.61×10^{-8}	8.30×10^{-7}	3.15×10^{-8}	6.78×10^{-7}	1.32×10^{-8}	4.11×10^{-7}	1.61×10^{-11}	5.78×10^{-8}
	0.4	4.82×10^{-8}	7.15×10^{-7}	2.75×10^{-8}	5.85×10^{-7}	1.13×10^{-8}	3.56×10^{-7}	1.38×10^{-11}	5.22×10^{-8}
	0.6	4.14×10^{-8}	6.12×10^{-7}	2.32×10^{-8}	5.00×10^{-7}	9.75×10^{-9}	3.03×10^{-7}	1.19×10^{-11}	4.24×10^{-8}
	0.8	3.56×10^{-8}	5.28×10^{-7}	1.99×10^{-8}	4.32×10^{-7}	8.38×10^{-9}	2.63×10^{-7}	1.02×10^{-11}	3.89×10^{-8}
	1	3.05×10^{-8}	4.487×10^{-7}	1.71×10^{-8}	3.66×10^{-7}	7.19×10^{-9}	2.21×10^{-7}	8.75×10^{-12}	2.83×10^{-8}
0.5	0.2	5.86×10^{-8}	5.60×10^{-7}	3.42×10^{-8}	4.47×10^{-7}	1.47×10^{-8}	2.25×10^{-7}	3.97×10^{-11}	9.45×10^{-8}
	0.4	5.03×10^{-8}	4.80×10^{-7}	2.93×10^{-8}	3.84×10^{-7}	1.27×10^{-8}	1.93×10^{-7}	3.41×10^{-11}	8.13×10^{-8}
	0.6	4.32×10^{-8}	4.16×10^{-7}	2.52×10^{-8}	3.33×10^{-7}	1.09×10^{-8}	1.69×10^{-7}	2.93×10^{-11}	6.65×10^{-8}
	0.8	3.71×10^{-8}	3.55×10^{-7}	2.16×10^{-8}	2.84×10^{-7}	9.34×10^{-9}	1.43×10^{-7}	2.52×10^{-11}	5.92×10^{-8}
	1	3.19×10^{-8}	3.06×10^{-7}	1.86×10^{-8}	2.45×10^{-7}	8.02×10^{-9}	1.24×10^{-7}	2.16×10^{-11}	4.98×10^{-8}

Table 3.2 shows that the RPSM achieves extremely small numerical errors (down to 10^{-14}) for all fractional orders α , with a consistent and clear superiority over NTDM [2] across all spatial and temporal grid points (x, t) . The results indicate that the difference between the two methods spans several orders of magnitude, confirming the high accuracy and efficiency of the proposed method. On the other hand, it is observed that the accuracy of RPSM improves as α increases. In particular, the results at $\alpha = 0.8$ are very close to those obtained at the classical case $\alpha = 1$, with only a negligible difference between them. This behavior indicates the rapid stabilization of the method and its fast convergence toward the classical solution over time. A similar general trend is also observed in Table 3.1, where RPSM consistently outperforms NTDM and the accuracy improves as α approaches 1. However, Table 3.2 exhibits smaller errors and stronger numerical stability, especially for larger values of α , which further confirms the robustness of the method across different applications.

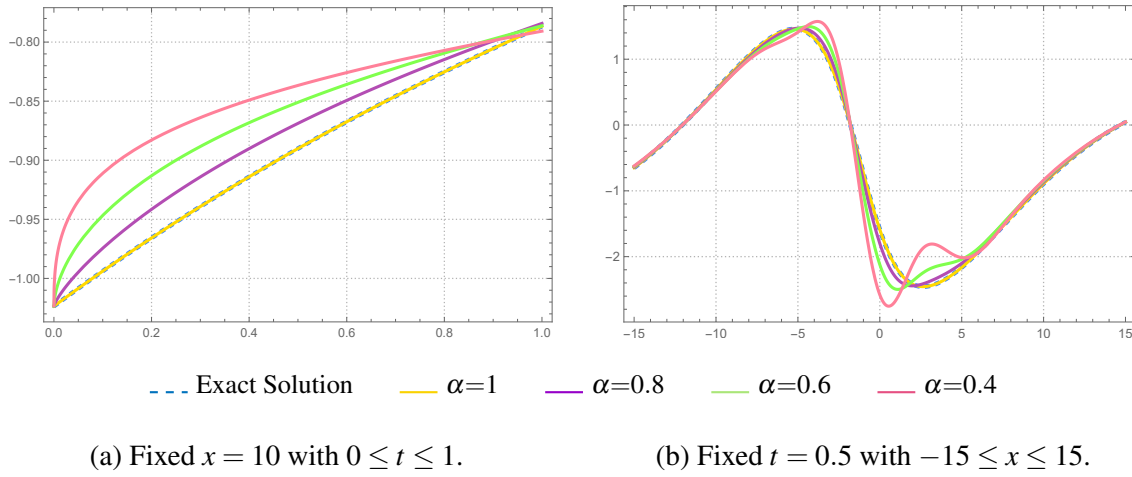


Figure 3.3: 2D comparison of the exact solution and the RPSM approximation for various fractional orders α ($\alpha = 1, \alpha = 0.8, \alpha = 0.6, \alpha = 0.4$) for Application 2.

Figure 3.3 presents a 2D comparison between the exact solution and the RPSM approximation for the fractional orders $\alpha = 1, 0.8, 0.6$, and 0.4 . Panel (a) shows the time evolution at the fixed spatial point $x = 10$ over the interval $0 \leq t \leq 1$, while panel (b) illustrates the spatial distribution at the fixed time $t = 0.5$ over $-15 \leq x \leq 15$. The results clearly indicate that the agreement between the exact and approximate solutions improves as α approaches 1, with a perfect match observed at $\alpha = 1$. This is expected since the fractional derivative reduces to the classical integer-order derivative in this case, allowing the RPSM to recover the exact solution exactly. For smaller values such as $\alpha = 0.4$, slight deviations appear due to the stronger memory effects inherent in fractional operators. Nevertheless, the RPSM approximation maintains high accuracy across both the temporal and spatial domains, confirming its efficiency and reliability in solving fractional differential models.

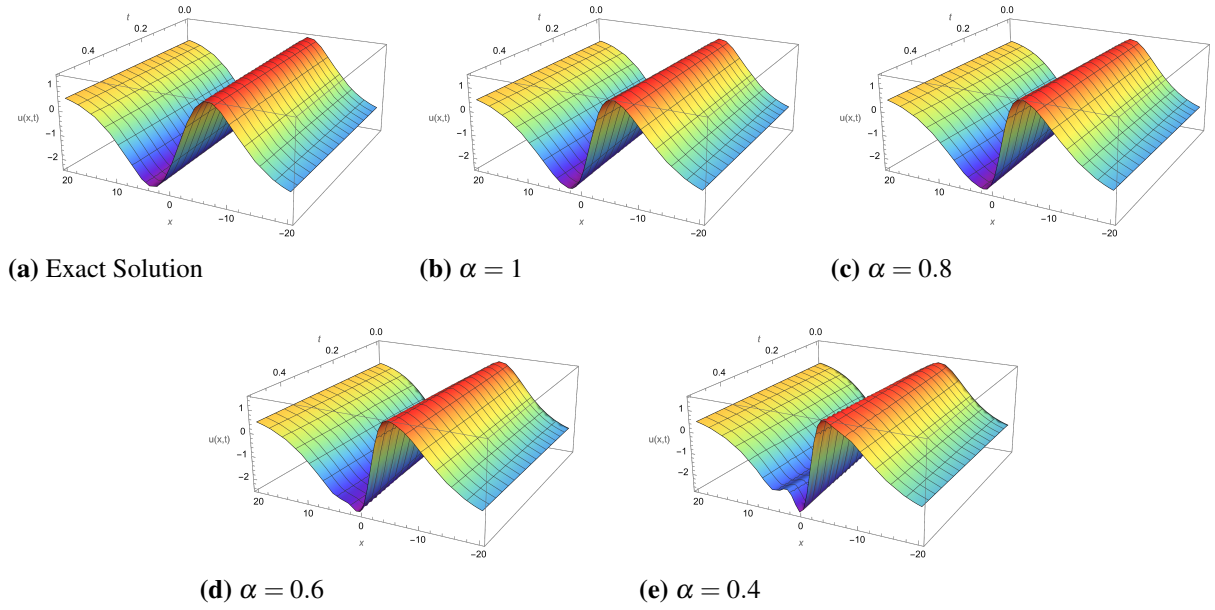


Figure 3.4: 3D surface plots of the exact solution and the RPSM approximation for various fractional orders α ($\alpha = 1, \alpha = 0.8, \alpha = 0.6, \alpha = 0.4$) for Application 2.

Figure 3.4 presents 3D surface plots comparing the exact solution with the RPSM approximation for different fractional orders α ($\alpha = 1, 0.8, 0.6, 0.4$). The horizontal axis represents the spatial variable $x \in [-20, 20]$, while the other axis corresponds to the temporal variable $t \in [0, 0.5]$, and the vertical axis denotes the solution $u(x, t)$. The results show that the approximate solution nearly coincides with the exact solution when $\alpha = 1$, and this agreement remains very strong for values close to unity such as $\alpha = 0.8$. As α decreases to 0.6 and 0.4, slight discrepancies begin to appear in certain regions, which is associated with the increasing memory effect inherent in fractional operators. Nevertheless, the RPSM method consistently preserves the overall structure of the solution across all considered cases, confirming its efficiency and reliability in handling fractional differential equations of varying orders.

2.3 Application 3

By substituting $a = 1, b = 4, c = 1$. into the equation (3.1), we find :

$$\begin{cases} D_t^\alpha u(x, t) + u(x, t) \frac{\partial u}{\partial x}(x, t) + \frac{\partial^2 u}{\partial x^2}(x, t) + 4 \frac{\partial^3 u}{\partial x^3}(x, t) + \frac{\partial^4 u}{\partial x^4}(x, t) = 0, \\ x \in \mathbb{R}, t \in [0, T], 0 < \alpha \leq 1, \\ u(x, 0) = \mu + 9 - 15 (\tanh(\rho(x - \eta)) + \tanh^2(\rho(x - \eta)) - \tanh^3(\rho(x - \eta))) \end{cases} \quad (3.19)$$

where μ, ρ and η are real constants.

By taking $\alpha = 1$, we obtain the exact solution as [22]

$$u(x, t) = \mu + 9 - 15 (\tanh(\rho(x - \mu t - \eta)) + \tanh^2(\rho(x - \mu t - \eta)) - \tanh^3(\rho(x - \mu t - \eta)))$$

For: $\mu = 3, \rho = \frac{1}{2}, \eta = -13$ Applying the RPSM method to solve equation (3.19), Suppose the solution is written as the following power series.

$$u(x, t) = \sum_{n=0}^{\infty} \varphi_n(x) \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)} \quad (3.20)$$

We consider that $u_k(x, t)$ is k . truncated series of $u(x, t)$

$$u_k(x, t) = \varphi(x) + \sum_{n=1}^k \varphi_n(x) \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)}, \quad k = 1, 2, 3, \dots \quad (3.21)$$

where $u_0(x, t) = u(x, 0) = \varphi_0(x) = \varphi(x)$

To obtain the value of coefficients $\varphi_n(x), n = 1, 2, \dots, k$ in series expanded of Eq.(3.19), Residual function

Res is given by

$$\text{Res}(x, t) = D_t^\alpha u(x, t) + u(x, t) \frac{\partial u}{\partial x}(x, t) + \frac{\partial^2 u}{\partial x^2}(x, t) + \frac{\partial^4 u}{\partial x^4}(x, t). \quad (3.22)$$

and the k -th Residual functions of equation (3.19) is defined by:

$$\text{Res}_k(x, t) = D_t^\alpha u_k(x, t) + u_k(x, t) \frac{\partial u_k}{\partial x}(x, t) + \frac{\partial^2 u_k}{\partial x^2}(x, t) + \frac{\partial^4 u_k}{\partial x^4}(x, t). \quad (3.23)$$

to find the needed coefficients $\varphi_n(x)$ for $n = 1, 2, \dots, k$ in Eq.(3.21). To obtain $\varphi_1(x)$, we consider $k = 1$ in Eq. (3.23), Where, we know that $Res_1(x, t) = 0$ at $t = 0$ and thus,

$$\varphi_1(x) = \frac{360e^{13} \left(\cosh\left(\frac{x}{2}\right) - \sinh\left(\frac{x}{2}\right) \right) \left((-1 + 2e^{13}) \cosh\left(\frac{x}{2}\right) + (1 + 2e^{13}) \sinh\left(\frac{x}{2}\right) \right)}{\left((1 + e^{13}) \cosh\left(\frac{x}{2}\right) + (-1 + e^{13}) \sinh\left(\frac{x}{2}\right) \right)^4}$$

Likewise, to obtain the form of the second unknown coefficient $\varphi_2(x)$, we write $k = 2$ in Eq.(3.23), we know that $D_t^\alpha Res_2(x, t)|_{t=0} = 0$ and thus,

$$\varphi_2(x) = \frac{1080e^{13} \left(\cosh\left(\frac{x}{2}\right) - \sinh\left(\frac{x}{2}\right) \right) \left(-7e^{13} + (1 + 4e^{26}) \cosh(x) + (-1 + 4e^{26}) \sinh(x) \right)}{\left((1 + e^{13}) \cosh\left(\frac{x}{2}\right) + (-1 + e^{13}) \sinh\left(\frac{x}{2}\right) \right)^5}$$

Similarly to obtain $\varphi_3(x)$, we consider $k = 3$ in Eq. (3.23),

When $D_t^{2\alpha} Res_3(x, 0) = 0$ and thus,

$$\begin{aligned} \varphi_3(x) = & \frac{3240e^{13} \left(\cosh\left(\frac{x}{2}\right) - \sinh\left(\frac{x}{2}\right) \right)}{\Gamma[1 + \alpha]^2 \left((1 + e^{13}) \cosh\left(\frac{x}{2}\right) + (-1 + e^{13}) \sinh\left(\frac{x}{2}\right) \right)^9} \left[40e^{13} \Gamma[1 + 2\alpha] \right. \\ & \left(-18e^{26} + e^{13} (9 + 8e^{26}) \cosh(x) - \cosh(2x) - 9e^{13} \sinh(x) + 8e^{39} \sinh(x) + \sinh(2x) \right) \\ & + \Gamma[1 + \alpha]^2 \left(1402e^{39} - e^{26} (702 + 697e^{26}) \cosh(x) + (95e^{13} - 9e^{65}) \cosh(2x) \right. \\ & - \cosh(3x) + 8e^{78} \cosh(3x) + 702e^{26} \sinh(x) - 697e^{52} \sinh(x) - 95e^{13} \sinh(2x) \\ & \left. \left. - 9e^{65} \sinh(2x) + \sinh(3x) + 8e^{78} \sinh(3x) \right) \right] \end{aligned}$$

and

$$\begin{aligned}
u_3(x, t) = & -\frac{3}{\Gamma[1+\alpha]^2\Gamma[1+2\alpha]\Gamma[1+3\alpha]((1+e^{13})\cosh(x/2)+(-1+e^{13})\sinh(x/2))^9} \\
& \times \left[-120e^{13}t^\alpha\Gamma[1+\alpha]\Gamma[1+2\alpha]\Gamma[1+3\alpha](\cosh(x/2)-\sinh(x/2)) \right. \\
& \times ((1+e^{13})\cosh(x/2)+(-1+e^{13})\sinh(x/2))^5 \times ((-1+2e^{13})\cosh(x/2) \\
& + (1+2e^{13})\sinh(x/2)) - 43200e^{26}t^{3\alpha}\Gamma[1+2\alpha]^2(\cosh(x)-\sinh(x)) \\
& \times \left[-9e^{13}(-1+2e^{13})\cosh(x/2)+(-1+8e^{39})\cosh(3x/2)-9e^{13}\sinh(x/2) \right. \\
& \left. \left. - 18e^{26}\sinh(x/2)+\sinh(3x/2)+8e^{39}\sinh(3x/2) \right] \right. \\
& + \Gamma[1+\alpha]^2 \left[-360e^{13}t^{2\alpha}\Gamma[1+3\alpha](\cosh(x/2)-\sinh(x/2))((1+e^{13})\cosh(x/2) \right. \\
& + (-1+e^{13})\sinh(x/2))^4 \left(-7e^{13}+(1+4e^{26})\cosh(x)+(-1+4e^{26})\sinh(x) \right) \\
& + \Gamma[1+2\alpha] \left(\Gamma[1+3\alpha]((1+e^{13})\cosh(x/2)+(-1+e^{13})\sinh(x/2))^6 \right. \\
& \times \left(e^{13}(-37+3e^{13})\cosh(x/2)+(1+e^{39})\cosh(3x/2)+(-1+37e^{13}+3e^{26} \right. \\
& + e^{39}+2(-1+e^{39})\cosh(x))\sinh(x/2) \left. \left. \right) - 1080e^{13}t^{3\alpha}(\cosh(x/2)-\sinh(x/2)) \right. \\
& \times \left(1402e^{39}-e^{26}(702+697e^{26})\cosh(x)+(95e^{13}-9e^{65})\cosh(2x)-\cosh(3x) \right. \\
& + 8e^{78}\cosh(3x)+702e^{26}\sinh(x)-697e^{52}\sinh(x)-95e^{13}\sinh(2x)-9e^{65}\sinh(2x) \\
& \left. \left. \left. + \sinh(3x)+8e^{78}\sinh(3x) \right) \right) \right) \left. \right]
\end{aligned}$$

Table 3.3: Numerical comparison of approximate solutions of RPSM and q-HATM [22] at different fractional orders of α ($\alpha = 0.6, \alpha = 0.75, \alpha = 0.9$ and $\alpha = 1$) for Application 3.

x	t	$\alpha = 0.6$		$\alpha = 0.75$		$\alpha = 0.9$		$\alpha = 1$	
		RPSM	q-HATM	RPSM	q-HATM	RPSM	q-HATM	RPSM	q-HATM
0.2	1.00	10^{-9}	4.04×10^{-7}	1.93×10^{-10}	1.45×10^{-7}	8.31×10^{-11}	3.35×10^{-8}	9.30×10^{-12}	3.75×10^{-9}
	1.36	10^{-10}	9.87×10^{-7}	2.61×10^{-11}	3.65×10^{-7}	1.13×10^{-11}	4.02×10^{-8}	1.26×10^{-12}	8.16×10^{-8}
	1.84	10^{-11}	1.29×10^{-6}	3.53×10^{-12}	2.77×10^{-7}	1.52×10^{-12}	3.36×10^{-7}	1.74×10^{-13}	5.94×10^{-7}
	2.49	10^{-12}	1.13×10^{-7}	4.76×10^{-13}	1.48×10^{-6}	2.06×10^{-13}	2.43×10^{-6}	2.49×10^{-14}	2.87×10^{-6}
	3.36	10^{-13}	7.83×10^{-6}	6.39×10^{-14}	9.50×10^{-6}	2.84×10^{-14}	1.08×10^{-5}	2.22×10^{-15}	1.14×10^{-5}
0.2	2.45	10^{-9}	5.47×10^{-8}	8.32×10^{-10}	1.96×10^{-8}	9.97×10^{-11}	4.54×10^{-9}	2.02×10^{-10}	5.08×10^{-10}
	3.31	10^{-10}	1.34×10^{-7}	1.13×10^{-10}	4.93×10^{-8}	1.35×10^{-11}	5.44×10^{-9}	2.74×10^{-11}	1.10×10^{-8}
	4.48	10^{-11}	1.74×10^{-7}	1.52×10^{-11}	3.75×10^{-8}	1.83×10^{-12}	4.55×10^{-8}	3.71×10^{-12}	3.88×10^{-7}
	6.07	10^{-12}	1.54×10^{-8}	2.06×10^{-12}	2.01×10^{-7}	2.49×10^{-13}	3.29×10^{-7}	5.04×10^{-13}	1.55×10^{-6}
	8.22	10^{-13}	1.06×10^{-6}	2.77×10^{-13}	1.29×10^{-6}	3.55×10^{-14}	1.46×10^{-6}	6.57×10^{-14}	6.87×10^{-11}
0.2	3.19	10^{-9}	7.41×10^{-9}	2.95×10^{-9}	2.65×10^{-9}	8.34×10^{-10}	6.14×10^{-10}	1.47×10^{-9}	1.50×10^{-9}
	4.31	10^{-10}	1.81×10^{-8}	4.00×10^{-10}	6.68×10^{-9}	1.13×10^{-10}	7.37×10^{-10}	1.99×10^{-10}	1.09×10^{-8}
	5.84	10^{-11}	2.36×10^{-8}	5.41×10^{-11}	5.08×10^{-9}	1.53×10^{-11}	6.17×10^{-9}	2.70×10^{-11}	5.26×10^{-8}
	7.90	10^{-12}	2.09×10^{-9}	7.32×10^{-12}	2.72×10^{-8}	2.07×10^{-12}	4.46×10^{-8}	3.65×10^{-12}	2.10×10^{-7}
	1.07	10^{-12}	1.45×10^{-7}	9.91×10^{-13}	1.74×10^{-7}	2.81×10^{-13}	1.98×10^{-7}	4.94×10^{-13}	9.30×10^{-12}
0.2	2.83	10^{-10}	1.00×10^{-9}	1.00×10^{-8}	5.59×10^{-10}	6.03×10^{-9}	8.31×10^{-11}	7.12×10^{-9}	2.02×10^{-10}
	3.84	10^{-11}	2.45×10^{-9}	1.35×10^{-9}	9.04×10^{-10}	8.16×10^{-10}	9.97×10^{-11}	9.63×10^{-10}	1.47×10^{-9}
	5.19	10^{-12}	3.19×10^{-9}	1.83×10^{-10}	6.87×10^{-10}	1.10×10^{-10}	8.34×10^{-10}	1.30×10^{-10}	7.12×10^{-9}
	7.03	10^{-13}	2.83×10^{-10}	2.48×10^{-11}	3.68×10^{-9}	1.49×10^{-11}	6.03×10^{-9}	1.76×10^{-11}	2.84×10^{-8}
	9.41	10^{-14}	1.94×10^{-8}	3.35×10^{-12}	2.36×10^{-8}	2.02×10^{-12}	2.68×10^{-8}	2.39×10^{-12}	1.25×10^{-12}
0.2	1.94	10^{-8}	1.36×10^{-10}	3.34×10^{-8}	4.85×10^{-11}	2.68×10^{-8}	1.13×10^{-11}	2.84×10^{-8}	1.74×10^{-11}
	2.63	10^{-9}	3.31×10^{-10}	4.52×10^{-9}	1.12×10^{-10}	3.63×10^{-9}	1.35×10^{-11}	3.85×10^{-9}	1.99×10^{-10}
	3.56	10^{-10}	4.31×10^{-10}	6.12×10^{-10}	9.30×10^{-10}	4.91×10^{-10}	1.13×10^{-10}	5.20×10^{-10}	9.63×10^{-10}
	4.82	10^{-11}	3.84×10^{-11}	8.28×10^{-11}	4.98×10^{-10}	6.64×10^{-11}	3.63×10^{-9}	7.04×10^{-11}	3.85×10^{-9}
	6.52	10^{-12}	2.63×10^{-9}	1.12×10^{-11}	3.19×10^{-9}	8.99×10^{-12}	3.35×10^{-8}	9.53×10^{-12}	3.75×10^{-9}

Table 3.3 shows that the RPSM achieves very small numerical errors (ranging between 10^{-8} and 10^{-15}) for all fractional orders α studied, with a clear superiority over the q-HATM [22] method. Unlike the previous tables, where the accuracy of RPSM improved regularly and significantly as α increased towards unity, we observe here that the accuracy remains similar for all values of α (0.6, 0.75, 0.9, 1), with the best accuracy (10^{-15}) recorded only at $\alpha = 1$, and by a slight margin over the other orders. This behavior reflects the robustness and numerical stability of RPSM, as it remains capable of achieving very high accuracy even at low fractional orders, without requiring significant improvement as α increases. These results confirm that RPSM provides an accurate and reliable numerical approximation, and serves as an effective tool in this field regardless of the value of α .

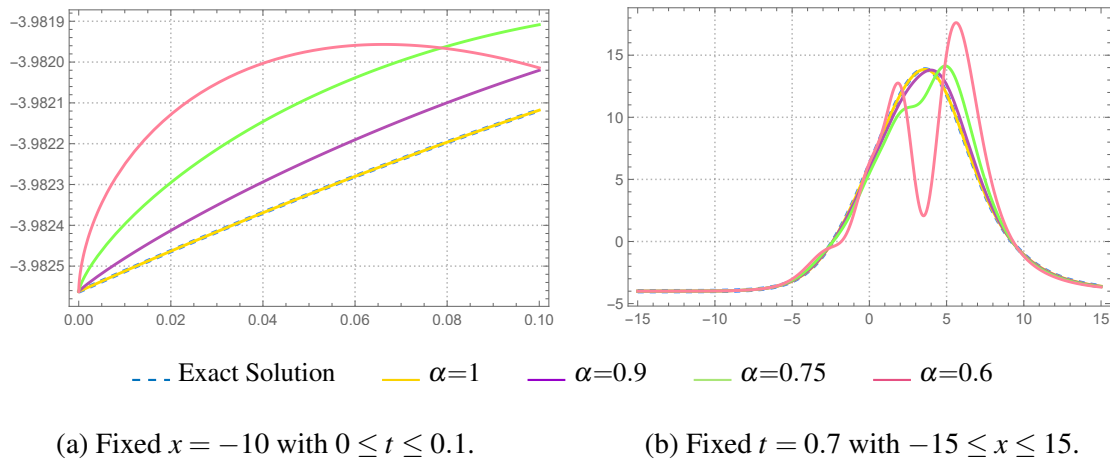


Figure 3.5: 2D comparison of the exact solution and the RPSM approximation for various fractional orders α ($\alpha = 1, \alpha = 0.9, \alpha = 0.75, \alpha = 0.6$) for Application 3.

Figure 3.5 presents a 2D comparison between the exact solution and the RPSM approximation for different fractional orders α ($\alpha = 1, 0.9, 0.75, 0.6$). Panel (a) shows the time evolution at the fixed spatial point $x = -10$ over the interval $0 \leq t \leq 0.1$, while panel (b) illustrates the spatial distribution at the fixed time $t = 0.7$ over the range $-15 \leq x \leq 15$. The results indicate that the agreement between the exact and approximate solutions improves as α approaches unity, reaching a perfect match at $\alpha = 1$. This is expected since, in this case, the fractional derivative reduces to the classical integer-order derivative, allowing the RPSM method to fully recover the exact solution. For smaller values such as $\alpha = 0.6$, slight deviations appear in some regions due to the stronger memory effects associated with fractional operators. Nevertheless, the RPSM approximation maintains high accuracy over both spatial and temporal domains, confirming its efficiency in analyzing fractional differential models.

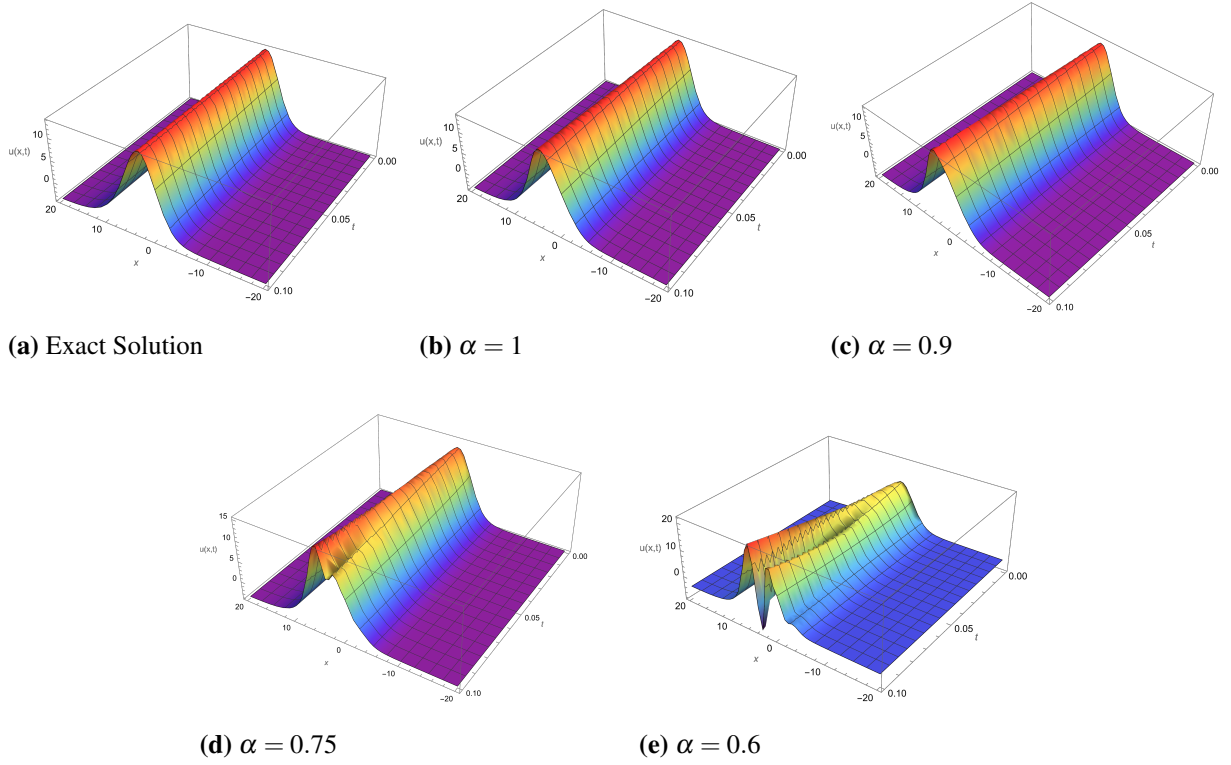


Figure 3.6: 3D surface plots of the exact solution and the RPSM approximation for various fractional orders α ($\alpha = 1, \alpha = 0.9, \alpha = 0.75, \alpha = 0.6$) for Application 3.

Figure 3.6 presents 3D surface plots comparing the exact solution with the RPSM approximation for different fractional orders α ($\alpha = 1, 0.9, 0.75, 0.6$). The horizontal axis represents the spatial domain $x \in [-20, 20]$, while the other axis corresponds to time $t \in [0, 0.1]$, and the vertical axis denotes the solution $u(x, t)$. The results indicate that the approximate solution almost perfectly matches the exact solution when $\alpha = 1$, and this strong agreement persists for values close to unity such as $\alpha = 0.9$. As α decreases to 0.75 and 0.6, slight discrepancies appear in certain regions due to the increasing memory effects inherent in fractional equations. Nevertheless, the RPSM method preserves the overall structure of the solution across all cases, confirming its effectiveness and reliability in solving fractional differential equations.

Conclusion

In this memoir, the semi-analytical solution of time-fractional nonlinear partial differential equations was presented using the residual power series method (RPSM), based on the Caputo time derivative. The method represents a direct and efficient tool that does not require linearization of the equation, decomposition of nonlinear terms, or complex transformations, which contributes to accelerating convergence and reducing computational effort without imposing additional initial conditions. The effectiveness of the method was verified by applying it to the time-fractional Kuramoto–Sivashinsky equation in three test cases with different coefficients and initial conditions. The results, represented by graphical plots and error tables, showed excellent agreement with known exact solutions, confirming the accuracy and reliability of the method. The comparison tables for absolute and relative errors demonstrated that the RPSM, even with a few series terms, provides highly accurate results and exhibits good numerical stability across various values of the fractional order α , reflecting competitive performance compared to other numerical and analytical methods applied to this type of models. The method is characterized by simplicity, flexibility, and high accuracy. Unlike some other methods that require recomputing the decomposition of nonlinear terms or employing integral transforms at each step to extract series coefficients, the RPSM computes these coefficients efficiently through direct repeated differentiation using Caputo derivative rules, thereby reducing the overall computational cost and making its implementation straightforward. Based on these results, the residual power series method (RPSM) can be considered an effective and promising tool for solving a wide spectrum of nonlinear fractional differential equations arising in physical and engineering models. In the future, this method can be extended to more complex systems of coupled or delayed fractional equations, applied to broader and more realistic dynamical problems, in addition to the possibility of combining it with acceleration transforms to enhance convergence efficiency.

References

- [1] Abdeljawad, T., Sher, M., Shah, K., Sarwar, M., Amacha, I., Alqudah, M., & Al-Jaser, A. (2024). Analysis of a class of fractal hybrid fractional differential equation with application to a biological model. *Scientific Reports*, 14(1), 18937.
- [2] Alshehry, S. A., Imran, M., Khan, A., Shah, R., & Weera, W. (2022). Fractional view analysis of Kuramoto–Sivashinsky equations with non-singular kernel operators. *Symmetry*, 14(7), 1463.
- [3] Arqub, O. A. (2013). Series solution of fuzzy differential equations under strongly generalized differentiability. *Journal of Advanced Research in Applied Mathematics*, 5(1), 31–52.
- [4] Arqub, O. A., El-Ajou, A., & Momani, S. (2015). Constructing and predicting solitary pattern solutions for nonlinear time-fractional dispersive partial differential equations. *Journal of Computational Physics*, 293, 385–399.
- [5] Barbero, G., Evangelista, L. R., Zola, R. S., Lenzi, E. K., & Scarfone, A. M. (2024). A brief review of fractional calculus as a tool for applications in physics: Adsorption phenomena and electrical impedance in complex fluids. *Fractal and Fractional*, 8(7), 369.
- [6] Bekhouche, F., Batiha, I., Boulakour, O., Batyha, R. M., Bouchenak, A., & Momani, S. (2026). Enhanced investigation of Caputo, Caputo–Fabrizio, and Atangana–Baleanu fractional derivatives in the Laplace Adomian decomposition method for nonlinear time fractional models. *Advanced Mathematical Models and Applications*, 11(1), 124–148.
- [7] Boulakour, O., Merad, A., Batiha, I., Bekhouche, F., Anakira, N., Shatnawi, M. T., & Attia, N. (2026). An efficient analysis of fractional derivatives for solving the nonlinear time-fractional Kuramoto–Sivashinsky equation using the Laplace transform and Adomian decomposition method. *International Journal of Robotics and Control Systems*, 6(1), 793–818.

- [8] Cheow, Y. H., Ng, K. H., Phang, C., & Ng, K. H. (2024). The application of fractional calculus in economic growth modelling: An approach based on regression analysis. *Heliyon*, 10(15).
- [9] Daftardar-Gejji, V., & Jafari, H. (2006). An iterative method for solving nonlinear functional equations. *Journal of Mathematical Analysis and Applications*, 316(2), 753–763.
- [10] El-Ajou, A., Abu-Arqoub, O., Al-Zhour, Z., & Momani, S. (2013). New results on fractional power series: Theories and applications. *Entropy*, 15, 5305–5323.
- [11] Elsamani, S. A., Mohmmmed, A. A., Kogan, K. M., Salih, S. Y. M., & Ahmed, I. A. (2024). Fractional calculus applications in engineering: Insights into four-dimensional chaotic systems. *Journal of Theory, Mathematics and Physics*, 3(5), 75–84.
- [12] Farraj, G., Maayah, B., Khalil, R., & Beghami, W. (2023). An algorithm for solving fractional differential equations using conformable optimized decomposition method. *International Journal of Advances in Soft Computing and Applications*, 15(1), 187–196.
- [13] Haider, J. A., Ahmad, S., & Nadeem, S. (2024). Dynamical properties of nonlinear oscillators by the variational iteration method. *Scientia Iranica*.
- [14] Hosseiny, R. M., Allahviranloo, T., Abbasbandy, S., & Babolian, E. (2022). Reproducing kernel method to solve non-local fractional boundary value problem. *Mathematical Sciences*, 16(3), 261–268.
- [15] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations* (North-Holland Mathematics Studies, Vol. 204). Elsevier.
- [16] Liu, Y., & Zhao, X. (2024, October). Application of homotopy perturbation method in solving fractional Cahn-Allen equation. In *2024 3rd International Conference on Cloud Computing, Big Data Application and Software Engineering (CBASE)* (pp. 339–342). IEEE.
- [17] Mohamed, M. Z., Yousif, M., & Hamza, A. E. (2022). Solving nonlinear fractional partial differential equations using the Elzaki transform method and the homotopy perturbation method. *Abstract and Applied Analysis*, 2022, 1–9.
- [18] Oldham, K. B., & Spanier, J. (1974). *The fractional calculus: Theory and applications of differentiation and integration to arbitrary order* (Vol. 111). Elsevier.
- [19] Podlubny, I. (1999). *Fractional differential equations*. Academic Press.

-
- [20] Prakasha, D. G., Veeresha, P., & Baskonus, H. M. (2019). Residual power series method for fractional Swift–Hohenberg equation. *Fractal and Fractional*, 3(1), 9.
- [21] Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). *Fractional integrals and derivatives: Theory and applications*. Gordon and Breach.
- [22] Veeresha, P., & Prakasha, D. G. (2021). An efficient computational technique for fractional Kuramoto-Sivashinsky equation. *International Journal of Applied and Computational Mathematics*, 7, 1–22.
- [23] Viera-Martin, E., Gómez-Aguilar, J. F., Solís-Pérez, J. E., Hernández-Pérez, J. A., & Escobar-Jiménez, R. F. (2022). Artificial neural networks: A practical review of applications involving fractional calculus. *The European Physical Journal Special Topics*, 231(10), 2059–2095.
- [24] Zada, L., Ali, N., Jamshed, W., Iqbal, A., Hussain, S. M., & Ahmad, H. (2024). Analyzing fractional effects on solutions of the generalized perturbed Zakharov-Kuznetsov equation using a residual series method with Caputo derivatives. *International Journal of Modelling and Simulation*, 1–18.