



Exam in Analysis 2



Exercise 1 (10 points)

Calculate the following integrals



$$\int \frac{\cos^2(x)}{1 - \sin(x)} dx, \quad \int \frac{\sin x + \cos x}{\sin x - \cos x} dx, \quad \int \frac{1}{\sin^2 x \cos^2 x} dx$$
$$\int \frac{1}{\cos(x) \sin(x)} dx, \quad \int \frac{1}{\sin x} dx$$



$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{1 + 2x^2} dx$$



Calculate the area of $\mathcal{A}$ defined as follow:

$$\mathcal{A} = \left\{ (x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2\pi, \quad \frac{1}{8} \leq y \leq \cos^3 x \right\}$$



Exercise 2 (10 points)

Solve the the following Cauchy problems

$$\begin{cases} y'(x) + (3x^2 - 1)y(x) = x^2 e^x \\ y(0) = -1 \end{cases}$$

$$\begin{cases} y''(x) + 9y(x) = 108x \cos(3x) + x e^x + x^2 \\ y(0) = -\frac{2}{81} \\ y'(0) = -\frac{1}{50} \end{cases}$$



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Correction the exam of Analysis 2

Exercise 1



$$\int \frac{\cos^2(x)}{1 - \sin(x)} dx = \int \frac{1 - \sin^2(x)}{1 - \sin(x)} dx = \int \frac{(1 - \sin(x))(1 + \sin(x))}{1 - \sin(x)} dx = x - \cos(x) + c$$

•

$$\int \frac{\sin x + \cos x}{\sin x - \cos x} dx = \ln(\sin x - \cos x) + c$$

•

$$\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^2(x) + \cos^2(x)}{\sin^2 x \cos^2 x} dx = \int \left[\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} \right] dx = -\frac{1}{\tan(x)} + \tan(x) + c$$

•

$$\int \frac{1}{\sin x \cos x} dx = \int \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} dx = \int \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} dx = \ln |\sin(x)| - \ln |\cos(x)| + c$$

•

$$\begin{aligned} \int \frac{1}{\sin x} dx &= \frac{1}{2} \int \frac{1}{\sin \frac{x}{2} \cos \frac{x}{2}} dx \\ &= \frac{1}{2} \int \frac{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}}{\sin \frac{x}{2} \cos \frac{x}{2}} dx \\ &= \frac{1}{2} \left[\ln \left| \sin \left(\frac{x}{2} \right) \right| - \ln \left| \cos \left(\frac{x}{2} \right) \right| \right] + c \end{aligned}$$



We calculate the following integrals

$$\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{dx}{1 + 2x^2} = \frac{1}{\sqrt{2}} \int_{-1}^1 \frac{dt}{1 + t^2} = \frac{\pi}{2\sqrt{2}}$$



We calculate the area of $\mathcal{A}$ defined as follow:

$$\mathcal{A} = \left\{ (x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2\pi, \frac{1}{8} \leq y \leq \cos^3 x \right\}$$

$\cos^3(x) = \frac{1}{8} = (\frac{1}{2})^3 \Leftrightarrow x \in \{\frac{\pi}{3}, \frac{5\pi}{3}\}$ then

$$\begin{aligned}
 \text{Area } (\mathcal{A}) &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\cos^3(x) - \frac{1}{8} \right) dx \\
 &\stackrel{(\cos^3(x) - \frac{1}{8}) \text{ is even function}}{=} 2 \int_0^{\frac{\pi}{3}} \left(\cos^3(x) - \frac{1}{8} \right) dx \\
 &= 2 \int_0^{\frac{\pi}{3}} \cos^3(x) dx - \frac{\pi}{6} \\
 &\stackrel{\cos^2(x) = 1 - \sin^2(x)}{=} 2 \int_0^{\frac{\pi}{3}} (1 - \sin^2(x)) \cos(x) dx - \frac{\pi}{6} \\
 &= 2 \int_0^{\frac{\pi}{3}} \cos(x) - \cos(x) \sin^2(x) dx - \frac{\pi}{6} \\
 &\stackrel{t = \sin x, \quad dt = \cos x dx}{=} \sqrt{3} - 2 \int_0^{\frac{\sqrt{3}}{2}} t^2 dt - \frac{\pi}{6} \\
 &= \sqrt{3} - \frac{2}{3} \cdot \frac{3\sqrt{3}}{8} - \frac{\pi}{6} \\
 &= \sqrt{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} \\
 &= \frac{9\sqrt{3} - \pi}{12}
 \end{aligned}$$



Exercise 2

We solve the following Cauchy problems

$$\begin{cases} y'(x) + (3x^2 - 1)y(x) = x^2 e^x \\ y(0) = -1 \end{cases}$$



This $y'(x) + (3x^2 - 1)y(x) = x^2 e^x$ is a **non-homogeneous** linear first-order ODE, its solution is therefore of the type

$$y_g(x) = y_h(x, C) + y_p(x)$$



We seek all the solutions of the homogeneous ODE. We have

$$\begin{aligned}
 y' + (3x^2 - 1)y &= 0 \Rightarrow \frac{dy}{y} = -(3x^2 - 1) dx \Rightarrow \int \frac{dy}{y} = - \int (3x^2 - 1) dx \\
 \ln |x| &= -x^3 + x + c \Rightarrow y = C e^{-x^3 + x}, \quad C \in \mathbb{R}
 \end{aligned}$$

So,

$$y_h(x, C) = C e^{-x^3 + x}, \quad C \in \mathbb{R}^*$$



We are looking for a particular solution. For $y_p(x) = C(x)e^{-x^3+x}$ we have

$$\begin{aligned} C'(x)e^{-x^3+x} + C(x)(-3x^2+1)e^{-x^3+x} + (3x^2+1)C(x)e^{-x^3+x} &= x^2e^x \\ C'(x)e^{-x^3+x} &= x^2e^x \\ C'(x) &= x^2e^{x^3} \end{aligned}$$

Then,

$$C(x) = \int x^2 e^{x^3} dx = \frac{1}{3} \int 3x^2 e^{x^3} dx = \frac{1}{3} e^{x^3}$$

So,

$$y_p(x) = \frac{1}{3} e^{x^3} e^{-x^3+x} = \frac{1}{3} e^x.$$



The general solutions of the EDO are therefore the functions

$$y_g(x) = y_h(x, C) + y_p(x) = C e^{-x^3+x} + \frac{1}{3} e^x$$

By imposing the $y(0) = -1$ we obtain $C = \frac{-4}{3}$ hence the unique solution to the Cauchy problem

$$y(x) = \frac{-4}{3} e^{-x^3+x} + \frac{1}{3} e^x$$

We solve the following Cauchy problem

$$\begin{cases} y''(x) + 9y(x) = 108x \cos(3x) + xe^x + x^2 \\ y(0) = -\frac{2}{81} \\ y'(0) = -\frac{1}{50} \end{cases}$$



Since the ODE $y'' + 9y = 108x \cos(3x) + xe^x + x^2$ is not homogeneous, we seek its solutions in the form

$$y_g(x, C_1, C_2) = y_h(x, C_1, C_2) + y_p(x)$$

- * We seek all the solutions of the homogeneous ODE $y'' + 9y = 0$. The characteristic polynomial associated with the ODE is $\lambda^2 + 9 = 0$ which has discriminant $\Delta = (3i)^2 < 0$. So, we have two complex conjugate solutions

$$\lambda_{1,2} = \pm 3i$$

As $\alpha = 0$ and $\beta = 3$, all solutions are then functions

$$y(x, C_1, C_2) = e^{\alpha x} (C_1 \cos(\beta x) + C_2 \sin(\beta x)) = C_1 \cos 3x + C_2 \sin 3x \quad C_1, C_2 \in \mathbb{R}$$

- * We are looking for particular solutions. As the source term is $g_1(x) = 108x \cos(3x)$, we have

$$n = 1, \quad \mu = 0, \quad \theta = 3$$

As $\Delta < 0$, $\theta = \beta = 3$ and $\mu = \alpha = 0$ so $m = 1$. The particular solution will be of the form

$$y_{p1}(x) = x((ax+b) \cos 3x + (cx+d) \sin 3x) = (ax^2+bx) \cos 3x + (cx^2+dx) \sin 3x$$

which gives $a = 0$, $b = 3$, $c = 9$, $d = 0$.

As the source term is $g_2(x) = xe^x$, we have

$$n = 1, \quad \mu = 1, \quad \theta = 0 \Rightarrow m = 0.$$

The particular solution will be of the form

$$y_{p_2}(x) = e^x(ax + b)$$

So,

$$y'_{p_2}(x) = e^x(ax + b + a), \quad y''_{p_2}(x) = e^x(ax + b + 2a)$$

which gives $a = \frac{1}{10}$, $b = -\frac{1}{50}$. We conclude that

$$y_{p_2}(x) = e^x\left(\frac{1}{10}x - \frac{1}{50}\right)$$

As the source term is $g_3(x) = x^2$, we have

$$n = 2, \quad \mu = 0, \quad \theta = 0 \Rightarrow m = 0.$$

The particular solution will be of the form

$$y_{p_3}(x) = ax^2 + bx + c$$

So,

$$y'_{p_3}(x) = 2ax + b, \quad y''_{p_3}(x) = 2a$$

which gives $a = \frac{1}{9}$, $b = 0$, $c = -\frac{2}{81}$. We conclude that

$$y_{p_3}(x) = \frac{1}{9}x^2 - \frac{2}{81}$$

In conclusion, all solutions are the functions

$$y_g(x, C_1, C_2) = C_1 \cos 3x + C_2 \sin 3x + e^x\left(\frac{1}{10}x - \frac{1}{50}\right) + \frac{1}{9}x^2 - \frac{2}{81} \quad C_1, C_2 \in \mathbb{R}$$



Since $y(0) = -\frac{2}{81}$ then $C_1 = \frac{1}{50}$. Since $y'(0) = -\frac{1}{50}$ then $C_2 = -\frac{1}{30}$. We conclude that the solution to the Cauchy problem is

$$y(x) = \frac{1}{50} \cos 3x - \frac{1}{30} \sin 3x + e^x\left(\frac{1}{10}x - \frac{1}{50}\right) + \frac{1}{9}x^2 - \frac{2}{81}$$



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