Abbés Laghrour University of Khenchela
Faculty of Sciences and Technology
Department of Mathematics
Analysis "4" Module, Licence 2 Mathematics
May 15, 2025 Duration: $1^h,30$

Final Examination

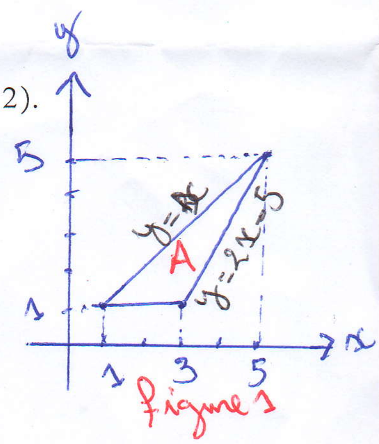
Exercise 1: The following statements are "**true**" or "**false**" ?

a- The range of $f(x,y) = \sqrt{x^2+y^2-16}$ is $[0,4]$.
b- $f_{xy} = f_{yx}$ Verified for all functions.
c- $\int chx.dx = (chx)' = shx$
d- $(D_u f)(x,y) = \lim\limits_{h\to 0} \frac{f(x-ha,y-hb)-f(x,y)}{h}$, $\vec{u} = a.\vec{i} + b.\vec{j}$

Exercise 2: A. $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & if \ (x,y) \neq (0,0) \\ 0 & at \quad (0,0) \end{cases}$

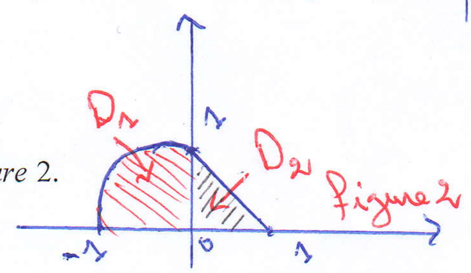
i) What is the domaine of f, $D_f = ?$
ii) Using two methods, calculate $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$
iii) Where is f continuous ?

B. $z = f(x,y) = x^2 + xy + 1$
1) Find the cartesian equation of the tangent plane at $(1,0,2)$.
2) If $x = \cos(t)$, $y = t^2$ Find $\frac{dz}{dt}$ Where $t = 0$.
3) Find $D_u f(1,0)$, $\vec{u} = \vec{i} - \vec{j}$
4) Find and classify the critical points of f



Exercise 3: *I.* By using multiple integrals, find the area A of the triange with vertices at $(1,1)$, $(3,1)$ and $(5,5)$, as shown in *figure* 1.

II. Evaluate $I = \iint_D (x-2y).dxdy$
$D = D_1 \cup D_2$ as shown in *figure* 2.



III. Calculate $J = \iiint_G yz\sin(xy).dxdydz$ over the region G defined by:

$0 \le x \le 1$; $0 \le y \le \frac{\pi}{2}$; $-1 \le z \le 1$

*** **** ***

(2 points + 9 points + 9 points) = 20 points
Good luck ... Module Manager
Dr. A. Abdelkrim

Model Answer

Analysis 4

Exercise 1: True / False

a → False

b → False (2)

c → True

d → False

Exercise 2: A –

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

A-i) Domaine of f ?

$D_f=(\mathbb{R}^2-\{(0,0)\})\cup\{(0,0)\}$.

$D_f=\mathbb{R}^2$. # (1)

ii) $\lim\limits_{(0,0)}\dfrac{x^2-y^2}{x^2+y^2}=?$

Method 1: $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$

$(x,y)\to(0,0)\Rightarrow r\to 0,\ \theta\in\mathbb{R}$

$$\lim_{(0,0)}\frac{x^2-y^2}{x^2+y^2}=\lim_{r\to0}\frac{\cancel{r^2}\cos^2\theta-\cancel{r^2}\sin^2\theta}{\cancel{r^2}(\underbrace{\cos^2\theta+\sin^2\theta}_{1})}$$

$=\cos^2\theta-\sin^2\theta,\ \forall\theta\in\mathbb{R}$ (1)

$\Rightarrow$ The limit does not exist.

Method 2:

* With $x=0$:

$$\lim_{(0,0)}\frac{x^2-y^2}{x^2+y^2}=\lim_{\substack{x=0\\y\to0}}\frac{x^2-y^2}{x^2+y^2}=\lim_{y\to0}\frac{-y^2}{y^2}$$

$=-1$.

* With $y=0$:

$$\lim_{(0,0)}\frac{x^2-y^2}{x^2+y^2}=\lim_{\substack{y=0\\x\to0}}\frac{x^2-y^2}{x^2+y^2}$$ (1)

$=\lim\limits_{x\to0}\dfrac{x^2}{x^2}=1\neq-1$

then: The limit does not exist.

A-iii) Where is f continuous?

We have $\dfrac{x^2-y^2}{x^2+y^2}$ continuous in $\mathbb{R}^2-\{0,0\}$, but in $(0,0)$

$f(0,0)=0\neq\lim\limits_{(0,0)}\dfrac{x^2-y^2}{x^2+y^2}$

then f is not continuous at $(0,0)$. (1)

Result: f continuous in $\mathbb{R}^2-\{0,0\}$

B – $z=f(x,y)=x^2+xy+1$

1) Equation of tangent plane? at $(1,0,2)$

$(x_0,y_0)=(1,0)$

$z=f_x(x_0,y_0)\cdot(x-x_0)+f_y(x_0,y_0)\cdot(y-y_0)$
$+f(x_0,y_0)$.

$f_x = 2x + y \Rightarrow f_x(1,0) = 2$

$f_y = x \Rightarrow f_y(1,0) = 1$

$f(x_0, y_0) = f(1,0) = 2$

1

<u>Then</u>: $z = 2(x-1) + 1.(y-0) + 2$

$\boxed{z = 2x + y}$ #

B-2) $\frac{dz}{dt}(0) = ?$ $\quad x = \cos t, \; y = t^2$

By The chaine rule:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$\frac{dx}{dt} = -\sin t$

$\frac{dy}{dt} = 2t$

$$\frac{dz}{dt} = (2x+y) \cdot (-\sin t) + x.(2t)$$

$$= (2\cos t + t^2)(-\sin t) + \cos t . 2t$$

1

if $t = 0$, $\frac{dz}{dt}(0) = 0$.

B-3) $D_u f(1,0)$, $\vec{u} = \vec{i} - \vec{j}$

The unit vector $\vec{v} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{\vec{u}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$.

1,25

$D_u f(1,0) = \nabla f_{(1,0)} \cdot \vec{v}$

$$\nabla f(1,0) = \begin{pmatrix} f_x(1,0) \\ f_y(1,0) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$D_u f(1,0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}. \quad \#$$

<u>Method 2</u>:

$$D_u f(1,0) = \lim_{h \to 0} \frac{f\left(1 + \frac{h}{\sqrt{2}}, \frac{-h}{\sqrt{2}}\right) - 2}{h}$$

$$= \lim_{h \to 0} \frac{\left(1 + \frac{h}{\sqrt{2}}\right)^2 - \frac{h}{\sqrt{2}}\left(1 + \frac{h}{\sqrt{2}}\right) - 1}{h} = \frac{1}{\sqrt{2}} \quad \#$$

<u>Remark</u>: Removing the indeterminate form using L'Hopital's rule.

B-4) Critical points?

(x,y) is a critical point if $\nabla f(x,y)=0$

$$\nabla f(x,y)=\begin{pmatrix} f_x \\ f_y \end{pmatrix}=0 \Rightarrow \begin{cases} f_x=0 \\ f_y=0 \end{cases}$$

$$\Rightarrow \begin{cases} 2x+y=0 \\ x=0 \end{cases} \Rightarrow (x,y)=(0,0) \text{ The unique critical point.}$$ 1

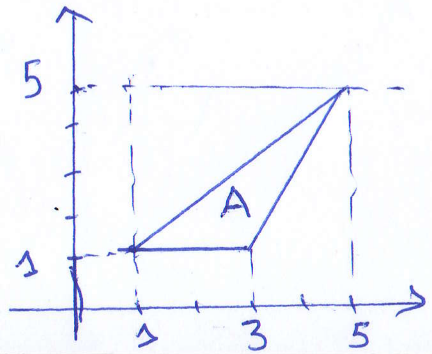
$$* \ \text{Hess} f=\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}=\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det \text{Hess} f=\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix}=-1$$

$\det \text{Hess} f(0,0)=-1<0$, Then $(0,0)$ is a saddle point. 0,75

Exercise 3:

I- $\cancel{A}$ $= \iint_A 1.\,dxdy$



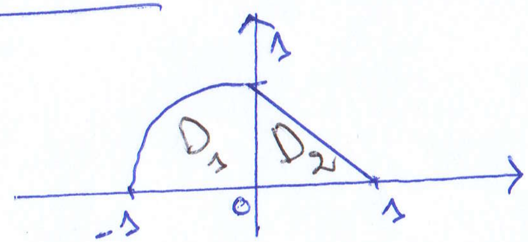
$$A=\int_1^5\int_y^{\frac{y+5}{2}} 1.\,dxdy=\int_1^5 \left(x\Big|_y^{\frac{y+5}{2}}\right)dy$$

$$=\int_1^5\left(\frac{y+5}{2}-y\right)dy=\int_1^5\left(\frac{-y}{2}+\frac{5}{2}\right)dy$$ 3

$$=\left[\frac{-y^2}{4}+\frac{5}{2}y\right]_1^5=\frac{-25}{4}+\frac{25}{2}+\frac{1}{4}-\frac{5}{2}=4$$

$\boxed{A=4\,cm^2}$ #

II-



$$I=\iint_D (x-2y)\,dxdy$$

$$I_1=\iint_{D_1}(x-2y)\,dxdy \ , \quad x=r\cos\theta, \ y=r\sin\theta, \quad dxdy=rdrd\theta$$

$$(x,y)\in D_1 \Rightarrow 0\leq r\leq 1, \ \pi/2\leq\theta\leq\pi$$ 3

$$\Rightarrow I_1=\int_{\pi/2}^{\pi}\int_0^1 (r\cos\theta-2r\sin\theta)\,r\,drd\theta$$

$$=\int_{\pi/2}^{\pi}\left(\frac{1}{3}\cos\theta-\frac{2}{3}\sin\theta\right)d\theta=\left[\frac{\sin\theta}{3}+\frac{2}{3}\cos\theta\right]_{\pi/2}^{\pi}$$

$$=-1, \quad I_1=|I_1|!$$

$$I_2 = \iint_{D_2} (x-2y)\,dx\,dy = \int_0^1 \int_0^{-x+1} (x-2y)\,dy\,dx$$

$$= \int_0^1 \left[xy - y^2 \right|_0^{-x+1} dx$$

$$= \int_0^1 \left[-x^2 + x - (-x+1)^2 \right] dx = \int_0^1 (-2x^2+3x-1)\,dx$$

$$= \left[\frac{-2}{3}x^3 + \frac{3x^2}{2} - x \right|_0^1 = \frac{-2}{3} + \frac{3}{2} - 1 = \frac{-1}{6}$$

$$I = I_1 + I_2 \quad \#$$

III - $J = \int_{-1}^{1} \int_0^{\pi/2} \left(\int_0^1 yz \sin(xy)\,dx \right) dy\,dz$

$$J = \int_{-1}^{1} \int_0^{\pi/2} \left[-z \cos xy \right|_0^1 dy\,dz$$

$$= \int_{-1}^{1} \left(\int_0^{\pi/2} (-z\cos y + z)\,dy \right) dz$$

(3)

$$= \int_{-1}^{1} \left[-z \sin y + zy \right|_0^{\pi/2} dz$$

$$= \int_{-1}^{1} \left(-z + z.\frac{\pi}{2} \right) dz$$

$$= \left[\frac{-z^2}{2} + \frac{\pi}{4} z^2 \right|_{-1}^{1}$$

$$= \left(\frac{-1}{2} + \frac{\pi}{4} \right) - \left(\frac{-1}{2} + \frac{\pi}{4} \right)$$

$$= \cancel{\frac{-1}{2}} + \cancel{\pi/4} + \cancel{1/2} - \cancel{\pi/4}$$

$$= \boxed{0}. \quad \#$$

End