Complex Analysis Exam

Exercise 1: 2pts + 2pts

- Find all solutions to the equation $\sinh z = i$.
- Determine the conditions on the real constants a, b, c, and d such that the function

$$f(x) = ax + by + i(cx + dy)$$

is holomorphic.

Exercise 2: 2pts + 1.5pts + 1.5pts + 1.5pts + 1.5pts
Let C denote the unit circle oriented positively. Compute

$$\int_C \frac{(z + \frac{1}{z})^{2n}}{z^2}\, dz.$$

Deduce the values of the following integrals:

$$\int_{-\pi}^{\pi} \sin^{2n}(t)\, dt, \quad \int_{-\pi}^{\pi} \cos^{2n}(t)\, dt, \quad \int_{-\pi}^{\pi} \sin^{2n+1}(t)\, dt, \quad \int_{-\pi}^{\pi} \cos^{2n+1}(t)\, dt.$$

Exercise 3: 2pts
Let f be an entire function such that there exist three real numbers a, b, and c (not all zero) satisfying

$$a\,\Re(f(z)) + b\,\Im(f(z)) \leq c, \quad \forall z \in \mathbb{C}.$$

Show that f is constant. (Hint: consider the function $e^{\alpha f(z)}$ with $\alpha = a + ib$.)

Exercise 4: 3pts + 3pts
Let $0 < r \neq \pi/6$. Let C be the circle parametrized by $z(t) = re^{it}$ for $0 \leq t \leq 2\pi$. Compute

$$\int_C \frac{\cos^6(z)}{(z - \frac{\pi}{6})(z - \frac{\pi}{3})^2}\, dz.$$

Evaluate the integral using residue calculus:

$$\int_0^{2\pi} \frac{\cos t}{2 + \cos t}\, dt.$$