

# Complex Analysis - Solutions

## Exercise 1

1. Find all solutions to the equation  $\sinh z = i$ .
2. Determine the conditions on the real constants  $a$ ,  $b$ ,  $c$ , and  $d$  such that the function

$$f(x, y) = ax + by + i(cx + dy)$$

is holomorphic.

### Solution:

1. We want to solve:

$$\sinh z = i.$$

Recall that  $\sinh z = \frac{e^z - e^{-z}}{2}$ . Setting this equal to  $i$ , we get:

$$\frac{e^z - e^{-z}}{2} = i \quad \Rightarrow \quad e^z - e^{-z} = 2i.$$

Multiply both sides by  $e^z$ :

$$e^{2z} - 2ie^z - 1 = 0.$$

Let  $w = e^z$ , so the equation becomes:

$$w^2 - 2iw - 1 = 0.$$

Solving this quadratic equation:

$$w = \frac{2i \pm \sqrt{(2i)^2 + 4}}{2} = \frac{2i \pm \sqrt{-4 + 4}}{2} = \frac{2i \pm 0}{2} = i.$$

Thus,  $e^z = i$ , and therefore:

$$z = \ln i = i \left( \frac{\pi}{2} + 2k\pi \right), \quad k \in \mathbb{Z}.$$

So, the solutions are:

$$z = i \left( \frac{\pi}{2} + 2k\pi \right), \quad k \in \mathbb{Z}.$$

2. Let  $f(x, y) = u(x, y) + iv(x, y)$ , where:

$$u(x, y) = ax + by, \quad v(x, y) = cx + dy.$$

To ensure  $f$  is holomorphic, the Cauchy-Riemann equations must be satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Compute the partial derivatives:

$$u_x = a, \quad u_y = b, \quad v_x = c, \quad v_y = d.$$

The Cauchy-Riemann equations become:

$$a = d \quad \text{and} \quad b = -c.$$

So the necessary and sufficient conditions are:

$$a = d \quad \text{and} \quad c = -b.$$

Under these conditions, the function becomes:

$$f(z) = (a + ic)(x + iy) = (a + ic)z.$$

Hence,  $f$  is holomorphic.

## Exercise 2

Let  $C$  denote the unit circle oriented positively. Compute

$$\int_C \frac{(z + \frac{1}{z})^{2n}}{z^2} dz.$$

Deduce the values of the following integrals:

$$\int_{-\pi}^{\pi} \sin^{2n}(t) dt, \quad \int_{-\pi}^{\pi} \cos^{2n}(t) dt, \quad \int_{-\pi}^{\pi} \sin^{2n+1}(t) dt, \quad \int_{-\pi}^{\pi} \cos^{2n+1}(t) dt.$$

**Solution:** By the binomial theorem and Cauchy's theorem,

$$\int_C \frac{(z + \frac{1}{z})^{2n}}{z^2} dz = \sum_{k=0}^{2n} \binom{2n}{k} \int_C z^{2n-2k-1} dz = \binom{2n}{n} 2\pi i.$$

Parametrizing,

$$\int_{-\pi}^{\pi} 2^{2n} \cos^{2n}(t) i dt = \binom{2n}{n} 2\pi i$$

and hence

$$\int_{-\pi}^{\pi} \cos^{2n}(t) dt = \int_{-\pi}^{\pi} \sin^{2n}(t) dt = \frac{\binom{2n}{n} 2\pi}{2^{2n}}.$$

Since

$$\sin(t) = \cos\left(t - \frac{\pi}{2}\right),$$

and since  $\sin^{2n+1}(t)$  is an odd function,

$$\int_{-\pi}^{\pi} \cos^{2n+1}(t) dt = \int_{-\pi}^{\pi} \sin^{2n+1}(t) dt = 0.$$

## Exercise 3

Let  $f$  be an entire function such that there exist three real numbers  $a$ ,  $b$ , and  $c$  (not all zero) satisfying

$$a\Re(f(z)) + b\Im(f(z)) \leq c, \quad \forall z \in \mathbb{C}.$$

Show that  $f$  is constant. (Hint: consider the function  $e^{\alpha f(z)}$  with  $\alpha = a + ib$ .)

**Solution:** Let  $\beta = a - ib$  and consider the entire function  $e^{\beta f(z)}$ . Its modulus is bounded by  $e^c$ , hence by Liouville's theorem, the function must be constant. Therefore,  $f$  is constant.

## Exercise 4

**Solution:**

**Problem 1:** Compute  $\oint_C \frac{\cos^6(z)}{(z - \frac{\pi}{6})(z - \frac{\pi}{3})^2} dz$

The integrand is  $f(z) = \frac{\cos^6(z)}{(z - \frac{\pi}{6})(z - \frac{\pi}{3})^2}$ . The singularities of  $f(z)$  are simple pole at  $z_1 = \frac{\pi}{6}$  and a pole of order 2 at  $z_2 = \frac{\pi}{3}$ . The contour  $C$  is a circle centered at the origin with radius  $r$ , given by  $z(t) = re^{it}$  for  $0 \leq t \leq 2\pi$ .

We need to consider the location of these singularities relative to the circle  $C$ .

**Case 1:**  $r < \frac{\pi}{6}$

In this case, neither  $z_1 = \frac{\pi}{6}$  nor  $z_2 = \frac{\pi}{3}$  are inside the circle  $C$ . Since  $f(z)$  is analytic inside and on  $C$ , by Cauchy's Integral Theorem:

$$\oint_C \frac{\cos^6(z)}{(z - \frac{\pi}{6})(z - \frac{\pi}{3})^2} dz = 0$$

**Case 2:**  $\frac{\pi}{6} < r < \frac{\pi}{3}$

In this case, only  $z_1 = \frac{\pi}{6}$  is inside the circle  $C$ . We calculate the residue at  $z_1 = \frac{\pi}{6}$ :

$$\begin{aligned} \operatorname{Res}\left(f(z), \frac{\pi}{6}\right) &= \lim_{z \rightarrow \frac{\pi}{6}} \left(z - \frac{\pi}{6}\right) \frac{\cos^6(z)}{\left(z - \frac{\pi}{6}\right)\left(z - \frac{\pi}{3}\right)^2} \\ &= \frac{\cos^6\left(\frac{\pi}{6}\right)}{\left(\frac{\pi}{6} - \frac{\pi}{3}\right)^2} = \frac{\left(\frac{\sqrt{3}}{2}\right)^6}{\left(-\frac{\pi}{6}\right)^2} = \frac{\frac{27}{64}}{\frac{\pi^2}{36}} = \frac{27}{64} \cdot \frac{36}{\pi^2} = \frac{243}{16\pi^2} \end{aligned}$$

By the Residue Theorem:

$$\oint_C \frac{\cos^6(z)}{\left(z - \frac{\pi}{6}\right)\left(z - \frac{\pi}{3}\right)^2} dz = 2\pi i \cdot \operatorname{Res}\left(f(z), \frac{\pi}{6}\right) = 2\pi i \cdot \frac{243}{16\pi^2} = \frac{243i}{8\pi}$$

**Case 3:**  $r > \frac{\pi}{3}$

In this case, both  $z_1 = \frac{\pi}{6}$  and  $z_2 = \frac{\pi}{3}$  are inside the circle  $C$ . The residue at  $z_1 = \frac{\pi}{6}$  is  $\frac{243}{16\pi^2}$ . Now, we calculate the residue at  $z_2 = \frac{\pi}{3}$  (pole of order 2):

$$\begin{aligned} \operatorname{Res}\left(f(z), \frac{\pi}{3}\right) &= \lim_{z \rightarrow \frac{\pi}{3}} \frac{d}{dz} \left[ \left(z - \frac{\pi}{3}\right)^2 \frac{\cos^6(z)}{\left(z - \frac{\pi}{6}\right)\left(z - \frac{\pi}{3}\right)^2} \right] \\ &= \lim_{z \rightarrow \frac{\pi}{3}} \frac{d}{dz} \left[ \frac{\cos^6(z)}{z - \frac{\pi}{6}} \right] \end{aligned}$$

Let  $g(z) = \frac{\cos^6(z)}{z - \frac{\pi}{6}}$ . Then  $g'(z) = \frac{6 \cos^5(z)(-\sin z)(z - \frac{\pi}{6}) - \cos^6(z)}{\left(z - \frac{\pi}{6}\right)^2}$ .

$$\begin{aligned} \operatorname{Res}\left(f(z), \frac{\pi}{3}\right) &= \frac{6 \cos^5\left(\frac{\pi}{3}\right) \left(-\sin \frac{\pi}{3}\right) \left(\frac{\pi}{3} - \frac{\pi}{6}\right) - \cos^6\left(\frac{\pi}{3}\right)}{\left(\frac{\pi}{3} - \frac{\pi}{6}\right)^2} \\ &= \frac{i}{8\pi} \left[ 234 - 9\sqrt{3}\pi \right] = \frac{9i}{8\pi} (26 - \sqrt{3}\pi) \end{aligned}$$

**Problem 2: Evaluate the integral using residue calculus:**  $\int_0^{2\pi} \frac{\cos t}{2 + \cos t} dt$

We use the substitution  $z = e^{it}$ . Then  $dt = \frac{dz}{iz}$ . Also,  $\cos t = \frac{e^{it} + e^{-it}}{2} = \frac{z + z^{-1}}{2}$ . The contour of integration is the unit circle  $C: |z| = 1$ .

$$\int_0^{2\pi} \frac{\cos t}{2 + \cos t} dt = \oint_C \frac{\frac{z+z^{-1}}{2}}{2 + \frac{z+z^{-1}}{2}} \frac{dz}{iz} = \frac{1}{i} \oint_C \frac{z^2 + 1}{z(z^2 + 4z + 1)} dz$$

Let  $f(z) = \frac{z^2 + 1}{z(z^2 + 4z + 1)}$ . The singularities are the roots of the denominator:

- $z = 0$
- $z^2 + 4z + 1 = 0 \implies z = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm \sqrt{12}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$

So the poles are  $z_1 = 0$ ,  $z_2 = -2 + \sqrt{3}$ , and  $z_3 = -2 - \sqrt{3}$ . We check which poles are inside the unit circle  $|z| = 1$ :

- $|z_1| = |0| = 0 < 1$  (inside)
- $|z_2| = |-2 + \sqrt{3}| \approx |-2 + 1.732| = |-0.268| < 1$  (inside)
- $|z_3| = |-2 - \sqrt{3}| \approx |-2 - 1.732| = |-3.732| > 1$  (outside)

We calculate the residues at  $z_1 = 0$  and  $z_2 = -2 + \sqrt{3}$ . Both are simple poles.

**Residue at  $z_1 = 0$ :**

$$\text{Res}(f(z), 0) = \lim_{z \rightarrow 0} z \cdot \frac{z^2 + 1}{z(z^2 + 4z + 1)} = \frac{0^2 + 1}{0^2 + 4(0) + 1} = \frac{1}{1} = 1$$

**Residue at  $z_2 = -2 + \sqrt{3}$ :**

Let  $z_0 = -2 + \sqrt{3}$ .

$$\begin{aligned} \text{Res}(f(z), z_0) &= \lim_{z \rightarrow z_0} (z - z_0) \frac{z^2 + 1}{z(z - z_0)(z - (-2 - \sqrt{3}))} \\ &= \frac{(-2 + \sqrt{3})^2 + 1}{(-2 + \sqrt{3})((-2 + \sqrt{3}) + 2 + \sqrt{3})} = -\frac{2\sqrt{3}}{3} \end{aligned}$$

By the Residue Theorem:

$$\begin{aligned} \oint_C f(z) dz &= 2\pi i \left[ \text{Res}(f(z), 0) + \text{Res}(f(z), -2 + \sqrt{3}) \right] \\ &= 2\pi i \left[ 1 - \frac{2\sqrt{3}}{3} \right] \end{aligned}$$

Finally, substitute this back into the original integral:

$$\int_0^{2\pi} \frac{\cos t}{2 + \cos t} dt = \frac{1}{i} \oint_C f(z) dz = 2\pi \left( \frac{3 - 2\sqrt{3}}{3} \right)$$