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ABBES LAGHROUR- KHENCHELA UNIVERSITY

Faculty of Sciences and Technology

Department of Mathematics and Computer Science

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Some existence results via the measure of noncompactness and applications

Presented by: **Laiche Amel**
Boudjemaa Hadjer

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Dr. N. Hamdi **President**
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2022-2023

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N° de série :……

Mémoire de fin d’études

Pour l’obtention du diplôme de **Master**

Filière: **Mathématiques**
Spécialité: **Mathématiques Appliquées**

Intitulé

Quelques résultats d’existence via la mesure de non-compacité et applications.

Réalisé par : **Laiche Amel**
Boudjemaa Hadjer

Dirigé par : **Dr. H. Ramoul**

Membres de jury :

Dr. N. Hamdi	**Président**
DR. H. Ramoul	**Encadrant**
Dr. O. Zahi	**Examinateur**

2022-2023

Hadjer’s Dedication

To the one who taught me to stand up and how to start the journey of a thousand (1000) miles with a step, to my right hand, to the one who taught me to climb with his eyes watching me....... My dear father to the one to whom the Almighty has placed paradise under her feet, to the generous chest and affectionate heart to the one who was by my side in all the stages that went by, she relished suffering for me and was a candle Burn to light up my Derby.... My beloved mom
To those who in my days sowed joy and lit the flames of fun, to those whose presence is a flavor and their absence is lost to those who in my self had an echo...... My sisters
To the lights that illuminate my darkness when the days and circumstances turn them off, to the clouds that mislead and Water me without wanting to return the favor, to the hands that help me when I stumble and push me to resist
Everything that calls for a fall......... My brothers
"To those I rely on in every small and Big my esteemed brother...... "Nassim
to my friends and acquaintances whom I respect and respect to my professors at the Faculty of mathematics and automated information
I dedicate my research to you

Amel’s Dedication

Sacrificed everything for my future, my dear mother Meriem. To the one who worked hard for me and spent his life for us, my dear father Mohamed, To the person who taught me that patience is the Horia key to success, my dear sister
To my dear sisters: Zahra, Hadiya, Maisam. To my dear brothers: Salim, Kamal, Osama. To the Abrar. To my respected teacher ,Watin ,Iyad, Meral ,Tinhinan ,children of my siblings: Said, Muath Hisham Ramoul. To my colleague who helped me complete the dissertation: Hajar. To all my friends: Computer and Mathematics the Rokia, Manal, Rania, Marwa, Zainab, and Shaimaa. To my friends in Science

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Abstract

In this dissertation, we consider the Proinov fixed point of Darbo type to prove the existence of solutions of integral equations. Roughly speaking, we use measure of noncompactness argument in the framework of bounded and continuous functions on $\mathbb{R}^+$ to obtain sufficient assumptions in order to prove the existence of solutions for some nonlinear integral equations.

Introduction

This dissertation is devoted to establish the existence of solutions for some nonlinear volterra integral equations. Through the measure of noncompactness, we define a new notion denoted : Proinov fixed point of Darbo-type which led us to improve and generalize some results existing in the literature.

The manuscript is organized as follows.

The first chapter is devoted to collect some many necessary notions and tools concerning the measure of noncompactness and fixed point theory.

Chapter 2 is devoted to state and prove some results about measure of noncompactness using the notion of Proinov fixed point of Darbo-type.

In chapter 3, we utilize the results obtained in the previous chapter to establish sufficient conditions to solve some integral equations. An example is given at end of this chapter to illustrate the validity of our results.

Chapter 1

Contents

Preliminaries

Abstract

In this chapter, we recall the main properties regarding the measure of noncompactness needed for establishing many fixed point results in the next chapters.

1.1 Metric space

We first recall the definition of a metric space, as follows.

Definition 1.1.1. Let X be a nonempty set. A mapping $d : X \times X \to [0,\infty)$ is said to be a metric if, for all $x, y, z \in X$, the following conditions hold:
(b_1) $d\left(x,y\right) = 0$ if and only if $x = y$;
(b_2) $d\left(x,y\right) = d\left(y,x\right)$;
(b_3) $d\left(x,z\right) \leq \left[d\left(x,y\right) + d\left(y,z\right)\right]$.
The pair (X,d) is called a metric space.

1.2 Background on measure of noncompactness

Henceforth, X is a Banach space and M is a nonempty subset of X. We denote by $\overline{Y}$ and $co\left(M\right)$ the closure of Y and the convex hull of M, respectively. In addition, we denote by $B\left(X\right)$ and $RC\left(X\right)$ the family of all nonempty bounded subsets of X and all relatively compact subsets of X, respectively.

Definition 1.2.1. (See [4]) A mapping $\mu : B\left(X\right) \longrightarrow [0,\infty)$ is said to be a measure of noncompactness in X if it satisfies the following conditions:

1) The family $\ker\mu = \{A \in B\left(X\right) : \mu(A) = 0\}$ is nonempty and $\ker\mu \subseteq RC\left(X\right)$.

2) $Z \subseteq Y \Longrightarrow \mu(Z) \leqslant \mu(Y)$.

3) $\mu(\overline{Z}) = \mu(Z)$.

4) $\mu(co\left(A\right)) = \mu(A)$.

5) $\mu(\lambda Z + (1-\lambda)Y) \leqslant \lambda\mu(Z) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.

6) If $\{M_n\}$ is a sequence of closed sets from $B\left(X\right)$ such that $M_{n+1} \subseteq M_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\longrightarrow\infty} \mu(M_n) = 0$, then $M_\infty := \bigcap\limits_{n=1}^{\infty} X_n$ is nonempty.

Remark 1.2.1. Let us observe that $M_\infty := \bigcap\limits_{n=1}^{\infty} X_n$ belongs to $\ker\mu$ as such $\mu\left(M_\infty\right) \leq \mu\left(M_n\right)$ for all $n \in \mathbb{N}$ and we thus obtain $\mu\left(M_\infty\right) = 0$. Therefore, it follows that $M_\infty \in \ker\mu$.

Theorem of Schauder is a fundamental result needed for proving many results in fixed point theory. The aforementioned theorem is given below.

Theorem 1.2.1 (Schauder)**.** *Let M be a nonempty bounded closed and convex subset of a Banach space X. If $T : M \to M$ is a self-map on M. Then T has a fixed point in M.*

1.3 Some fixed point results via measure of noncompactness

1.3.1 Darbo fixed point

Definition 1.3.1. Let M be a nonempty, bounded, closed, and convex subset of a Banach space X. A self-mapping $T : M \to M$ is said to be a μ-contraction if there exists some constant $k \in (0,1)$ such that

$$\mu(TX) \leq k\mu(X), \tag{1.1}$$

Darbo's theorem is stated as follows (see [4]).

Theorem 1.3.1. *Let M be a nonempty, bounded, closed, and convex subset of a Banach space X and let $T : M \to M$ be a continuous operator. If T is a μ-contraction, then T has at least one fixed point in M.*

1.3.2 F-contractions

In 2012, Wardowski [11] defined the so-called F-contraction as follows:

Definition 1.3.2. Let (X,d) be a metric space. A mapping $T : X \to X$ is called an F-contraction if there exist $F \in \mathscr{F}$ and $\tau > 0$ such that for all $x, y \in X$,

$$d\left(Tx,Ty\right) > 0 \Rightarrow \tau + F\left(d\left(Tx,Ty\right)\right) \leq F\left(d\left(x,y\right)\right), \tag{1.2}$$

where $\mathscr{F}$ is the family of all functions $F : (0,\infty) \to \mathbb{R}$ satisfying the following conditions:
(F_1) F is strictly increasing;
(F_2) For each sequence $\{\alpha_n\}$ of positive numbers, the following holds:

$$\lim_{n\to\infty} \alpha_n = 0 \textit{ iff } \lim_{n\to\infty} F\left(\alpha_n\right) = -\infty;$$

(F_3) There exists $k \in (0,1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F\left(\alpha\right) = 0$.

Example 1.3.1. (See [11]) Let $\alpha \in (0,\infty)$. The following functions $F_1\left(\alpha\right) = \ln\alpha$, $F_2\left(\alpha\right) = \ln\alpha + \alpha$, $F_3\left(\alpha\right) = \dfrac{-1}{\sqrt{\alpha}}$ and $F_4\left(\alpha\right) = \ln\left(\alpha^2 + \alpha\right)$ belong to the family $\mathscr{F}$.

Remark 1.3.1. Taking $F\left(\alpha\right) = \ln\alpha$ in (1.2), one can get a Banach contraction (see [11, example 2.1]).

Wardowski established the following result.

Theorem 1.3.2. *([11, Theorem 2.1]) Let (X,d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\left\{T^n x_0\right\}$ converges to x^*.*

In 2018, Wardowski [12] fine-tuned the class of contractions $\mathscr{F}$ by introducing the concept of $\left(\chi,F\right)$-contraction on a metric space. The author substitute a function χ for the positive constant τ and relaxed some assumptions on the mapping F.

Definition 1.3.3. (See [12]) Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a (χ, F)-contraction if there exist two functions $F : (0,\infty) \to \mathbb{R}$ and $\chi : (0,\infty) \to (0,\infty)$ satisfying the following conditions:

1. F satisfies (F_1);
2. (F_2'): $\lim_{t\to 0^+} F(t) = -\infty$;
3. (H_0): $\lim\inf_{t\to \varepsilon^+} \chi(t) > 0$ for all $\varepsilon \geq 0$;
4. $\chi(d(x,y)) + F(d(Tx,Ty)) \leq F(d(x,y))$ for all $x, y \in X$ with $Tx \neq Ty$.

Additionally, Wardowski [12] proved the following theorem.

Theorem 1.3.3. *([12, Theorem 2.1]) On a complete metric space (X,d), every (χ, F)-contraction mapping has a unique fixed point.*

Henceforth, we denote by $\mathscr{L}$ the family of all functions $\chi : (0,\infty) \to (0,\infty)$ which satisfy the following condition:

$$\liminf_{t\to\eta^+} \chi(t) > 0 \quad \text{for all } \eta > 0. \tag{H}$$

Example 1.3.2. (See [6, Example 3.3] and [9, Example 2.2])
(a) Let $\chi > 0$ be a fixed real number and $\chi_1(t) = \chi$ for all $t \in (0,\infty)$. Then $\chi_1 \in \mathscr{L}$.
(b) Let $\chi_2(t) = \delta t$ for all $t \in (0,\infty)$, where $\delta > 0$. Then $\chi_2 \in \mathscr{L}$.
(c) Let $\chi_3(t) = e^t$ for all $t \in (0,\infty)$. Then $\chi_3 \in \mathscr{L}$.

Remark 1.3.2. It is easy to see that condition (H) is slightly weaker than condition (H_0) given in Definition 1.3.3. For instance, we can observe that χ_2 does not satisfy condition (H_0) since $\liminf_{t\to 0^+} \chi_2(t) = 0$.

Definition 1.3.4. (See [10]) Let M be a nonempty, bounded, closed, and convex subset of a Banach space X. A continuous self-operator T on M is called an F-contraction of Darbo-type if there exists F and τ that satisfies (F_2) and (H_0), respectively and such that the following holds

$$\tau(\mu(X)) + F(\mu(TX)) \leqslant F(\mu(X)) \quad \text{for any} \quad X \subset M \quad \text{with} \quad \mu(X), \mu(TX) > 0. \tag{1.3}$$

Consistent with [10, Remark 7.3], we have the following theorem:

Theorem 1.3.4. *(See [10]) Let M be a nonempty, bounded, closed, and convex subset of a Banach space X. If T is an F-contraction of Darbo-type, then T has a fixed point in M.*

Remark 1.3.3. By inspecting the proof of the above theorem, we have observed that hypothesis (H_0) cannot be assumed because the sequence $\{\mu(M_n)\}$ is only decreasing and therefore we do not necessarily have

$$\lim_{n\to\infty} \mu(M_n) = r^+.$$

Actually, the suitable hypothesis (stronger than (H_0)) to make Theorem 1.3.4 valid is given by

$$\liminf_{t\to r} \tau(t) > 0 \quad \text{for every } r > 0. \tag{H_1}$$

1.3.3 θ-contractions

Let Θ be the family of all functions $\theta:(0,\infty)\to(1,\infty)$ satisfying the the following condition:

(θ_2): For each sequence $\{t_n\}\subset(0,\infty)$,

$$\lim_{n\to\infty}\theta(t_n)=1\Leftrightarrow\lim_{n\to\infty}t_n=0.$$

The authors in [8] obtained the result below.

Theorem 1.3.5. *Let M be a nonempty, bounded, closed and convex subset of a Banach space X and let $T:M\to M$ be a continuous operator such that for every subset Z of M*

$$\theta\left(\mu(TZ)\right)\leq\left[\theta\left(\mu(Z)\right)\right]^k,\quad \textit{with}\quad \mu(TZ),\mu(TZ)>0,$$

where $\theta\in\Theta$ and $k\in(0,1)$. Then T has at least one fixed point.

The following lemmas are needed in the sequel (see [7])

Lemme 1.3.1. *Let $\psi:(0,\infty)\to\mathbb{R}$. Then the following conditions are equivalent:*

1) $\inf_{t>r}\psi(t)>-\infty$ *for any* $r>0$;
2) $\lim_{n\to\infty}\psi(t_n)=-\infty$ *implies* $\lim_{n\to\infty}t_n=0$

Lemme 1.3.2. *Let $\varphi:(0,\infty)\to(0,\infty)$. Then the following two conditions are equivalent:*

1) *If* $\lim_{n\to\infty}\varphi(t_n)=0$ *for a bounded sequence* $\{t_n\}$, *then* $\lim t_n=0$.
2) $\liminf_{t\to r}\varphi(t)>0$ *for every* $r>0$.

Chapter 2

Contents

Proinov contraction of Darbo-type

Abstract

In this chapter, we generalize Darbo fixed point using the Proinov contraction and the measure of noncompactness. Our results improve some results existing in the literature.

2.1 Introduction and settings

In this chapter, we introduce the notion of Proinov contraction of Darbo-type through the measure of noncompactness μ and we prove the existence of the fixed point in some bounded, closed and convex subset of a Banach space X.

Definition 2.1.1. Let M be a nonempty, bounded, closed and convex subset of a Banach space X. A continuous self- operator $T : M \to M$ is said to be a *Proinov contraction of Darbo-type* if there exist two mappings $\varphi, \psi : (0, \infty) \to \mathbb{R}$ such that for every nonempty subset Z of M, the following holds

$$\psi\left(\mu(TZ)\right) \leq \varphi\left(\mu(Z)\right) \quad \text{with} \quad \mu(Z), \mu(TZ) > 0. \tag{2.1}$$

2.2 Main results

Theorem 2.2.1. *Let M be a nonempty, bounded, closed and convex subset of a Banach space X. Assume that $T : M \to M$ be a continuous Proinov contraction of Darbo-type and at least one of the three following conditions holds:*

(i)

$$\limsup_{t\to r} \varphi(t) < \liminf_{t\to r} \psi(t) \quad \textit{for every} \quad r > 0;$$

(ii) $\varphi(t) < \psi(t)$ *for any* $t > 0$, ψ *nondecreasing and*

$$\limsup_{t\to r^+} \varphi(t) < \psi\left(r^+\right) \quad \textit{for every} \quad r > 0;$$

(iii) $\varphi(t) < \psi(t)$ *for any* $t > 0$ *and the following conditions are satisfied:*

(a)

$$\inf_{t>r} \psi(t) > -\infty \quad \textit{for every} \quad r > 0,$$

(b) *If* $\left\{\psi(t_n)\right\}$ *and* $\left\{\varphi(t_n)\right\}$ *are two convergent sequences with the same limit for a bounded sequence* $\{t_n\}$*, then* $t_n \to 0$ *as* $n \to \infty$.

Then T has at least one fixed point in M.

Proof. Let us construct a sequence $\{M_n\}$ such that $M_0 = M$ and

$$M_n = \overline{co(TM_{n-1})}, \quad \text{for } n \geq 1.$$

We are going to prove that

$$M_{n+1} \subseteq M_n \quad \text{and } TM_n \subseteq M_n \quad \text{for any } n \in \mathbb{N}.$$

For the first inclusion, we use the mathematical induction. From the fact that $M_0 = M$ and M is convex and closed with $T : M \to M$, we obtain

$$M_1 = \overline{co(TM_0)} = \overline{co(TM)} \subseteq M = M_0.$$

Now suppose that $M_n \subseteq M_{n-1}$ for $n \geq 1$. It follows that

$$M_{n+1} = \overline{co\,(TM_n)} \subseteq \overline{co\,(TM_{n-1})} = M_n$$

and we are done.

From the first inclusion $M_{n+1} \subseteq M_n$, we immediately get

$$TM_n \subseteq \overline{co\,(TM_n)} = M_{n+1} \subseteq M_n.$$

Hence the second inclusion is proved.

Next, we distinguish two cases:

Case 1. If there exists $n_0 \in \mathbb{N}$ such that $\mu\left(M_{n_0}\right) = 0$. Thus, M_{n_0} is a compact set in X. Since $TM_{n_0} \subseteq M_{n_0}$, the theorem of Schauder yields that T has a fixed point in $M_{n_0} \subseteq M$.

Case 2. If $\mu\,(M_n) > 0$ for any $n \in \mathbb{N}$. Through the contractive inequality (2.1) and the properties of the measure of noncompactness, we obtain

$$\begin{aligned} \psi\left(\mu\,(M_{n+1})\right) &= \psi\left(\mu\left(\overline{co\,(TM_n)}\right)\right) \\ &= \psi\left(\mu\,(TM_n)\right) \leq \varphi\left(\mu\,(M_n)\right), \quad \text{for any } n \in \mathbb{N}. \end{aligned} \tag{2.2}$$

On the other hand, we have $0 \leq \mu\,(M_{n+1}) \leq \mu\,(M_n)$, which implies that the sequence $\left\{\mu\,(M_n)\right\}$ is a decreasing sequence and bounded below. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \mu\,(M_n) = r. \tag{2.3}$$

Now let us prove that $r = 0$. Assume on the contrary, i.e., $r > 0$.

Suppose that condition (i) holds. From (2.2) and (2.3), we get

$$\begin{aligned} \liminf_{t\to r} \psi\,(t) &\leq \liminf_{n\to\infty} \psi\left(\mu\,(M_n)\right) \\ &\leq \limsup_{n\to\infty} \varphi\left(\mu\,(M_n)\right) \\ &\leq \limsup_{t\to r} \varphi\,(t), \end{aligned}$$

a contradiction. Hence $r = 0$ and $\lim\limits_{n\to\infty} \mu\,(M_n) = 0$.

If condition (ii) holds. In view of the fact that $\varphi\,(t) < \psi\,(t)$ for any $t > 0$, the monotonicity of ψ and (2.2), we obtain that $\left\{\mu\,(M_n)\right\}$ is a strictly decreasing sequence and boubded below. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \mu\,(M_n) = r^+.$$

Again from (2.3), we have

$$\begin{aligned} \psi\left(r^+\right) &= \lim_{n\to\infty} \psi\left(\mu\,(M_n)\right) \\ &\leq \limsup_{n\to\infty} \varphi\left(\mu\,(M_n)\right) \\ &\leq \limsup_{t\to r^+} \varphi\,(t), \end{aligned}$$

which is a contradiction. Therefore $\lim\limits_{n\to\infty} \mu\,(M_n) = 0$.

Now assume that condition (iii) holds. In this case, we consider the following subcases.

Subcase 1. If $\{\psi(\alpha_n)\}$ is not bounded below, where $\alpha_n := \mu(M_n)$. It follows that

$$\lim_{n\to\infty} \psi(\alpha_n) = -\infty.$$

Due to Lemma 1.3.1, we obtain $\lim\limits_{n\to\infty} \alpha_n = \lim\limits_{n\to\infty} \mu(M_n) = 0$.

Subscase 2. If $\{\psi(\alpha_n)\}$ is bounded below. Utilizing the fact that $\varphi(t) < \psi(t)$ for any $t > 0$, we get

$$\psi(\alpha_{n+1}) \leq \varphi(\alpha_n) < \psi(\alpha_n).$$

This implies that $\{\psi(\alpha_n)\}$ is strictly decreasing. Then, $\{\psi(\alpha_n)\}$ is a convergent sequence and therefore $\{\varphi(\alpha_n)\}$ is also a convergent sequence with the same limit. Consequently, we obtain

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \mu(M_n) = 0.$$

In conclusion, in each case, we have obtained that $\lim\limits_{n\to\infty} \mu(M_n) = 0$. It follows from property (6) in Definition 1.2.1, that

$$M_\infty = \bigcap_{n=1}^{\infty} M_n$$

is nonempty and compact subset. Furthermore, we have

$$\begin{aligned} TM_\infty &= T\left(\bigcap_{n=1}^{\infty} M_n\right) \subseteq \bigcap_{n=1}^{\infty} TM_n \\ &\subseteq \bigcap_{n=1}^{\infty} M_n = M_\infty. \end{aligned}$$

Finally, by virtue of Schauder's fixed point theorem, we conclude that the operator $T : M_\infty \to M_\infty$ has a fixed point in $M_\infty \subset M$. This completes the proof of the theorem. □

Remark 2.2.1. Taking $\psi = Id$ and $\varphi(t) = kt$, with $0 \leq k < 1$, we recover Theorem 1.3.1.

Setting $\psi(t) = t$ in Theorem 2.2.1, we obtain the following result (Boyd-Wong's type):

Corollary 2.2.1. *Let M be a nonempty, bounded, closed and convex subset of a Banach space X. Assume that $T : M \to M$ be a continuous operator satisfying*

$$\mu(TZ) \leq \varphi\big(\mu(Z)\big) \quad \textit{with} \quad \mu(Z), \mu(TZ) > 0.$$

Moreover, we assume that the following assumption holds:

(i) *$\varphi(t) < t$ for any $t > 0$ and*

$$\limsup_{t\to r^+} \varphi(t) < r \quad \textit{for every} \quad r > 0.$$

Then T has at least one fixed point in M.

Corollary 2.2.2. *Let* M *be a nonempty, bounded, closed and convex subset of a Banach space* X*. Assume that* $T : M \to M$ *be an operator and* $\varphi : (0,\infty) \to \mathbb{R}$ *a nondecreasing and upper semicontinuous from the right with* $\varphi(t) < t$ *for any* $t > 0$ *such that*

$$\|Tx - Ty\| \leq \varphi(\|x-y\|) \quad \textit{for } Tx \neq Ty \textit{ and } x, y \in X.$$

Then T *has at least one fixed point in* M*.*

Proof. Let $\mu : B(X) \to [0,\infty)$ be a set quantity given by

$$\mu(Z) = diamZ = \sup\{\|x-y\|, x, y \in Z\}.$$

It easy to see that $diamZ$ (the diameter of X) is a measure on noncompactness in the space X following the sense of Definition . Taking into account the assumptions of the Corollary, we get

$$\sup_{x,y\in Z} \|Tx - Ty\| \leq \sup_{x,y\in Z} \varphi(\|x-y\|) \leq \varphi\left(\sup_{x,y\in Z} \|x-y\|\right),$$

which implies that

$$\mu(TZ) \leq \varphi(\mu(Z)), \quad \text{with } \mu(TZ) > 0.$$

In addition, φ is right continuous since φ is nondecreasing and upper semicontinuous from the right. Hence, we obtain

$$\limsup_{t\to r^+} \varphi(t) = \varphi(r) < r; \quad \text{for every } r > 0.$$

Therefore, all the hypotheses of Corollary 2.2.1 are satisfied and then T has at least one fixed point in M. ☐

Let Δ be the set of all functions $\beta : (0,\infty) \to (0,1)$ satisfying the following condition:

$$\limsup_{t\to r^+} \beta(t) < 1 \quad \text{for any} \quad r > 0,$$

Let $\beta \in \Delta$. Taking $\varphi(t) = \beta(t)\psi(t)$ in Theorem 2.2.1, we get the result below (Geraghty's-type).

Corollary 2.2.3. *Let* M *be a nonempty, bounded, closed and convex subset of a Banach space* X*. Assume that* $T : M \to M$ *be a continuous operator such that*

$$\psi(\mu(TZ)) \leq \beta(\mu(Z))\psi(\mu(Z)) \quad \textit{with} \quad \mu(Z), \mu(TZ) > 0.$$

Then T *has at least one fixed point in* M*.*

Let $\alpha : (0,\infty) \to (0,1)$ and $\gamma : (0,\infty) \to (0,\infty)$. Setting $\varphi(t) = \alpha(t)\gamma(t)$ in Theorem 2.2.1, we obtain the following result.

Corollary 2.2.4. *Let* M *be a nonempty, bounded, closed and convex subset of a Banach space* X*. Suppose that* $T : M \to M$ *be a continuous operator such that*

$$\psi(\mu(TZ)) \leq \alpha(\mu(Z))\gamma(\mu(Z)) \quad \textit{with} \quad \mu(Z), \mu(TZ) > 0,$$

where ψ *is nondecreasing,* γ *is a right continuous function with* $\gamma(t) < \psi(t)$*, for all* $t > 0$ *and*

$$\limsup_{t\to r^+} \alpha(t) < \frac{\psi(r^+)}{\gamma(r^+)} \quad \textit{for any} \quad r > 0. \tag{2.4}$$

Then T *has at least one fixed point in* M*.*

Remark 2.2.2. Corollary 2.2.3 is an improvement of [2, Corollary 3.10]. Indeed, the condition $\beta \in \Delta$ is replaced by the weaker condition (2.4) since $\frac{\psi(r^+)}{\gamma(r^+)} > 1$. Moreover, the continuity of γ is weaken to the continuity from the right.

Corollary 2.2.5. *Let M be a nonempty, bounded, closed and convex subset of a Banach space X. Assume that $T : M \to M$ be a continuous Proinov contraction of Darbo-type such that $\varphi(t) < \psi(t)$ for any $t > 0$. Furthermore, if at least one of the following conditions holds:*

(i) *ψ is lower semicontinuous and φ is upper semicontinuous;*

(ii) *ψ nondecreasing and φ is upper semicontinuous from the right.*

Then T has at least one fixed point in M.

Proof. Assume first that (i) holds. Then, we get

$$\limsup_{t\to r}\varphi(t) \le \varphi(r) < \psi(r) \le \liminf_{t\to r}\psi(t) \quad \text{for every } r > 0.$$

If (ii) holds. In this case, we have

$$\limsup_{t\to r^+}\varphi(t) \le \varphi(r) < \psi(r) \le \psi\left(r^+\right) \quad \text{for every } r > 0;$$

□

Setting $\varphi = \psi - \tau$, where $\tau : (0,\infty) \to (0,\infty)$ in Theorem 2.2.1, we obtain the following result

Corollary 2.2.6. *Let M be a nonempty, bounded, closed and convex subset of a Banach space X. Assume that $T : M \to M$ be a continuous mapping satisfying*

$$\psi\left(\mu(TZ)\right) \le \psi\left(\mu(Z)\right) - \tau\left(\mu(Z)\right) \quad \textit{with} \quad \mu(Z), \mu(TZ) > 0.$$

Moreover, we assume that at least one of the three following conditions holds:

(i)

$$\liminf_{t\to r}\tau(t) > \limsup_{t\to r}\psi(t) - \liminf_{t\to r}\psi(t) \quad \textit{for every} \quad r > 0;$$

(ii) *ψ nondecreasing and $\tau \in \mathscr{L}$;*

(iii) *The following conditions are satisfied:*

(a)

$$\inf_{t>r}\psi(t) > -\infty \quad \textit{for every} \quad r > 0,$$

(b) *If $\lim_{n\to\infty}\tau(t_n) = 0$ for a bounded sequence $\{t_n\}$, then $\lim t_n = 0$.*

Then T has at least one fixed point in M.

Remark 2.2.3. Combining Corollary 2.2.6 with Lemmas 1.3.1 and 1.3.2, assumption (iii) is replaced by the two following conditions:

(a') $\lim_{n\to\infty}\psi(t_n) = -\infty$ implies $\lim_{n\to\infty} t_n = 0$.

(b') τ satisfies (H_1).

Remark 2.2.4. It follows from the above remark that Corollary 2.2.6 is a generalization and an improvement of Theorem 1.3.4.

Corollary 2.2.7. *Let M be a nonempty, bounded, closed and convex subset of a Banach space X and let $T: M \to M$ be a continuous operator such that there exist $\theta: (0,\infty) \to (1,\infty)$ and for every subset Z of M*

$$\theta\left(\mu(TZ)\right) \le \left[\theta\left(\mu(Z)\right)\right]^{\chi(\mu(Z))} \quad \textit{with} \quad \mu(Z), \mu(TZ) > 0,$$

Moreover, we assume that the following conditions hold:

(a)

$$\inf_{t>r}\theta(t) > 1 \quad \textit{for every} \quad r > 0,$$

(b) *If* $\lim_{n\to\infty}\chi(t_n) = 1$ *for a bounded sequence* $\{t_n\}$, *then* $\lim t_n = 0$.

Then T has at least one fixed point in M.

Proof. The desired result follows straightforward from Corollary 2.2.6 with $\psi(t) = \ln\ln\theta(t)$ and $\tau(t) = -\ln\chi(t)$. ☐

Remark 2.2.5. Combining Remark 2.2.3 with Corollary 2.2.7, assumptions (a) and (b) are replaced by the two following conditions:

(a') $\lim_{n\to\infty}\theta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$.

(b') $\chi \in \Delta$.

Remark 2.2.6. It follows from the above remark that Corollary 2.2.7 is a generalization and an improvement of Theorem 1.3.5.

Chapter 3

Contents

An application to a nonlinear integral equation

Abstract

In this chapter, we prove the existence of various integral equations using the results obtained in the previous chapter.

3.1 Measure of noncompactness in $BC(\mathbb{R}^+)$

In this section, we focus on the existence of solution for the the integral equation of Volterra type:

$$u(t) = f(t, u(t)) + g(t, u(t)) \int_0^t K(t, s, u(s))ds, \quad t \in \mathbb{R}^+ = [0, \infty), \tag{3.1}$$

where $K : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $f, g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are suitable functions.

Roughly speaking, we use the notion of the measure of noncompactness to prove at least one solution of (3.1) in the space $BC(\mathbb{R}^+)$ consisting of all bounded and continuous real functions defined on $\mathbb{R}^+$. The space $BC(\mathbb{R}^+)$ is equipped with the standard supremum norm given as follows:

$$\| u \| = \sup \left\{ | u(t) | : \quad t \in \mathbb{R}^+ \right\}.$$

Now, let us define the measure of noncompactness in the space $BC(\mathbb{R}^+)$ (see [5]). Let us consider Z be a nonempty, bounded subset of $BC(\mathbb{R}^+)$ and a positive number $L > 0$. For $u \in Z$ and $\varepsilon > 0$, let us denote $\omega^L(u, \varepsilon)$ the modulus of continuity of the functional u on the interval $[0, L]$, that is

$$\omega^L(u, \epsilon) = \sup \{ | u(t) - u(s) | : t, s \in [0, L], | t - s | \leq \varepsilon \}.$$

Moreover, we put

$$\omega^L(Z, \varepsilon) = sup \left\{ \omega^L(u, \varepsilon) : u \in Z \right\},$$

$$\omega_0^L(Z) = \lim_{\varepsilon \to 0} \omega^L(Z, \varepsilon),$$

$$\omega_0(Z) = \lim_{L \to \infty} \omega_0^L(Z).$$

In addition, for a fixed number $t \in \mathbb{R}^+$, we denote

$$Z(t) = \{ u(t) : u \in Z \}.$$

Henceforth, we can define the measure of noncompactness μ on $B(BC(\mathbb{R}^+))$, as follows:

$$\mu(Z) = \omega_0(Z) + \limsup_{t \to +\infty} diam Z(t), \tag{3.2}$$

where $diam Z(t)$ is the diameter of $Z(t)$.

3.2 Existence of solutions for integral equations via Proinov contractions

In this section we use Corollary 2.2.5 to establish the existence of the integral equation (3.1).

Let Φ be the set of all nondecreasing functions $\varphi : (0, \infty) \to (0, \infty)$ satisfying the following conditions:

(*a*)

$$\varphi(t) < t \quad \text{for any } t > 0;$$

(b) φ is superadditive, i.e.,

$$\varphi(t)+\varphi(s)\leq\varphi(t+s)\quad\text{for } t,s\in\mathbb{R}^{+}.$$

In the sequel, we consider equation (3.1) under the following assumptions:

(1) f and g are continuous functions such that both $t\rightarrow f(t,0)$ and $t\rightarrow g(t,0)$ are elements of the space $BC\left(\mathbb{R}^{+}\right)$;

(2) There exists an upper semicontinuous function $\varphi\in\Phi$ and a constant $k\in\left(0,\frac{1}{2}\right)$ such that for any $x,y\in\mathbb{R}$ and $t\in\mathbb{R}^{+}$, we have

$$\left|f(t,x)-f\left(t,y\right)\right|\leq k\varphi\left(\left|x-y\right|\right);$$

(3) There exists a continuous function $p:\mathbb{R}^{+}\rightarrow\mathbb{R}^{+}$ such that for all $x,y\in\mathbb{R}$ and $t\in\mathbb{R}^{+}$, we have

$$\left|g(t,x)-g\left(t,y\right)\right|\leq p(t)\varphi\left(\left|x-y\right|\right);$$

(4) K is a continuous function and there exist continuous functions $a,b:\mathbb{R}^{+}\rightarrow\mathbb{R}^{+}$ such that

$$\lim_{t\rightarrow\infty}a(t)\int_{0}^{t}b(s)\,ds=0,$$

$$\lim_{t\rightarrow\infty}p(t)\,a(t)\int_{0}^{t}b(s)\,ds=0$$

and

$$|K(t,s,x)|\leq a(t)\,b(s)\quad\text{for all } x\in\mathbb{R}\text{ and } t,s\in\mathbb{R}^{+};$$

(5)

$$p(t)\,a(t)\int_{0}^{t}b(s)\,ds\leq k\quad\text{for any } t\in\mathbb{R}^{+};$$

(6) There exists a positive solution r_0 of the inequality

$$\varphi(r)+A\leq r,\quad r>0,$$

where

$$A=\sup_{t\geq0}\left\{\left|f(t,0)\right|+\left|g(t,0)\right|a(t)\int_{0}^{t}b(s)\,ds\right\}.$$

Now, we are ready to state and prove our existence result.

Theorem 3.2.1. *Under assumptions* (1) – (6), *integral equation* (3.1) *has at least one solution* $u=u(t)$ *belonging to* $BC\left(\mathbb{R}^{+}\right)$.

Proof. Let us consider the operator on the space $BC\left(\mathbb{R}^{+}\right)$ as follows

$$(Tu)(t)=f(t,u(t))+g(t,u(t))\int_0^t K(t,s,u(s))\,ds,\quad t\in\mathbb{R}^{+},u\in BC\left(\mathbb{R}^{+}\right).$$

By virtue of the imposed conditions, we easily observe that Tu is continuous on $\mathbb{R}^{+}$ for each function $u\in BC\left(\mathbb{R}^{+}\right)$, then T is well- defined.

Further, through our assumptions, we get the following inequality

$$\begin{aligned}
|(Tu)(t)| &\leq \left|f(t,u(t))\right|+\left|g(t,u(t))\right|\int_0^t |K(t,s,u(s))|\,ds\\
&\leq \left|f(t,u(t))\right|+\left|g(t,u(t))\right|a(t)\int_0^t b(s)\,ds\\
&\leq \left|f(t,u(t))-f(t,0)\right|+\left|f(t,0)\right|\\
&\quad+\left|g(t,u(t))-g(t,0)\right|a(t)\int_0^t b(s)\,ds+\left|g(t,0)\right|a(t)\int_0^t b(s)\,ds\\
&\leq k\varphi(|u(t)|)+\left|f(t,0)\right|+\varphi(|u(t)|)\,p(t)\,a(t)\int_0^t b(s)\,ds+\left|g(t,0)\right|a(t)\int_0^t b(s)\,ds\\
&\leq 2k\varphi(|u(t)|)+\left|f(t,0)\right|+\left|g(t,0)\right|a(t)\int_0^t b(s)\,ds\\
&\leq \varphi(|u(t)|)+A.
\end{aligned}$$

Since φ is nondecreasing, we obtain

$$\|Tu\|\leq\varphi(\|u\|)+A. \tag{3.3}$$

Since $A<\infty$ (by assumptions (1) and (4)), we infer that Tu is bounded on $\mathbb{R}^{+}$. Hence, T maps the space $BC\left(\mathbb{R}^{+}\right)$ into itself. In addition, $T\left(B_{r_0}\right)\subseteq B_{r_0}$ where

$$B_{r_0}=\left\{u\in BC\left(\mathbb{R}^{+}\right),\quad \left\|x-y\right\|\leq r_0\right\}.$$

Indeed, in view of(3.3) and assumption (6), we deduce

$$\|Tu\|\leq\varphi(r)+A\leq r.$$

Now, we show that T is continuous on B_{r_0}. Let us fix an arbitrary $\varepsilon>0$. and take $u,v\in B_{r_0}$ such that

$\|u-v\| \le \varepsilon$. Therefore, for $t \in \mathbb{R}^+$, we get

$$
\begin{aligned}
|(Tu)(t)-(Tv)(t)| &\le \left|f(t,u(t))-f(t,v(t))\right| \\
&\quad + \left|g(t,u(t))\int_0^t K(t,s,u(s))\,ds - g(t,v(t))\int_0^t K(t,s,u(s))\,ds\right| \\
&\quad + \left|g(t,v(t))\int_0^t K(t,s,u(s))\,ds - g(t,v(t))\int_0^t K(t,s,v(s))\,ds\right| \\
&\le k\varphi(|u(t)-v(t)|) + \left|g(t,u(t))-g(t,v(t))\right|\int_0^t |K(t,s,u(s))|\,ds \\
&\quad + \left|g(t,v(t))\right|\int_0^t |K(t,s,u(s))-K(t,s,v(s))|\,ds \\
&\le k\varphi(|u(t)-v(t)|) + \varphi(|u(t)-v(t)|)\,p(t)\,a(t)\int_0^t b(s)\,ds \\
&\quad + \left[\left|g(t,v(t))-g(t,0)\right| + \left|g(t,0)\right|\right]\int_0^t |K(t,s,u(s))-K(t,s,v(s))|\,ds \\
&\le 2k\varphi(|u(t)-v(t)|) + \left[p(t)\varphi(|u(t)|) + \left|g(t,0)\right|\right]\int_0^t |K(t,s,u(s))-K(t,s,v(s))|\,ds \\
&\le \varphi(\varepsilon) + 2\left[p(t)\varphi(r_0) + \left|g(t,0)\right|\right]a(t)\int_0^t b(s)\,ds. \qquad (3.4)
\end{aligned}
$$

Taking into account assumptions (1) and (4), we deduce that there exists $L>0$ such that for $t \ge L$, the following inequalities hold

$$2p(t)\varphi(r_0)\,a(t)\int_0^t b(s)\,ds \le \frac{\varepsilon}{2}$$

and

$$2\left|g(t,0)\right|a(t)\int_0^t b(s)\,ds \le B\,a(t)\int_0^t b(s)\,ds \le \frac{\varepsilon}{2},$$

where

$$B := \sup_{t\ge 0}\left\{\left|g(t,0)\right|\right\} < \infty.$$

Consequently, we obtain

$$|(Tu)(t)-(Tv)(t)| \le \varphi(\varepsilon) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \le 2\varepsilon \quad \text{for all } t \ge L.$$

Next, we discuss the case $0 \le t \le L$. In this case, we consider the quantity

$$\omega^L(K,\varepsilon) = \sup\left\{\left|K(t,s,x)-K\left(t,s,y\right)\right| : t,s \in [0,L],\, x,y \in [-r_0,r_0],\, \left|x-y\right| \le \varepsilon\right\}.$$

From the fact that $K(t,s,x)$ is uniformly continuous on the $[0,L]\times[0,L]\times[-r_0,r_0]$, we deduce that

$$\lim_{\varepsilon\to 0}\omega^L(K,\varepsilon) = 0.$$

Coming back to (3.4), we obtain for $t \in [0,L]$

$$
\begin{aligned}
|(Tu)(t)-(Tv)(t)| &\le \varphi(\varepsilon) + \left[p(t)\varphi(|u(t)|) + \left|g(t,0)\right|\right]\int_0^L \omega^L(K,\varepsilon)\,ds \\
&\le \varphi(\varepsilon) + \left[\varphi(r_0)C_1 + C_2\right]L\omega^L(K,\varepsilon),
\end{aligned}
$$

where

$$C_1 := \sup_{t \le L} \{p(t)\} < \infty$$

and

$$C_2 := \sup_{t \le L} \{|g(t,0)|\} < \infty.$$

From, we conclude that T is continuous on the closed ball B_{r_0}.

Let us consider now a nonempty set $Z \subset B_{r_0}$. Then, for $u, v \in Z$ and for a fixed $t \in \mathbb{R}^+$ and following the same steps as those used in (3.4), one gets

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &\le \varphi(|u(t) - v(t)|) + 2p(t)\varphi(r_0)\, a(t) \int_0^t b(s)\, ds \\ &\quad + 2|g(t,0)|\, a(t) \int_0^t b(s)\, ds. \end{aligned}$$

Hence, the above estimate turns into

$$\begin{aligned} diam(TZ)(t) &\le \varphi(diam(Z)(t)) + 2p(t)\varphi(r_0)\, a(t) \int_0^t b(s)\, ds \\ &\quad + 2|g(t,0)|\, a(t) \int_0^t b(s)\, ds. \end{aligned}$$

Using assumption (4) and the upper semicontinuity of the function φ, we find

$$\limsup_{t\to\infty} diam(TZ)(t) \le \varphi\left(\limsup_{t\to\infty} diam(Z)(t)\right). \tag{3.5}$$

On the other hand, let us fix $\varepsilon > 0$ and $L > 0$ and take arbitrarily $t, s \in [0, L]$ such that $|t - s| \le \varepsilon$. Without loss of generality, we may suppose that $s < t$. Thus, for $u \in Z$ and through our assumptions, we get

$$\begin{aligned} |(Tu)(t) - (Tu)(s)| &\le |f(t,u(t)) - f(s,u(s))| \\ &\le \left| g(t,u(t)) \int_0^t K(t,\tau,u(\tau))\, d\tau - g(s,u(s)) \int_0^t K(t,\tau,u(\tau))\, d\tau \right| \\ &\quad + \left| g(s,u(s)) \int_0^t K(t,\tau,u(\tau))\, d\tau - g(s,u(s)) \int_0^s K(s,\tau,u(\tau))\, d\tau \right| \\ &\le |f(t,u(t)) - f(t,u(s))| + |f(t,u(s)) - f(s,u(s))| \\ &\quad + |g(t,u(t)) - g(s,u(s))| \int_0^t |K(t,\tau,u(\tau))|\, d\tau \\ &\quad + |g(s,u(s))| \left| \int_0^t K(t,\tau,u(\tau))\, d\tau - \int_0^s K(s,\tau,u(\tau))\, d\tau \right| \\ &\le k\varphi(|u(t) - u(s)|) + \omega_{r_0}^L(f,\varepsilon) \\ &\quad + \left[|g(t,u(t)) - g(t,u(s))| + |g(t,u(s)) - g(s,u(s))| \right] a(t) \int_0^t b(\tau)\, d\tau \\ &\quad + \left[|g(s,u(s)) - g(s,0)| + |g(s,0)| \right] \\ &\quad \times \left[\int_s^t |K(t,\tau,u(\tau))|\, d\tau + \int_0^s |K(t,\tau,u(\tau)) - K(s,\tau,u(\tau))|\, d\tau \right], \end{aligned}$$

where

$$\omega_{r_0}^L\left(f,\varepsilon\right)=\sup\left\{\left|f\left(t,x\right)-f\left(s,x\right)\right|:t,s\in[0,L],x\in[-r_0,r_0],|t-s|\leq\varepsilon\right\}.$$

It follows that

$$\begin{aligned}|(Tu)(t)-(Tu)(s)| &\leq k\varphi(|u(t)-u(s)|)+\omega_{r_0}^L\left(f,\varepsilon\right)\\ &\quad+\left[p(s)\varphi(|u(t)-u(s)|)+\omega_{r_0}^L\left(g,\varepsilon\right)\right]a(t)\int_0^t b(\tau)\,d\tau\\ &\quad+\left[p(s)\varphi(|u(s)|)+\left|g(s,0)\right|\right]\\ &\quad\times\left[\int_s^t|K(t,\tau,u(\tau))|\,d\tau+\int_0^s|K(t,\tau,u(\tau))-K(s,\tau,u(\tau))|\,d\tau\right],\end{aligned}$$

where

$$\omega_{r_0}^L\left(g,\varepsilon\right)=\sup\left\{\left|g\left(t,x\right)-g\left(s,x\right)\right|:t,s\in[0,L],x\in[-r_0,r_0],|t-s|\leq\varepsilon\right\}.$$

This yields that

$$\begin{aligned}|(Tu)(t)-(Tu)(s)| &\leq 2k\varphi(|u(t)-u(s)|)+\omega_{r_0}^L\left(f,\varepsilon\right)+\omega_{r_0}^L\left(g,\varepsilon\right)a(t)\int_0^t b(\tau)\,d\tau\\ &\quad+\left[p(s)\varphi(r_0)+\left|g(s,0)\right|\right]a(t)\int_s^t b(\tau)\,d\tau\\ &\quad+\left[p(s)\varphi(r_0)+\left|g(s,0)\right|\right]L\omega_{r_0}^L(K,\varepsilon),\end{aligned}\tag{3.6}$$

where

$$\omega_{r_0}^L(K,\varepsilon)=\sup\left\{\left|K(t,\tau,x)-g(s,\tau,x)\right|:t,s,\tau\in[0,L],x\in[-r_0,r_0],|t-s|\leq\varepsilon\right\}.$$

Hence, we obtain

$$\begin{aligned}\omega^L(TX,\varepsilon) &\leq \varphi\left(\omega^L(X,\varepsilon)\right)++\omega_{r_0}^L\left(f,\varepsilon\right)+L\omega_{r_0}^L\left(g,\varepsilon\right)D_1\\ &\quad+\varepsilon\left[p(s)\varphi(r_0)+\left|g(s,0)\right|\right]D_1\\ &\quad+L\omega_{r_0}^L(K,\varepsilon)D_2,\end{aligned}$$

where

$$D_1:=\sup_{0\leq\tau,t\leq L}\{a(t)b(\tau)\}<\infty$$

and

$$D_2:=\sup_{0\leq t\leq L}\left\{p(s)\varphi(r_0)+\left|g(s,0)\right|\right\}<\infty.$$

Furthermore, from the uniform continuity of f,g on $[0,L]\times[-r_0,r_0]$ as well as the function K on $[0,L]\times[0,L]\times[-r_0,r_0]$, we infer that

$$\lim_{\varepsilon\to0}\omega_{r_0}^L\left(f,\varepsilon\right)=0,\quad\lim_{\varepsilon\to0}\omega_{r_0}^L\left(g,\varepsilon\right)=0,\quad\text{and }\lim_{\varepsilon\to0}\omega_{r_0}^L(K,\varepsilon)=0.\tag{3.7}$$

In view of (3.6) and (3.7), it follows that

$$\omega_0^L(TZ)\leq\lim_{\varepsilon\to0}\varphi\left(\omega^L(Z,\varepsilon)\right).$$

Again, through the upper semicontinuity of φ, we get

$$\omega_0^L(TZ) \leq \varphi\left(\omega_0^L(Z)\right),$$

which yields

$$\omega_0(TZ) \leq \varphi(\omega_0(Z)). \tag{3.8}$$

Combining (3.5), (3.8) and the superadditivity of φ, we get

$$\omega_0(TZ) + \limsup_{t\to\infty} diam(TZ)(t) \leq \varphi\left(\omega_0(Z) + \limsup_{t\to\infty} diam(Z)(t)\right),$$

or, equivalently

$$\mu(TZ) \leq \varphi\left(\mu(Z)\right).$$

Hence, due to Corollary 2.2.5-(i) with $\psi(t) = t$, we conclude that T has at least one fixed point in B_{r_0} and the proof of the theorem is finished. □

Remark 3.2.1. In view of the paper [1], the hypothesis of superadditivity of the function φ can be replaced by the following one:

"φ is a concave function."

3.3 Example

Let us consider the following functional integral equation

$$u(t) = \frac{t}{2(1+t)}\ln(1+|u(t)|) + \frac{e^{-t}}{1+t^2}\ln(1+|u(t)|)\int_0^t \frac{se^{-t}\sin x(s)}{1+|x(s)|}ds, \quad t \in \mathbb{R}^+. \tag{3.9}$$

Let us observe that

$$f(t,x) = \frac{t}{2(1+t)}\ln(1+|x|),$$

$$g(t,x) = \frac{e^{-t}}{1+t^2}\ln(1+|x|)$$

and

$$K(t,s,x) = \frac{se^{-t}\sin x}{1+|x|}.$$

It easy to see that integral equation (3.9) is a special case of Integral equation (3.1). Indeed, f, g and K on their suitable domains. Also, if we take

$$\varphi(t) = \ln(1+t),$$

we have $\varphi(t) < t$ for all $t > 0$ and φ nondecreasing and concave on $\mathbb{R}^+$. In addition, for arbitrary $x, y \in \mathbb{R}$ such that $|x| \leq |y|$ (or $|y| \leq |x|$) and we get

$$\left|f(t,x) - f(t,y)\right| \leq \frac{1}{2}\ln\left(1+|x-y|\right) = k\varphi\left(|x-y|\right),$$

where $k = \frac{1}{2}$.

Further, we have

$$\left|g(t,x) - g\left(t,y\right)\right| \le p(t) \ln\left(1 + |x-y|\right),$$

where $p(t) = \dfrac{e^{-t}}{1+t^2}$ and

$$|K(t,s,x)| \le e^{-t} s.$$

Hence, we can take $a(t) = e^{-t}$ and $b(s) = s$. So, we obtain

$$\lim_{t\to\infty} a(t) \int_0^t b(s)\, ds = \lim_{t\to\infty} e^{-t} \int_0^t s ds = 0,$$

$$\lim_{t\to\infty} p(t)\, a(t) \int_0^t b(s)\, ds = \lim_{t\to\infty} \frac{e^{-2t}}{1+t^2} \int_0^t s ds = 0$$

and

$$p(t)\, a(t) \int_0^t b(s)\, ds = \frac{e^{-2t}}{1+t^2} \frac{t^2}{2} \le \frac{1}{2} = k \quad \text{for any } t \in \mathbb{R}^+.$$

Moreover, there exists a positive solution r_0 of the inequality

$$\ln(1+r) + A \le r,$$

since

$$A = \sup_{t\ge 0} \left\{ f(t,0) + \left|g(t,0)\right| a(t) \int_0^t b(s)\, ds \right\} = 0 < \infty.$$

Conclusion

In this dissertation, we use have used the measure of noncompactness in Banach spaces which is a forceful tool to prove the existence of solutions for nonlinear integral equations. We have obtained some improvements and generalizations in the theory part regarding the Proinov contraction of Darbo-type. The application part is a generalization of the examples given in [1].

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