

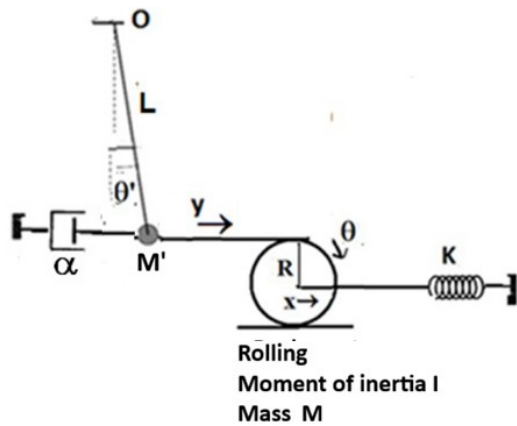
Exercise 1

We consider the system formed by:

- a pendulum of length L and mass M' ,
- a wheel of radius R , mass M , and moment of inertia I rolling on a horizontal plane,
- a string wound around the wheel,
- and a spring of stiffness K .

The mass M' is subjected to damping with a friction coefficient α .

1. Determine the relationships between the coordinates x , y , θ , θ' .
2. Calculate the kinetic energy and the potential energy as a function of θ .
3. Determine the differential equation of motion as a function of θ and its solution $\theta(t)$ in the case of a pseudo-periodic oscillatory motion.



Exercise 2

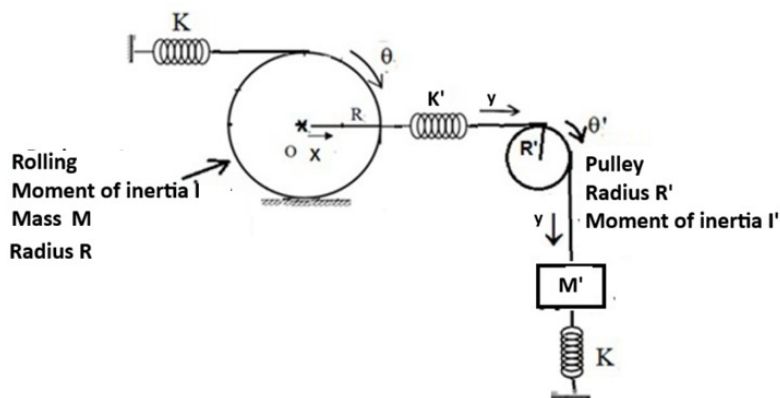
We consider the following system with two degrees of freedom:

- a mass M' and a wheel of radius R , mass M , and moment of inertia I rolling on a horizontal plane,
- a pulley of radius R' and moment of inertia I' ,
- 3 springs of stiffness K , K' , K .

1. Using Lagrange's formula, determine the differential equations of motion as a function of (x, y) .
2. To simplify the equations, let $\mu = M + I/R^2$, $\mu' = M' + I'/R'^2$, and assume $\mu = \mu'$ and $K = K'$.

Rewrite the two differential equations in simplified form.

Determine the two natural frequencies (ω' , ω'') of the system.



Exercise 1 - Solution (9 pts)	
1. Relationships between coordinates: Non-slip rolling: $x = R\theta$, $y = L\theta' = 2x \rightarrow \theta' = 2R\theta/L$	1
2. Calculation of kinetic and potential energies	
$E_c = E_c$ (wheel rotation) + E_c (wheel translation) + E_c (pendulum)	
$E_c = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M'\dot{y}^2 = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}MR^2\dot{\theta}^2 + \frac{1}{2}M'(4R^2\dot{\theta}^2)$	1.5
$E_c = \frac{1}{2}a\dot{\theta}^2$ with $a = I + MR^2 + M'4R^2$	0.5
$E_p = \frac{1}{2}M'gL\dot{\theta}^2 + \frac{1}{2}Kx^2 = \frac{1}{2}M'gL\left(\frac{2R\theta}{L}\right)^2 + \frac{1}{2}K(R\theta)^2 = \frac{1}{2}b\theta^2$	
$b = M'g\frac{4R^2}{L} + KR^2$	1.5
Lagrangian : $L = E_c - E_p$	
Equation of LAGRANGE:	
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = -\frac{\partial D}{\partial \dot{\theta}}$	0.5
Dissipation : $D = \alpha V^2/2$ avec $D = \frac{1}{2}\alpha V^2 = \frac{1}{2}\alpha(4R^2\dot{\theta}^2) \rightarrow \frac{\partial D}{\partial \dot{\theta}} = 4\alpha R^2\dot{\theta}$	1
Differential equation : $a\ddot{\theta} + 4\alpha R^2\dot{\theta} + b\theta = 0 \rightarrow \ddot{\theta} + 2\gamma\dot{\theta} + \omega_0^2\theta = 0$	
With $2\gamma = 4\alpha R^2/a$, and $\omega_0^2 = b/a$	1.5
Condition for the existence of a pseudo-periodic oscillatory motion: $\gamma < \omega_0$	0.5
Oscillation pseudo-pulsation: $\omega_a = \sqrt{\omega_0^2 - \gamma^2}$	0.5
Solution of the differential equation:	
$x(t) = C \cdot \exp(-\gamma t) \cdot \cos(\omega_a t + \varphi)$	0.5
with C , φ = integration constants determined from initial conditions (at $t=0$)	

Exercise 2 – Solution (11 pts)	
Relationships between coordinates: $x = R\theta$, $y = R'\theta'$	0.5
1. Differential equations	
$E_c = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I'\dot{\theta}'^2 + \frac{1}{2}M'\dot{y}^2 = \frac{1}{2}(I + MR^2)\dot{\theta}^2 + \frac{1}{2}(I' + M'R'^2)\dot{\theta}'^2 = \frac{1}{2}(M + I/R^2)\dot{x}^2 + \frac{1}{2}(M' + I'/R'^2)\dot{y}^2$	1.5
Let $\mu = M + I/R^2$, $\mu' = M' + I'/R'^2 \rightarrow E_c = \frac{1}{2}\mu\dot{x}^2 + \frac{1}{2}\mu'\dot{y}^2$	
$E_p = \frac{1}{2}K(2x)^2 + \frac{1}{2}K'(x - y)^2 + \frac{1}{2}Ky^2 = \frac{1}{2}K(4x^2) + \frac{1}{2}K'(x^2 + y^2 - 2xy) + \frac{1}{2}Ky^2$	1.5
$= \frac{1}{2}(K' + 4K)x^2 + \frac{1}{2}(K' + K)y^2 - K'xy$	
Lagrangian : $L = E_c - E_p$	
Equations of LAGRANGE:	
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$	1
By differentiating, we obtain :	
$\mu\ddot{x} + (K' + 4K)x - K'y = 0$	
$\mu'\ddot{y} + (K' + K)y - K'x = 0$	1.5
Suppose $\mu' = \mu$, $K = K'$ and let $a = K/\mu = K'/\mu'$	
\rightarrow Hence a system of 2 second-order differential equations:	
$\ddot{x} + 5ax - ay = 0$	
$\ddot{y} + 2ay - ax = 0$	1
2. Natural frequencies	
- First step of the solution: Searching for particular sinusoidal solutions with the same frequency ω :	
$x = A_1 \cos(\omega t + \varphi_1)$, $y = A_2 \cos(\omega t + \varphi_2) \rightarrow \ddot{x} - \omega^2 x = 0$, $\ddot{y} - \omega^2 y = 0$	1

Substituting into equations (1) and (2), we obtain :	
$(-\omega^2+5a)x-ay = 0$	1
$-ax +(-\omega^2+2a)y = 0$	
This system of equations has non-zero solutions $(x, y) \neq (0,0)$ if the determinant $D = 0$	
$D = (\omega^2-5a)(\omega^2-2a)-a^2 = 0 \rightarrow \omega^4-7a\omega^2+9a^2 = 0$	0.5
Discriminant $\Delta = 49a^2-36 a^2 = 13a^2$	0.5
Hence the 2 natural frequencies ω' and ω'' (with $\omega' < \omega''$):	
$\omega'^2 = a(7-13^{1/2})/2$ and $\omega''^2 = a(7+13^{1/2})/2$	1