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Solving Fractional Differential Equations by Analysis Method, Time-Discretisation Method

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Aknowledjment

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BEKAKRA YOUCEF

Abstract

The main objective of this work is to discuss some existence and uniqueness results for a classes of fractional differential equations involving Caputo derivative of order between zero and one.

Our results will be achieved by means of method of Rothe. We develop this technique in such a way that a new sequence is introduced that we will call the $\alpha-$ Rothe sequence.

It must be pointed out that the constructed L1 scheme is designed to approximate the Caputo fractional derivative mentioned in the problems.

Keywords:

fractional differential equation, Caputo derivative,
Rothe's method, Apriori estimates

Résumé

L'objectif principal de ce travail est de discuter de certains résultats d'existence et d'unicité pour des classes d'équations différentielles fractionnaires impliquant la dérivée de Caputo d'ordre compris entre zéro et un.
Nos résultats seront obtenus au moyen de la méthode de Rothe. Nous développons cette technique de manière à introduire une nouvelle séquence que nous appellerons la séquence $\alpha-$ Rothe.

Il convient de souligner que le schéma L1 construit est conçu pour approcher la dérivée fractionnaire de Caputo mentionnée dans les problèmes.

ملخص

إن الهدف الرئيسي من هذا العمل هو مناقشة بعض النتائج المتعلقة بوجود ووحدانية أنواع معينة من المعادلات التفاضلية الكسرية التي تتضمن مشتقة كابوتو , من الرتبة بين صفر وواحد. سيتم تحقيق نتائجنا من خلال طريقة روث. قمنا بتطوير هذه التقنية من خلال إرفاق تسلسل جديد سنسميه تسلسل ألفا روث. تجدر الإشارة الى ان مخطط L1 تم استعماله لتقريب مشتقة كابوتو الكسرية المستعملة في المسائل.

Contents

Notations

- Δu the Laplacian of u.
- E_2 2-dimensional Euclidean space.
- Ω is a bounded domaine in multiply connected with a lipschitz boundary $\partial\Omega$.
- Q is a domain $(Q = \Omega \times (0, T))$.
- α is a real number between zero and one.
- ${}_cD^{\alpha}_{0,t}$ refers to the Caputo fractional derivative with order α.
- $u(x)$ brief notation for $u(x_1, x_2)$.
- $L^2(\Omega)$ denotes the space of measurable square-integrable functions on Ω.
- $(.,.)$ usual scalar product in $L_2(\Omega)$.
- $|i|$ the module of a multi-index $(|i| = i_1 + i_2)$.
- $\Gamma(.)$ is the classical gamma function.
- T is a given positive real number.
- h is the time step $(h = \dfrac{T}{p})$.
- $y_j(x)$ represents the approximation solution of the exact one at the point (x, t_j) .
- $\rightharpoonup$ denotes the weak convergence.
- $C(I)$ denotes the space of continuous functions on I.
- $AC(I)$ denotes the space of absolutely continuous functions on I.

- $a.a.$ almost all.
- $\forall$ for all.
- $S - C$ Cauchy-Schwartz inequality

Introduction

Fractional calculus is the branch of mathematical analysis that studies the generalization of the notions of derivation and integration to arbitrary orders (real or complex). Introduced in 1695, this concept of differentiation for non-integer numbers was first noticed. Since then, many mathematicians have contributed to this development of fractional calculus like P.S. Laplace [1], Laurent [2], Liouville [3], Letnikov [4], Riesz [5] and so forth. However, it wasn't until 1884 that the theory of the generalised differentiation operator reached a level of development sufficient to serve as a starting point for the modern mathematician.. At that time, the theory was extended to D^v operators, where v can be rational or irrational, positive or negative, real or complex. So the impact and inspiration of fractional calculus in applied science and theory have grown considerably. In fact, with many results found in mathematical physics, biophysics, control theory and statistics, fractional calculus is a useful tool in applied mathematics to examine a multitude of problems from many domains of research and engineering [6–11]. It should be mentioned that from the viewpoint of these applications it was undoubtedly the book written by K. U. Oldham and J. Spanier [12] which played an outstanding role in the development of the subject which can be called applied fractional calculus. Moreover, it was the first book which was entirely devoted to a systematic presentation of the ideas, methods, and applications of the fractional calculus. On the other hand, fractional derivatives have several definitions, such as Caputo, Riemann-Liouville, Hadamard, and Riesz derivatives. Investigations of the history of the fractional derivative theory may be found in [13–16]. Among these definitions, we can pick the caputo derivative, which is widely popularized in the literature. It is worth noting that recently, algorithms have received attention and increased interest in approximating the Caputo derivative. To our knowledge, Lubich is the leader in the concept of discretized fractional calculus using the linear multistep method. The L1 method discussed in [17] is the widely used approximation technique for the Caputo derivative.

In the last couple decades, there's been an observable growth in researchers' attention to fractional partial differential equations (FPDEs) as a generalization of the integer-order differential equation. This increase can be attributed to the numerous effective uses of fractional calculus in a range of engineering and scientific fields [18–22]. The time-fractional diffusion equation (TFDE) is developed as a particular case of FPDEs; it is a representation of a linear integro-differential equation. This equation can be especially useful in different fields, such as chemistry [23], the economy [8], and many others [24, 25]. Despite the fact that numerous methods for solving fractional differential equations have been presented, many scholars point out that their solutions cannot always be obtained analytically, however, they only work for particular cases.

As a result of this, numerical techniques are frequently utilized, and there has been significant interest in researching these techniques, namely, a plentiful array of numerical methods is developed for classical differential equations, while accurate numerical methods for fractional differential equations (FDEs) are relatively poor. In spite of that, researchers are currently trying to develop many methods.

In this work, we are mainly focusing on the Rothe time method. Originally, this method is developed in [26, 27] as a partial differential equation discretization technique. Several authors have employed this method to many types of initial boundary value problem, see for instance [28–32].

It's worth noting that Rothe's method has also been successful in resolving (FPDEs) [33, 34].

One of the main aims of this thesis is to develop this method by creating a new function called the α-Rothe function throughout the approximation of Caputo derivative by the L1 approximation. As an application we investigate the existence and uniqueness of weak solutions for fractional equations

Thesis scheme

Throughout this work, $u = u(x,t)$ is a Lipschitz function with respect to t, $\dfrac{\partial^\alpha u}{\partial t^\alpha}$ refers to the Caputo fractional derivative with order α (between zero and one), Ω is a bounded domaine in multiply connected with a lipschitz boundary $\Gamma, (0,T)$ is a bounded interval, $f \in L_2(\Omega)$ and A is a given linear operator.

This thesis is arranged as follows:

In **Chapter** 1, we look at some definitions and preliminary facts for functional analysis. We also present the essential theoretical tools about fractional calculus that will be helpful in developing and understanding the chapters that follow.

In **Chapter** 2, we discuss the existence and uniqueness of weak solution for a time-fractional parabolic partial differential equation with homogeneous conditions:

$$\frac{\partial^\alpha_{0,t} u}{\partial t^\alpha} - \Delta u = f, \quad in \quad Q = \Omega \times (0,T), \tag{0.0.1}$$

$$u(x;0) = 0, \tag{0.0.2}$$

$$Au = 0, \quad on \quad \partial\Omega \times (0;T), \tag{0.0.3}$$

In **Chapter** 3, we are interested in the same problem as the one dealt with in the previous chapter, where the conditions are non-homogeneous in this case.

In **Chapter** 4, we deal with the existence and uniqueness of a time fractional equation involving an integral operator, this chapter presents the main results of our contribution [35].

Chapter 1

Preliminaries

To develop this work, it is important to introduce all the materials that we are going to use throughout the whole thesis. More specifically, after presenting some facts about functional analysis, some topics regarding fractional calculus will be defined.

1.1 Basic definitions and functional spaces

We start this section by presenting some classical-based facts from such many subjects of analysis. We also introduce some functional spaces that we will utilize in the upcoming chapters.

Let $I = [a; b]$ be an interval of $\mathbb{R}$.

Definition 1.1.1 (Norm) *Let H be a real vector space, a mapping*

$$\|.\| : H \to [0, +\infty[$$
$$u : \to \|u\|$$

is called a norm if it satisfies the following statements

- $\|u\| = 0$ *iff* $u = 0$.
- $\|ku\| = |k|\|u\|$ *for all* $u \in H$ *and* $k \in \mathbb{R}$.
- $\|u + v\| \leqslant \|u\| + \|v\|$ *for all* $u, v \in H$.

The space H is called a normed linear space.

Definition 1.1.2 *A normed linear space H is called a Banach space if every Cauchy sequence in H converges in H.*

Definition 1.1.3 (Inner product) *A mapping*

$$
\begin{aligned}
(.,.) : H \times H &\to \mathbb{R} \\
u, v : &\to (u, v)
\end{aligned}
$$

is called inner product if

- $(u, v) = (v, u)$ *for all* $u, v \in H$.
- $v \to (u, v)$ *is linear for each* $v \in H$.
- $(u, u) \geqslant 0$ *for all* $u \in H$.
- $(u, u) = 0$ *iff* $u = 0$.

Definition 1.1.4 *Let $(.,.)$ be an inner product, the associated norm is defined as*

$$\|u\| := \sqrt{(u, u)}, \ \ u \in H.$$

Lemma 1.1.1 (Cauchy-Schwartz inequality) *Let $(.,.)$ be an inner product . Then*

$$(u, v) \leqslant \|u\|\|v\|, \ \ u, v \in H.$$

Definition 1.1.5 *A Hilbert space H is a complex inner product space that is complete under the associated norm.*

Definition 1.1.6 *Let E be a Banach space provided with the norm $\|.\|_E$ and μ a measure on I, a function $f : I \to E$ is Bochner integrable on I if it is strongly measurable and such that*

$$\lim_{n\to\infty} \int_a^b \|f(t) - \phi_n(t)\|_E d\mu(t) = 0,$$

for some sequence of simple functions $\{\phi_n\}$. In this case the Bochner integral on I is defined by :

$$\int_a^b f(t)d\mu(t) = \lim_{n\to\infty} \int_a^b \phi_n(n)d\mu(t).$$

Lemma 1.1.2 (Gronwall's lemma [36]) *Let* $b_1, \dots, b_k$ *be positive numbers satisfying* $b_1 \leqslant B$, $b_j \leqslant B + Ch\sum_{l=1}^{j-1} b_l, \quad \forall j = 2, \dots, k$, *where* B, C, h *are nonnegative constants. Then*

$$b_j \leqslant B\exp\left[C(j-1)h\right], \quad j = 1, 2 \dots, k.$$

Definition 1.1.7 $C(I) := C(I, R)$ *is the Banach space of continuous functions* u *with*

$$\|u\| = \sup_{t\in I} |u(t)|.$$

Definition 1.1.8 *A function* $f : I \to \mathbb{R}$ *is said to be absolutely continuous on* I *if for all* $\epsilon > 0$ *there is some* $\delta > 0$ *such that* $\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \epsilon$ *for every number of disjoint intervals* $(a_i, b_i), i = 1, \dots, n$, *with* $[a_i, b_i] \subset I$ *and* $\sum_{i=1}^{n}(a_i, b_i) < \delta$.

The space of all absolutely continuous functions $f : I \to \mathbb{R}$ *is denoted by* $AC([a; b])$. *It is known (see [37], p. 338) that* $AC(I)$ *coincides with the space of primitives of Lebesgue summable functions:*

$$f(x) \in AC(I) \Longleftrightarrow f(x) = c + \int_a^x \varphi(t)dt \quad (\varphi(t) \in L(I), \tag{1.1.1}$$

and therefore an absolutely continuous function $f(x)$ *has a summable derivative* $f'(x) = \varphi(x)$ *almost everywhere on* I. *Thus* (1.1.1) *yields*

$$\varphi(t) = f'(t) \text{ and } c = f(a).$$

Definition 1.1.9 *We denote by* $L_2(I)$ *the space of all functions* $u(x)$ *square integrable in* I *(thus the integral* $\int_I u(x)dx$ *exists), in the Lebesgue sense, and is finite. On which there are defined:*

the scalar product

$$(u, v) = \int_I u(x)v(x)dx,$$

the norm

$$\|.\| = \sqrt{(u, u)} = \sqrt{\int_I u^2(x)dx}.$$

Definition 1.1.10 *[36] The space* $L_2(I, \Phi)$ *is the space of all functions* $\varphi \in L_2(I)$ *Bochner* *integrable and satisfying*

$$\int_I \|\varphi(t)\|_\Phi^2 \ \mathrm{d}t < +\infty,$$

equipped with the scalar product

$$(\varphi_1, \varphi_2)_{L_2(I,\Phi)} = \int_I (\varphi_1(t), \varphi_2(t))_\Phi \ \mathrm{d}t. \tag{1.1.2}$$

Note that the space Φ is a Hilbert space, and it can be proven that $L_2(I, \Phi)$ is also a Hilbert space.

Definition 1.1.11 *We define the Sobolev space $H^1(I)$ by*

$$H^1(I) = \left\{u \in L_2(I) : D^i u \in L_2(I) \ \ for \ \ 0 \leqslant |i| \leqslant 1\right\},$$

here $D^i u$ is the weak partial derivative,

$$(u, v)_{H^1(I)} = \sum_{|i| \leqslant 1} (D^i u, D^i v)$$

where

$$(u, v) = \int u(t)v(t)dt$$

is the inner product in $L_2(I)$.

Definition 1.1.12 *[36] Let A be a given linear operator, we define the space ϑ as:*

$$\vartheta = \left\{\mu; \mu \in H^1(\Omega), A\mu = 0, in\ the\ sense\ of\ traces\right\}. \tag{1.1.3}$$

Definition 1.1.13 *[36]*

A sequence $\{x_n\}$ in a normed linear space X converge weakly to $x \in X$ if

$$f(x_n) \to f(x), \tag{1.1.4}$$

where f is any bounded functional from X.

To distinguish between notations, we write $x_n \to x$ for convergence in norm, $x_n \rightharpoonup x$ for weak convergence.

Proposition 1.1.1 *[36] The strong convergence implies the weak one.*

1.2 Fractional calculus

In this section, we recall some basic facts concerning fractional calculus.

1.2.1 Special functions

We present here the definitions of the gamma function and the Mittag-Leffler function.

Definition 1.2.1 (Gamma function) *The Euler gamma function $\Gamma(t)$ is defined by the so-called Euler integral of the second kind:*

$$\Gamma(t) = \int_0^\infty e^{-z} z^{t-1} dz, \qquad (Re(t) > 0).$$

Definition 1.2.2 (Mittag-Leffler function) *[38]*

For $z \in \mathbb{R}$ and $\alpha > 0$, the one-parameter Mittag-Leffler function is defined by :

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

Definition 1.2.3 *[39] For $z \in \mathbb{R}$ and $\alpha > 0$, the two-parameter Mittag-Leffler function is defined by :*

$$E_{\alpha,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \alpha)}.$$

1.2.2 Elements from fractional calculus theory

Definition 1.2.4 *[9] Let $\alpha > 0$, the Riemann-Liouville fractional integral ($D_{0,t}^{-\alpha}$) of order α of function φ is defined as:*

$$D_{0,t}^{-\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) \mathrm{d}s \quad (t > 0).$$

Definition 1.2.5 *[9] The Caputo derivative of order α $(0 < \alpha < 1)$ of a function $v(.,t)$ is defined as:*

$$ {}_cD_{0,t}^{\alpha} v(.,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial v(.,s)}{\partial s} \mathrm{d}s. \tag{1.2.1}$$

Theorem 1.2.1 *[40] If $v(t) \in C^0[0,T]$ and $\alpha \in \mathbb{R}^{*+}$, then*

$$D_{0,t}^{-\alpha} v(0) = 0.$$

Theorem 1.2.2 *[40] If $v(t) \in C^1[0,T]$ and $0 < \alpha < 1$, then*

$$_cD_{0,t}^{\alpha} D_{0,t}^{-\alpha} v(t) = v(t).$$

Theorem 1.2.3 *Let E be a Hilbert space, if $\{u_n\} \in E$ is a weakly convergent sequence to an $u \in E$, then*

$$_cD_{0,t}^{\alpha} u_n \rightharpoonup {_cD_{0,t}^{\alpha}} u \quad \text{in } E, \tag{1.2.2}$$

$$D_{0,t}^{-\alpha} u_n \rightharpoonup D_{0,t}^{-\alpha} u \quad \text{in } E. \tag{1.2.3}$$

Proof. (1.2.2)- Since $_cD_{0,t}^{\alpha}$ is a bounded linear functional, it follows from

$$u_{n_k} \rightharpoonup u,$$

that

$$_cD_{0,t}^{\alpha} u_{n_k} \to {_cD_{0,t}^{\alpha}} u,$$

by (1.1.4). Finally, by proposition 1.1.1 we complete the proof.

(1.2.3)- The proof is almost the same as in (1.2.2). ■

Lemma 1.2.1 *[41] Let $u(t) \in AC([0,T])$, then*

$$\left(u, {_cD_{0,t}^{\alpha}} u\right) \geqslant \frac{1}{2}\, {_cD_{0,t}^{\alpha}} \|u\|^2, \qquad 0 < \alpha < 1.$$

Lemma 1.2.2 *[41] Let $y(t) \in AC([0,T])$ be nonnegative and satisfy*

$$_cD_{0,t}^{\alpha} y(t) \leq g_1 y(t) + g_2(t), \quad 0 < \alpha \leq 1,$$

for a. a. t in $[0,T]$, where $g_1 > 0$ and $g_2(t)$ is an integrable nonnegative function on $[0,T]$. Hence

$$y(t) \leq y(0) E_\alpha\left(c_1 t^\alpha\right) + \Gamma(\alpha) E_{\alpha,\alpha}\left(g_1 t^\alpha\right) D_{0t}^{-\alpha} g_2(t)$$

where $E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}\Gamma(\alpha n+1)$ *and* $E_{\alpha,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\mu)}$ *are the Mittag-Leffler functions.*

1.3 Functional instruments

In this section, we devote our attention to certain aspects of functional analysis that will be of significant relevance and utility later. Let $(.,.)$ denotes the usual scalar product in $L_2(0,T)$ and $\|.\|$ the corresponding norm.

Definition 1.3.1 *We denote $b_\zeta(.,.)$ by the bilinear form associated with $-\Delta u$:*

$$b_\zeta(u,v) = \sum_{i=1}^{2} \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx. \tag{1.3.1}$$

Definition 1.3.2 *[36] The form (1.3.1) is supposed to be bounded in $H^1(\Omega)$ if a constant ξ can be found such that*

$$|b_\zeta(u,v)| \leqslant \xi \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \quad \forall u,v \in H^1(\Omega). \tag{1.3.2}$$

Definition 1.3.3 *[36] The form (1.3.1) is supposed to be $\vartheta-$elliptic if a certain constant η exists such that*

$$b_\zeta(v,v) \geqslant \eta\|v\|^2_{H^1(\Omega)} \quad \forall v \in \vartheta. \tag{1.3.3}$$

In the space $L_2\left(I,\vartheta\right)$, the scalar product is given by:

$$(u_1,u_2)_{L_2(I,\vartheta)} = \int_I \left(u_1(t),u_2(t)\right)_\vartheta \, \mathrm{d}t,$$

consequently, the norm is provided by:

$$\|u\|^2_{L_2(I,\vartheta)} = \int_I \|u(t)\|^2_\vartheta \, \mathrm{d}t. \tag{1.3.4}$$

1.4 The L1 scheme for Caputo derivative

We end this chapter by giving a finite difference approximation to discritize the Caputo derivative which is known by the L1 approximation. Let $t_j = j\Delta t,\ 0 \leqslant j \leqslant p,\quad$ where $\Delta t = \dfrac{T}{p}$ is the time step. We have from (1.2.1)

$$\frac{\partial^\alpha u\,(x,t_j)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} \frac{\partial u(x,\tau)}{\partial \tau} \frac{\mathrm{d}\tau}{(t_j-\tau)^\alpha}. \tag{1.4.1}$$

Use the one-sided difference approximation to discretize the time first-order derivative, (1.4.1) reads:

$$\begin{aligned}
\frac{\partial^\alpha u\,(x,t_j)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \frac{u\,(x,t_{k+1}) - u\,(x,t_k)}{\Delta t} \int_{t_k}^{t_{k+1}} \frac{\mathrm{d}\tau}{(t_j-\tau)^\alpha} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \frac{u\,(x,t_{k+1}) - u\,(x,t_k)}{\Delta t} \int_{t_{j-k-1}}^{t_{j-k}} \frac{\mathrm{d}t}{t^\alpha} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \frac{u\,(x,t_{j-k}) - u\,(x,t_{j-k-1})}{\Delta t} \int_{t_k}^{t_{k+1}} \frac{\mathrm{d}t}{t^\alpha} \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{j-1} \frac{u\,(x,t_{j-k}) - u\,(x,t_{j-k-1})}{\Delta t^\alpha} \left[(k+1)^{1-\alpha} - k^{1-\alpha}\right] \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{j-1} a_k \frac{u\,(x,t_{j-k}) - u\,(x,t_{j-k-1})}{\Delta t^\alpha},
\end{aligned} \tag{1.4.2}$$

where

$$a_k = \left[(k+1)^{1-\alpha} - k^{1-\alpha}\right] \quad k = 0,1,\ldots,j-1.$$

It is simple to verify that

$$a_k > 0, \quad k = 0,1,\ldots,j-1,$$

$$1 = a_0 > a_1 > \ldots a_{j-1}.$$

Chapter 2

Solvability for time fractional parabolic equation with homogeneous conditions.

2.1 Introduction

The problem treated in this chapter is inspired by the work of A. Raheem and D. Bahuguna [42]. Our idea is based on replacing $\frac{\partial u}{\partial t}$ by $\frac{\partial_{0,t}^{\alpha} u}{\partial t^{\alpha}}$, also, the Laplacian is considered, as a particular case of the operator A.

2.2 Statement of the problem and weak formulation

We are concerned with the existence and uniqueness of weak solution for the following time-fractional parabolic partial differential equation with homogeneous condition:

$$\frac{\partial_{0,t}^{\alpha} u}{\partial t^{\alpha}} - \Delta u = f, \quad in \quad Q = \Omega \times (0,T), \tag{2.2.1}$$

$$u(x;0) = 0, \tag{2.2.2}$$

$$Au = 0, \quad on \quad \Gamma \times (0;T), \tag{2.2.3}$$

Let $y_j(x)$ be an approximation of the exact solution $u(x,t_j)$, with a Lipschitz error, where $t_j = j\Delta t$, $0 \leqslant j \leqslant p$ and $\Delta t = \frac{T}{p}$.

Substituting (1.4.2) into equation (2.2.1), we get

$$-\Gamma(2-\alpha)\Delta y_j + \left(\sum_{k=0}^{j-1} \frac{(y_{j-k} - y_{j-k-1})}{\Delta t^{\alpha}} a_k\right) = \Gamma(2-\alpha) f \quad 0 < \alpha < 1.$$

For the sake of simplification let us $\Delta t = h$, then we can get because of $a_0 = 1$

$$-\Gamma(2-\alpha)\Delta y_j + \frac{y_j - y_{j-1}}{h^{\alpha}} = \Gamma(2-\alpha) f - \frac{1}{h^{\alpha}} \sum_{k=1}^{j-1} (y_{j-k} - y_{j-k-1})\, a_k \qquad 0 < \alpha < 1, \tag{2.2.4}$$

hence, we have to find the functions $y_j(x)$ $(j = 1, \ldots, p,)$, as solutions of the equations

$$-\Gamma(2-\alpha)\Delta y_j + \frac{y_j}{h^{\alpha}} = \Gamma(2-\alpha) f - \frac{1}{h^{\alpha}} \sum_{k=1}^{j-1} (y_{j-k} - y_{j-k-1})\, a_k + \frac{y_{j-1}}{h^{\alpha}} \qquad 0 < \alpha < 1, \tag{2.2.5}$$

with the boundary condition:

$$Ay_j = 0, \quad j = 1, \ldots, p. \tag{2.2.6}$$

The equation (2.2.4) is now multiplied by an arbitrary function v as well as integrate over Ω, we get the integral identities:

$$\begin{aligned} &\Gamma(2-\alpha) b_{\zeta}(v, y_j) + \frac{1}{h^{\alpha}} (v, y_j - y_{j-1}) \\ &= \left(v, \Gamma(2-\alpha) f - \frac{1}{h^{\alpha}} \sum_{k=1}^{j-1} (y_{j-k} - y_{j-k-1})\, a_k\right) \quad \forall v \in \vartheta, \quad 0 < \alpha < 1, \end{aligned} \tag{2.2.7}$$

where, $b_{\zeta}(.,.)$ is the bilinear form (1.3.1), i.e.

$$b_{\zeta}(v, u) = \int_{\Omega} \frac{\partial v}{\partial x_1} \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial v}{\partial x_2} \frac{\partial u}{\partial x_2} dx_2. \tag{2.2.8}$$

In the weak formulation, the problems (2.2.5)-(2.2.6) read:

Firstly we give a function $u_0(x) = 0$, and find the functions:

$$y_j \in \vartheta \quad \text{for } j = 1, \ldots, p, \tag{2.2.9}$$

where the integral identities are given by (2.2.7), i.e. by

$$\begin{aligned} &\Gamma(2-\alpha) b_\zeta(v, y_j) + \frac{1}{h^\alpha}(v, y_j - y_{j-1}) \\ &= \left(v, \Gamma(2-\alpha) f - \frac{1}{h^\alpha} \sum_{k=1}^{j-1} (y_{j-k} - y_{j-k-1})\, a_k \right) \quad \forall v \in \vartheta, \quad 0 < \alpha < 1. \end{aligned} \tag{2.2.10}$$

Definition 2.2.1 *Let j be fixed, if*

$$\left(\Gamma(2-\alpha) f - \frac{1}{h^\alpha} \sum_{k=1}^{j-1} (y_{j-k} - y_{j-k-1})\, a_k + \frac{y_{j-1}}{h^\alpha} \right) \in L_2(\Omega).$$

Then by a weak solution of the problem (2.2.5)-(2.2.6) we understand such a function $y_j \in \vartheta$ for which the integral identity

$$b_\zeta((v, u)) = \int_\Omega v \left(\Gamma(2-\alpha) f - \frac{1}{h^\alpha} \sum_{k=1}^{j-1} (y_{j-k} - y_{j-k-1})\, a_k + \frac{y_{j-1}}{h^\alpha} \right) dx, \qquad \forall v \in \vartheta \tag{2.2.11}$$

where

$$b_\zeta((v, u)) = b_\zeta(v, u) + \frac{1}{h^\alpha}(v, u) \quad 0 < \alpha < 1, \tag{2.2.12}$$

and $b_\zeta(v, u)$ given by (2.2.8).

Theorem 2.2.1 *Let h be fixed, $f \in L_2(\Omega)$, Let the form $b_\zeta(.,.)$ be bounded in $H^1(\Omega)$ and $\vartheta -$ elliptic. Then each of problems (2.2.5)-(2.2.6) solved successively for $j = 1, \ldots, p$, has precisely one weak solution, i.e. there exist precisely p functions $y_j \in \vartheta, j = 1, \ldots, p$, satisfying, subsequently, (2.2.11).*

For the proof we need the following lemmas.

Lemma 2.2.1 *Let j be fixed,*

$$\left(\Gamma(2-\alpha) f - \frac{1}{h^\alpha} \sum_{k=1}^{j-1} (y_{j-k} - y_{j-k-1})\, a_k + \frac{y_{j-1}}{h^\alpha} \right) \in L_2(\Omega).$$

Then if (1.3.1) *is bounded in* $H^1(\Omega)$ *and* $\vartheta -$ *elliptic, there exists one weak solution of the problem* (2.2.5)-(2.2.6), *i.e. precisely one function* $y_j \in \vartheta$ *satisfying* (2.2.11).

Proof. The proof is almost the same as that in [36, Theorem 3.1]. ■

Lemma 2.2.2 *Let* (1.3.1) *be bounded in* $H^1(\Omega)$ *and* $\vartheta -$ *elliptic. Then the same holds for the form* (2.2.12) *i.e. for*

$$b_\varsigma((v,u)) = b_\varsigma(v,u) + \frac{1}{h^\alpha}(v,u) \quad 0 < \alpha < 1.$$

Proof. First, in terms of S-C inequality, we have

$$|(v,u)| \leqslant \|u\|\|v\| \leqslant \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.$$

Thus, if

$$|b_\varsigma(u,v)| \leqslant \xi\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \quad \forall u,v \in H^1(\Omega),$$

it follows that

$$|b_\varsigma((v,u))| \leqslant |b_\varsigma(v,u)| + \frac{1}{h^\alpha}|(v,u)| \leqslant (1+\xi)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \quad \forall u,v \in H^1(\Omega).$$

Further, if

$$b_\varsigma(v,v) \geqslant \eta\|v\|^2_{H^1(\Omega)} \quad \forall v \in \vartheta$$

then the more

$$b_\varsigma((v,v)) \geqslant \eta\|v\|^2_{H^1(\Omega)} \quad \forall v \in \vartheta$$

because of $h > 0$. ■

Proof of Theorem 2.2.1

The form $b_\varsigma(.,.)$ is bounded and $\vartheta -$*elliptic*, this also applies to the form $b_\varsigma((v,u))$ according to Lemma 2.2.2. Further, $f \in L_2(\Omega)$, imply

$$\Gamma(2-\alpha)f \in L_2(\Omega).$$

Thus, according to Lemma 2.2.1, there exists one function $y_1 \in \vartheta$ satisfying (2.2.11) (for $j = 1$).

Now,

$$y_1 \in H^1(\Omega) \Rightarrow y_1 \in L_2(\Omega),$$

so that

$$\Gamma(2-\alpha)f + y_1\frac{(1-a_1)}{h^\alpha} \in L_2(\Omega),$$

thus, there exists one function $y_1 \in \vartheta$ satisfying (2.2.11) (for $j = 2$), etc and so proof is complete.

2.3 A priori estimates

Construct now the new function $u_1(x,t)$, that we will call α-Rothe function, defined for all $t \in I$ by:

$$u_1(x,t) = y_{j-1}(x) + \frac{y_j(x) - y_{j-1}(x)}{h^\alpha}(t - t_{j-1})^\alpha$$
$$\text{in } I_j = [t_{j-1}, t_j], j = 1, ..., p.$$

Instead of the division d_1, consider now the division $d_n, n = 2, 3, \ldots,$ subdividing the interval I into $2^{n-1}p$ equal subintervals $I_j^n = \left[t_{j-1}^n, t_j^n\right]$ of lengths $h_n = \dfrac{T}{2^{n-1}p}$, here $t_j^n = jh_n$, $\quad j = 0, 1, \ldots, 2^{n-1}p$. We obtain, in the same way as before, for each fixed n, the functions:

$$y_j^n \in \vartheta, \quad j = 1, \ldots, 2^{n-1}p,$$

satisfying the integral identities:

$$\begin{aligned} &\Gamma(2-\alpha)b_\varsigma(v, y_j^n) + \frac{1}{h^\alpha}\left(v, y_j^n - y_{j-1}^n\right) \\ &= \left(v, \Gamma(2-\alpha)f - \frac{1}{h^\alpha}\sum_{k=1}^{j-1}\left(y_{j-k}^n - y_{j-k-1}^n\right)a_k\right) \quad \forall v \in \vartheta, \quad 0 < \alpha < 1, \end{aligned} \tag{2.3.1}$$

considering that $y_0^n(x) \equiv 0$, therefore, we consider the function $u_n(x,t)$ defined for every $t \in I$ by:

$$u_n(x,t) = y_{j-1}^n + \frac{y_j^n - y_{j-1}^n}{h_n^\alpha}\left(t - t_{j-1}^n\right)^\alpha \quad \text{in} \quad I_j^n = \left[t_{j-1}^n, t_j^n\right] \tag{2.3.2}$$

$j = 1, \ldots, 2^{n-1}p$. Thus, we obtain a sequence:

$$\{u_n(x,t)\} \tag{2.3.3}$$

of functions (2.3.2); this sequence should converge (in a sense) to a function $u(x,t)$ that appears to be a solution to the problem (2.2.1)–(2.2.3). The investigations that follow are devoted to proving the above-mentioned assertions. Let's look into the first of the aforementioned issues, namely the convergence of the α-Rothe sequence (2.3.3). For this purpose, let us start by deriving some apriori estimates.

Lemma 2.3.1 *There exist two constants $c_1 > 0$ and $c_{2,\alpha} > 0$ such that*

$$\|y_j\| \leqslant c_1 \quad j = 1, \dots, p,$$

$$\|Y_j\| \leqslant c_{2,\alpha} \quad j = 1, \dots, p, \quad 0 < \alpha < 1,$$

where

$$Y_j(x) = \frac{y_j(x) - y_{j-1}(x)}{h^\alpha}.$$

Proof. Since u is Lipschitz, we have

$$\|y_j - y_{j-1}\| \leqslant \rho h \quad j = 1, \dots, p, \tag{2.3.4}$$

where ρ is the Lipschitz constant, (2.3.4) then gives

$$\|y_j\| \leqslant \|y_{j-1}\| + \rho h, \quad j = 1, \dots, p,$$

wherefrom

$$\|y_j\| \leqslant j\rho h, \quad j = 1, \dots, p,$$

because of $\|y_0\| = 0$. Now, $h = \dfrac{T}{p}$ implying finally

$$\|y_j\| \leqslant T\rho = c_1 \quad j = 1, \dots, p, \quad 0 < \alpha < 1.$$

Next, taking into account the fact that $0 < \alpha < 1$, then it follows immediately from (2.3.4) that

$$\|Y_j\| \leqslant \rho h^{1-\alpha} < \rho T^{1-\alpha} = c_{2,\alpha}, \quad j = 1, \dots, p,$$

and so our proof is complete. ■

Corollary 2.3.1 *What was deduced for division d_1, can be easily deduced for any arbitrary* *division d_n in a similar manner, i.e.*

$$\left\|y_j^n\right\| \leqq C_1, \quad 0<\alpha<1, \tag{2.3.5}$$

$$\left\|Y_j^n\right\| \leqslant C_{2,\alpha}, \quad 0<\alpha<1, \tag{2.3.6}$$

$j=1,\ldots,2^{n-1}p$, *where*

$$Y_j^n(x)=\frac{y_j^n(x)-y_{j-1}^n(x)}{h_n^\alpha}, \quad j=1,\ldots,p., \quad 0<\alpha<1. \tag{2.3.7}$$

Remark 2.3.1 *According to (2.3.5) and (2.3.6) we can conclude that the norms (in $L_2(\Omega)$) of the functions $y_j^n(x), Y_j^n(x)$ are bounded uniformly with regards to both j and n, as a result, regardless of division d_n.*

Corollary 2.3.2 *Even more easily, the same follows in the space $H^1(\Omega)$ for the norms of the functions y_j^n, namely there exist a constant $C_{3,\alpha}>0$ such that*

$$\left\|y_j^n\right\|_{H^1(\Omega)} \leqslant C_{3,\alpha} \quad 0<\alpha<1, \tag{2.3.8}$$

is valid for all j and n.

Proof. Let us write (2.3.1) in the form

$$\Gamma(2-\alpha)b_\zeta(v,y_j^n)+\left(v,Y_j^n\right)=(v,G^n) \quad \forall v\in v \quad 0<\alpha<1, \tag{2.3.9}$$

with

$$G^n=\Gamma(2-\alpha)f-\frac{1}{h_n^\alpha}\sum_{k=1}^{j-1}\left(y_{j-k}^n-y_{j-k-1}^n\right)a_k \quad 0<\alpha<1,$$

and Y_j^n given by (2.3.7). Put $v=y_j^n$ in (2.3.9), we get

$$\Gamma(2-\alpha)b_\zeta(y_j^n,y_j^n)+\left(y_j^n,Y_j^n\right)=\left(y_j^n,G^n\right) \quad 0<\alpha<1.$$

Now,

$$\left|\left(y_j^n,Y_j^n\right)\right| \leqq \left\|y_j^n\right\|\left\|Y_j^n\right\|, \quad \left|\left(y_j^n,G^n\right)\right| \leqslant \left\|y_j^n\right\|\left\|G^n\right\| \quad 0<\alpha<1, \tag{2.3.10}$$

$$\Gamma(2-\alpha)b_\zeta(y_j^n,y_j^n) \geqslant \eta\Gamma(2-\alpha)\left\|y_j^n\right\|_{H^1(\Omega)}^2 \quad 0<\alpha<1, \tag{2.3.11}$$

(according to (1.3.3)). From (2.3.10), (2.3.11), (2.3.5), and (2.3.6), the uniform boundedness of $\left\|y_j^n\right\|_{H^1(\Omega)}$, i.e. (2.3.8), immediately follows. ■

Instead of

$$\|v\|_{H^1(\Omega)}$$

the brief notation

$$\|v\|_{\vartheta} \tag{2.3.12}$$

will be used in the following text, where ϑ is the space defined by (1.1.3). If $v \in \vartheta$. Applying (2.3.12), (2.3.8) can be expressed in the form

$$\left\|y_j^n\right\|_{\vartheta} \leqslant C_{3,\alpha}, \quad 0 < \alpha < 1. \tag{2.3.13}$$

2.4 Convergence and existence

The α-Rothe functions (2.3.2) is an example of an abstract function (we frequently write to emphasize this fact

$$u_n(t) = y_{j-1}^n + \frac{y_j^n - y_{j-1}^n}{h_n^\alpha}\left(t - t_{j-1}^n\right)^\alpha \quad \text{in} \quad I_j^n = \left[t_{j-1}^n, t_j^n\right], \quad 0 < \alpha < 1, \tag{2.4.1}$$

$j = 1, \ldots, 2^{n-1}p$, instead of $u_n(x,t)$). In $L_2(I, \vartheta)$, the scalar product is given by (1.1.2), consequently, the norm given by

$$\|y\|_{L_2(I,\vartheta)}^2 = \int_I \|y(t)\|_{\vartheta}^2 \, \mathrm{d}t. \tag{2.4.2}$$

Here (cf .(2.3.12)), we write briefly, for $v, v_1, v_2 \in \vartheta$

$$(v_1, v_2)_{\vartheta}, \quad \text{or} \quad \|v\|_{\vartheta}$$

instead of

$$(v_1, v_2)_{H^1(\Omega)}, \quad \text{or} \quad \|v\|_{H^1(\Omega)}$$

respectively.

Corollary 2.4.1 *There exists $M_\alpha > 0$, depending only upon α such that*

$$\|u_n(t)\|_{L_2(I,\vartheta)} \leqslant M_\alpha, \quad 0 < \alpha < 1. \tag{2.4.3}$$

Proof. By (2.3.13), and since $0 \leqslant \left(\dfrac{t - t_{j-1}^n}{h_n}\right)^\alpha \leqslant 1$ in I_j^n, one has from (2.4.1) for an arbitrary $t \in I$ that

$$\begin{aligned}
\|u_n(t)\|_\vartheta &= \left\| y_{j-1}^n \left(1 - \frac{\left(t - t_{j-1}^n\right)^\alpha}{h_n^\alpha}\right) + y_j^n \frac{\left(t - t_{j-1}^n\right)^\alpha}{h_n^\alpha} \right\|_\vartheta \\
&\leqslant \left\| \left(1 - \frac{\left(t - t_{j-1}^n\right)^\alpha}{h_n^\alpha}\right) y_{j-1}^n \right\|_\vartheta + \left\| \frac{\left(t - t_{j-1}^n\right)^\alpha}{h_n^\alpha} y_j^n \right\|_\vartheta \\
&\leqslant \left(1 - \frac{\left(t - t_{j-1}^n\right)^\alpha}{h_n^\alpha}\right) C_{3,\alpha} + \frac{\left(t - t_{j-1}^n\right)^\alpha}{h_n^\alpha} C_{3,\alpha} = C_{3,\alpha}.
\end{aligned} \tag{2.4.4}$$

By (2.4.2) and (2.4.4), we then get, for $n = 1, 2, \ldots$

$$\|u_n(t)\|_{L_2(I,\vartheta)}^2 = \int_0^T \|u_n(t)\|_\vartheta^2 \, \mathrm{d}t \leqslant C_{3,\alpha}^2 T,$$

then, (2.4.3) holds with $M_\alpha = C_{3,\alpha}\sqrt{T}$. ■

Corollary 2.4.2 *the α-Rothe's sequence (2.3.3) admits a weakly convergent subsequence $\{u_{n_k}(t)\}$ in $L_2(I,\vartheta)$ to $u \in L_2(I,\vartheta)$ we write*

$$u_{n_k} \rightharpoonup u \quad in \quad L_2(I,\vartheta). \tag{2.4.5}$$

Proof. According to (2.4.3), the sequence (2.3.3) is bounded in $L_2(I,\vartheta)$. Because this is a Hilbert space, a subsequence

$$\{u_{n_k}(t)\}$$

can be found in this space that is weakly convergent to an abstract function

$$u \in L_2(I,\vartheta).$$

■

We'll show in the next section that this function is the required solution.

First, recall, the definition (2.3.7) of the functions Y_j^n, the functions (2.4.1) can be taken in the following format:

$$u_n(t) = y_{j-1}^n + Y_j^n \left(t - t_{j-1}^n\right)^\alpha \quad \text{in} \quad I_j^n. \tag{2.4.6}$$

Consider now the abstract functions

$$U_n^\alpha(t) : I \to L_2(\Omega), \quad n = 1, 2, \ldots$$

defined by

$$U_n^\alpha(t) = \begin{cases} \Gamma(\alpha+1) Y_1^n \;\; for \quad t = 0 \\ \Gamma(\alpha+1) Y_j^n \;\; for \quad t \in \tilde{I}_j^n = (t_{j-1}^n, t_j^n], \;\; j = 1, \ldots, 2^{n-1}p. \end{cases} \tag{2.4.7}$$

Corollary 2.4.3 *A subsequence* $\left\{U_{n_k}^\alpha(t)\right\}$ *and a function* $U^\alpha \in L_2\left(I, L_2(\Omega)\right)$ *can be found such that*

$$U_{n_k}^\alpha \rightharpoonup U^\alpha \quad in \quad L_2\left(I, L_2(\Omega)\right). \tag{2.4.8}$$

Proof. Due to (2.3.6), one immediately has that $\{U_n^\alpha(t)\}$ is bounded in the space $L_2\left(I, L_2(\Omega)\right)$, this space being a Hilbert space, consequently, there exists a subsequence $\left\{U_{n_k}^\alpha(t)\right\}$ weakly convergent to a function $U^\alpha \in L_2\left(I, L_2(\Omega)\right)$. ■

On the other hand, the link between the functions u and U^α is specified in the following result:

Lemma 2.4.1

$$_cD_{0,t}^\alpha u(t) = U^\alpha(t) \;\; in \;\; L_2(\Omega) \quad for\ a.\ a.\ t \in I. \tag{2.4.9}$$

Proof. By (2.4.6) and (2.4.7), one can check that

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} U_{n_k}^\alpha(\tau) \mathrm{d}\tau = u_{n_k}(t), \tag{2.4.10}$$

now, from Definition 1.2.4, (2.4.10) is none other than

$$D_{0,t}^{-\alpha} U_{n_k}^\alpha(t) = u_{n_k}(t),$$

wherefrom by theorem 1.2.2, we get

$$ {}_C D^{\alpha}_{0,t} u_{n_k}(t) = U^{\alpha}_{n_k}(t). \tag{2.4.11} $$

Applying now (1.2.2) to the sequence $\{u_{n_k}(t)\}$, we get in view of (2.4.5), (2.4.8), (2.4.11), and by the unicity of the weak limit the desired result. ■

Lemma 2.4.2 *If*

$$ u_{n_k} \rightharpoonup u \quad in \quad L_2(I, \vartheta), $$

then also

$$ \tilde{u}_{n_k} \rightharpoonup u \quad in \quad L_2(I, \vartheta), \tag{2.4.12} $$

where $\{\tilde{u}_n(t)\}$ *is the sequence defined by:*

$$ \tilde{u}_n(t) = \begin{cases} y^n_1 \;\; for \quad t = 0 \\ y^n_j \;\; in \;\; I^n_j = (t^n_{j-1}, t^n_j], j = 1, \dots, p. \end{cases} $$

Proof. Similar to the proof of [36, (11.141)], we omit it. ■

We are now ready to prove in which ways the function $u(t)$ satisfies (2.2.1).

Consider the integral identities (2.3.9) written for n_k

$$ \Gamma(2-\alpha) b_\zeta(v, y^{n_k}_j) + \left(v, Y^{n_k}_j\right) = (v, G^{n_k}) \quad \forall v \in v \quad 0 < \alpha < 1, $$

$j = 1, \dots, 2^{n_k - 1} p$. Consider an arbitrary function $v(t)$ from $L_2(I, \vartheta)$. Recalling the definition of the functions $\tilde{u}_n(t)$ and $U_n(t)$ we have

$$ \begin{gathered} \Gamma(2-\alpha) b_\zeta(v(t), \tilde{u}_{n_k}(t)) + \frac{1}{\Gamma(\alpha+1)} \left(v(t), U^{\alpha}_{n_k}(t)\right) = (v(t), G^{n_k}(t)), \\ 0 < \alpha < 1, \text{for a. a. } t \in [0, T]. \end{gathered} \tag{2.4.13} $$

Integrate (2.4.13) between 0 and T. We know

$$ \begin{aligned} &\Gamma(2-\alpha) \int_0^T b_\zeta(v(t), \tilde{u}_{n_k}(t)) \mathrm{d}t + \frac{1}{\Gamma(\alpha+1)} \int_0^T \left(v(t), U^{\alpha}_{n_k}(t)\right) \mathrm{d}t \\ &= \int_0^T (v(t), G^{n_k}(t)) \mathrm{d}t \quad 0 < \alpha < 1. \end{aligned} \tag{2.4.14} $$

Each of these integrals is well-defined since $v \in L_2(I,\vartheta)$ (and, consequently, also $v \in L_2(I, L_2(\Omega)))$,
$_{n_k} \in L_2(I,\vartheta), U^{\alpha}_{n_k} \in L_2(I, L_2(\Omega)), G^{n_k} \in L_2(I, L_2(\Omega))$. Now, for $n_k \to \infty$, we have

$$U^{\alpha}_{n_k} \rightharpoonup {}_C D^{\alpha}_{0,t} u \quad \text{in} \quad L_2(I, L_2(\Omega))$$

according to (2.4.8) and (2.4.9), so that

$$\frac{1}{\Gamma(\alpha+1)} \int_0^T \left(v(t), U^{\alpha}_{n_k}(t)\right) \mathrm{d}t \to \frac{1}{\Gamma(\alpha+1)} \int_0^T \left(v(t), {}_C D^{\alpha}_{0,t} u(t)\right) \mathrm{d}t$$
$$\text{for } n_k \to \infty.$$

Moreover, for $v \in L_2(I,\vartheta)$ be fixed

$$\int_0^T b_\zeta(v(t), s(t)) \mathrm{d}t \tag{2.4.15}$$

is a linear bounded function in $L_2(I,\vartheta)$. Linearity is obvious. Boundedness follows from (1.3.2), by which

$$\left(\int_0^T b_\zeta(v(t), s(t)) \mathrm{d}t\right)^2 \leqslant \xi^2 \left(\int_0^T \|v(t)\|_\vartheta \|s(t)\|_v \ \mathrm{d}t\right)^2$$
$$\leqslant \xi^2 \int_0^T \|v(t)\|^2_\vartheta \ \mathrm{d}t \cdot \int_0^T \|s(t)\|^2_\vartheta \ \mathrm{d}t = \xi^2 \|v\|^2_{L_2(I,\vartheta)} \|s\|^2_{L_2(I,\vartheta)}$$

(due to the Schwartz inequality), wherefrom (for $v(t)$ fixed)

$$\Gamma(2-\alpha) \int_0^T b_\zeta(v(t), s(t)) \mathrm{d}t \leqslant C_{4,\alpha} \|s\|_{L_2(I,\vartheta)}.$$

Thus, (2.4.15) is actually a bounded linear functional in $L_2(I,\vartheta)$. Consequently, from (2.4.12) it follows that

$$\Gamma(2-\alpha) \int_0^T b_\zeta(v(t), \tilde{u}_{n_k}(t)) \mathrm{d}t \to \Gamma(2-\alpha) \int_0^T b_\zeta(v(t), u(t)) \mathrm{d}t \ \ for \ \ n_k \to \infty.$$

In addition, the integral on the right side of (2.4.14) being equal to $\Gamma(2-\alpha) \displaystyle\int_0^T (v(t), f) \mathrm{d}t$, because $G^{n_k} \to \Gamma(2-\alpha) f$ for $n_k \to \infty$.

Summarizing, (2.4.14) yields, for $v \in L_2(I, \vartheta)$ and for $n_k \to \infty$

$$\begin{aligned} \Gamma(2-\alpha) \int_0^T b_\varsigma(v(t), u(t)) \mathrm{d}t + \frac{1}{\Gamma(\alpha+1)} \int_0^T \left(v(t), {}_C D_{0,t}^\alpha u(t)\right) \mathrm{d}t \\ = \Gamma(2-\alpha) \int_0^T (v(t), f) \mathrm{d}t. \end{aligned} \tag{2.4.16}$$

However in (2.4.14) $v(t)$ is chosen arbitrarily from $L_2(I, \vartheta)$. Thus, (2.4.16) is valid for all functions from $L_2(I, \vartheta)$, i.e.

$$\begin{aligned} \Gamma(2-\alpha) \int_0^T b_\varsigma(v(t), u(t)) \mathrm{d}t + \frac{1}{\Gamma(\alpha+1)} \int_0^T \left(v(t), {}_c D_{0,t}^\alpha u(t)\right) \mathrm{d}t \\ = \Gamma(2-\alpha) \int_0^T (v(t), f) \mathrm{d}t \quad \forall v \in L_2(I, \vartheta). \end{aligned}$$

In this sense, the differential equation (2.2.1) is satisfied by the function $u(t)$.

Let us review the above-mentioned properties of this function:

$$u \in L_2(I, \vartheta), \tag{2.4.17}$$

$${}_C D_{0,t}^\alpha u \in L_2\left(I, L_2(\Omega)\right), \tag{2.4.18}$$

$$\begin{aligned} \Gamma(2-\alpha) \int_0^T b_\varsigma(v(t), u(t)) \mathrm{d}t + \frac{1}{\Gamma(\alpha+1)} \int_0^T \left(v(t), {}_C D_{0,t}^\alpha u(t))\right) \mathrm{d}t \\ = \Gamma(2-\alpha) \int_0^T (v(t), f) \mathrm{d}t \quad \forall v \in L_2(I, \vartheta). \end{aligned} \tag{2.4.19}$$

Definition 2.4.1 *A function $u(t)$ with properties (2.4.17)–(2.4.19), is said to be a weak solution of (2.2.1)–(2.2.3).*
We've just demonstrated that such a weak solution exists.

2.5 Uniqueness

Consider two weak solutions u_1, u_2 to problem (2.2.1)–(2.2.3). Denote $u(t) = u_1(t) - u_2(t)$ their difference. $u\,(t)$ satisfies (2.4.17)–(2.4.19), due to the linearity of the considered spaces.

Further,

$$\Gamma\left(2-\alpha\right)\int_0^T b_\varsigma(v(t),u(t))dt+\frac{1}{\Gamma\left(\alpha+1\right)}\int_0^T\left(v\left(t\right),{}_CD_{0,t}^{\alpha}u(t)\right)dt=0$$
$$\forall v\in L_2\left(I,\vartheta\right). \tag{2.5.1}$$

Let $s\in I$ be arbitrary. For $v\left(t\right)$ in (2.5.1) choose the function

$$v\left(t\right)=\begin{cases}u\left(t\right) & \text{for}\quad 0\leqslant t\leqslant s\\ 0 & \text{for}\quad s\leqslant t\leqslant T.\end{cases} \tag{2.5.2}$$

By lemma 1.2.1, we have

$$\int_0^s\left(u\left(t\right),{}_CD_{0,t}^{\alpha}u(t)\right)dt\geqslant\frac{1}{2}\int_0^s {}_CD_{0,t}^{\alpha}\left\|u\right\|^2dt\geqslant 0. \tag{2.5.3}$$

Hence, we obtain by (2.5.1)–(2.5.3), and the fact that
$\displaystyle\int_0^a b_\varsigma(u(t),u(t))dt\geqslant 0$, that $\left\|u\left(s\right)\right\|=0$. However, s has been chosen arbitrarily. Thus

$$u\left(t\right)=0\quad\text{in } I,$$

which was to be proved.

Chapter 3

Solvability for time fractional parabolic equation with nonhomogeneous conditions.

We'll now turn to the more general problem of nonhomogeneous initial and boundary conditions:

$$\frac{\partial_{0,t}^{\alpha} u}{\partial t^{\alpha}} - \Delta u = f, \quad in \quad Q = \Omega \times (0, T), \tag{3.0.1}$$

$$u(x; 0) = u_0(x), \tag{3.0.2}$$

$$Au = 0, \quad on \quad \Gamma \times (0; T), \tag{3.0.3}$$

here, it is assumed that

$$f \in L_2(\Omega), \tag{3.0.4}$$

$$u_0 \in L_2(\Omega), \tag{3.0.5}$$

The problem (3.0.1)–(3.0.3) being linear, its solution can be found as the sum of the solutions of the following two problems:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \Delta u = f, \quad in \quad Q = \Omega \times (0,T), \tag{3.0.6}$$

$$u(x;0) = 0, \tag{3.0.7}$$

$$Au = 0, \quad on \quad \Gamma \times (0;T). \tag{3.0.8}$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \Delta u = 0, \quad in \quad Q = \Omega \times (0,T), \tag{3.0.9}$$

$$u(x;0) = u_0(x), \tag{3.0.10}$$

$$Au = 0, \quad on \quad \Gamma \times (0;T). \tag{3.0.11}$$

The problem (3.0.6)-(3.0.8) was treated in the previous chapter, so, we shall be concerned only with the problems (3.0.9)–(3.0.11).

3.1 Nonhomogeneous initial conditions

Let us consider the problem (3.0.9)–(3.0.11). Applying the method of discretization in time with step $h = \frac{T}{p}$, one has to find:

$$y_j \in \vartheta \tag{3.1.1}$$

for $j = 1, \ldots, p$, satisfying the integral identities

$$\begin{aligned} &\Gamma(2-\alpha) b_\varsigma(v, y_j) + \frac{1}{h^\alpha}(v, y_j - y_{j-1}) \\ &= -\frac{1}{h^\alpha}\left(v, \sum_{k=1}^{j-1} (y_{j-k} - y_{j-k-1})\, a_k\right) \quad \forall v \in \vartheta, \quad 0 < \alpha < 1, \end{aligned} \tag{3.1.2}$$

with

$$y_0 = u_0 \in L_2(\Omega). \tag{3.1.3}$$

Theorem 3.1.1 *Under the assumptions (1.3.2), (1.3.3), (3.0.5) each of the problems (3.1.1), (3.1.2) has exactly one solution.*

Proof. The proof is similar as in theorem 2.2.1 . ■

Similarly as in the previous chapter, consider the α-Rothe function $u_1(t) = u_1(x,t)$ defined by

$$u_1(t) = y_{j-1} + (t - t_{j-1})^{\alpha} Y_j^{\alpha} \quad \textbf{in } I_j = [t_{j-1}, t_j], j = 1, \ldots, p, \tag{3.1.4}$$

where

$$Y_j^{\alpha} = \frac{y_j(x) - y_{j-1}(x)}{h^{\alpha}}. \tag{3.1.5}$$

In the same way, we can create for the division d_n, the α-Rothe function $u_n(t) = u_n(x,t)$ defined by

$$u_n(t) = y_{j-1}^n + \left(t - t_{j-1}^n\right)^{\alpha} Y_j^{n,\alpha} \quad \textbf{in } I_j^n = \left[t_{j-1}^n, t_j^n\right], j = 1, \ldots, 2^{n-1}p \tag{3.1.6}$$

with

$$Y_j^{n,\alpha} = \frac{y_j^n(x) - y_{j-1}^n(x)}{h^{\alpha}}. \tag{3.1.7}$$

If

$$u_0 \in H^2(\Omega) \cap \vartheta, \tag{3.1.8}$$

it is natural to apply the subtitution

$$u = u_0 + y \tag{3.1.9}$$

and to convert the given problem into a problem with right-hand side $-(-\Delta u_0)$ and with zero initial condition.

It is also assumed that

$$y \in H^2(\Omega) \cap \vartheta, v \in \vartheta \Rightarrow -\Delta y \in L_2(\Omega), \quad b_{\varsigma}(v,y) = (v, -\Delta y). \tag{3.1.10}$$

In particular, for u_0 satisfying (3.1.8) this means that

$$-\Delta u_0 \in L_2(\Omega), \quad b_{\varsigma}(v,u_0) = (v, -\Delta u_0) \quad \forall v \in \vartheta. \tag{3.1.11}$$

Let us first consider (3.1.2) written for the division d_n

$$\begin{aligned} &\Gamma(2-\alpha) b_{\varsigma}(v, y_j^n) + \frac{1}{h^{\alpha}}\left(v, y_j^n - y_{j-1}^n\right) \\ &= -\frac{1}{h_n^{\alpha}}\left(\sum_{k=1}^{j-1}\left(v, y_{j-k}^n - y_{j-k-1}^n\right) a_k\right) \quad \forall v \in \vartheta, \quad 0 < \alpha < 1. \end{aligned} \tag{3.1.12}$$

Putting $v = y_j^n$ and noting that $b_\zeta(y_j^n, y_j^n) \geqslant 0$, we get

$$\|y_j^n\| \leqslant \|y_{j-1}^n\| + \sum_{k=1}^{j-1} \left(\|y_{j-k}^n\| + \|y_{j-k-1}^n\|\right) a_k \quad j = 1, ..., p. \tag{3.1.13}$$

wich yields

$$\|y_j^n\| \leqslant \sum_{k=1}^{j-1} \|y_{j-k}^n\| \left(a_k + a_{k-1}\right) + \|y_0^n\| \, a_{j-1} \quad j = 1, ..., p. \tag{3.1.14}$$

Since (3.1.14) yields

$$\|y_1^n\| \leqslant \|y_0^n\| = \|u_0\| \tag{3.1.15}$$

for j=1, we have

$$\|y_j^n\| \leqslant \lambda_\alpha \|u_0\| \tag{3.1.16}$$

(uniformly for j and n), where λ_α is a constant depending only on α. From (3.1.6) we can then get as in Corollary 2.4.1 that

$$\|u_n(t)\|_{L_2(I,\vartheta)} \leqslant \lambda_\alpha^{'} \|u_0\| \quad \forall t \in I, \tag{3.1.17}$$

where $\lambda_\alpha^{'} = \lambda_\alpha \sqrt{T}$. The inequality (3.1.17) represents the continuous dependence , in $C(I, L_2(G))$, of the norms of the $\alpha-$ Rothe functions $u_n(t)$ on $\|u_0\|$.

Since (1.3.1) is ϑ-elliptic, a set M exists [43], dense in ϑ and consequently in $L_2(\Omega)$, with the following property : If $s \in M$, then precisely one $g \in L_2(\Omega)$ exists such that

$$b_\zeta(v, s) = (v, g) \quad \forall v \in \vartheta \tag{3.1.18}$$

is fulfilled. First. Let

$$u_0 = s \in M. \tag{3.1.19}$$

In this case, the first of the integral identities (3.1.12) becomes

$$\Gamma(2-\alpha) b_\zeta(v, y_1^n) + \frac{1}{h_n^\alpha} (v, y_1^n - s) = 0 \quad \forall v \in \vartheta. \tag{3.1.20}$$

Substituting

$$y_1^n = \tilde{y}_1^n + s \tag{3.1.21}$$

and taking (3.1.18) into account, we get

$$\Gamma(2-\alpha)b_\zeta(v,\tilde{y}_1^n)+\frac{1}{h_n^\alpha}(v,\tilde{y}_1^n)=-\Gamma(2-\alpha)(v,g)\quad \forall v\in\vartheta. \tag{3.1.22}$$

Similarly, putting

$$y_j^n=\tilde{y}_j^n+s,\quad j=1,...,p, \tag{3.1.23}$$

the integral identities (3.1.12) become

$$\begin{aligned}&\Gamma(2-\alpha)b_\zeta(v,\tilde{y}_j^n)+\frac{1}{h_n^\alpha}\left(v,\tilde{y}_j^n-\tilde{y}_{j-1}^n\right)\\ &=\left(v,-\Gamma(2-\alpha)g-\frac{1}{h_n^\alpha}\sum_{k=1}^{j-1}\left(\tilde{y}_{j-k}^n-\tilde{y}_{j-k-1}^n\right)a_k\right)\quad \forall v\in\vartheta,\quad 0<\alpha<1.\end{aligned} \tag{3.1.24}$$

j=2,...,p. (3.1.22), (3.1.24) are precisely the integral identities (2.3.1) in previous chapter with $f\in L_2(\Omega)$ replaced by $-g\in L_2(\Omega)$, thus corresponding to the problem (reminding that $\tilde{y}_1^n=0$ by (3.1.19) and (3.1.21))

$$\frac{\partial^\alpha\tilde{u}}{\partial t^\alpha}-\Delta\tilde{u}=-g,\quad in\quad Q=\Omega\times(0,T), \tag{3.1.25}$$

$$\tilde{u}(x;0)=0, \tag{3.1.26}$$

$$A\tilde{u}=0,\quad on\quad \Gamma\times(0;T). \tag{3.1.27}$$

Applying the results obtained in the previous chapter, it is possible to assert that the sequence of abstract functions $\tilde{u}_n(t)$, defined in I_j^n by

$$\tilde{u}_n(t)=\tilde{y}_{j-1}^n+\frac{\tilde{y}_j^n-\tilde{y}_{j-1}^n}{h_n^\alpha}\left(t-t_{j-1}^n\right)^\alpha \tag{3.1.28}$$

converges weakly in $L_2(I,\vartheta)$ and strongly in $C(I,L_2(\Omega))$ to a (unique) function $\tilde{u}(t)$, the so-called weak solution of the problem (3.1.25)-(3.1.27). The function $\tilde{u}(t)$ satisfies (cf. (2.4.17)-(2.4.19))

$$\tilde{u}\in L_2(I,\vartheta), \tag{3.1.29}$$

$${}_cD_{0,t}^\alpha\tilde{u}\in L_2\left(I,L_2(\Omega)\right), \tag{3.1.30}$$

$$\tilde{u}(0)=0\quad \text{in}\quad C\left(I,L_2(\Omega)\right), \tag{3.1.31}$$

and fulfills the integral identity

$$\begin{aligned}\Gamma(2-\alpha)\int_0^T b_\varsigma(v(t),\tilde{u}(t))\mathrm{d}t+\frac{1}{\Gamma(\alpha+1)}\int_0^T\big(v(t),{}_cD_{0,t}^{\alpha}\tilde{u}(t)\big)\,\mathrm{d}t\\=-\Gamma(2-\alpha)\int_0^T(v(t),g)\mathrm{d}t\quad\forall v\in L_2(I,\vartheta).\end{aligned}\tag{3.1.32}$$

Then the function

$$u(t)=\tilde{u}(t)+s\tag{3.1.33}$$

is the weak limit in $L_2(I,\vartheta)$ and strong limit in $C\left(I,L_2(\Omega)\right)$, of the α-Rothe sequence $\{u_n(t)\}$ defined by (3.1.6), has the properties (3.1.29)-(3.1.30), with

$$u(0)=s\quad in\quad C\left(I,L_2(\Omega)\right)\tag{3.1.34}$$

instead of (3.1.31), and satisfies the integral identity

$$\begin{aligned}\Gamma(2-\alpha)\int_0^T b_\varsigma(v(t),u(t)-s)\mathrm{d}t+\frac{1}{\Gamma(\alpha+1)}\int_0^T\big(v(t),{}_cD_{0,t}^{\alpha}\left(u(t)-s\right)\big)\,\mathrm{d}t\\=-\Gamma(2-\alpha)\int_0^T(v(t),g)\mathrm{d}t\quad\forall v\in L_2(I,\vartheta),\end{aligned}\tag{3.1.35}$$

i.e., the integral identity

$$\Gamma(2-\alpha)\int_0^T b_\varsigma(v(t),u(t))\mathrm{d}t+\frac{1}{\Gamma(\alpha+1)}\int_0^T\big(v(t),{}_cD_{0,t}^{\alpha}(u(t)\big)\,\mathrm{d}t=0\tag{3.1.36}$$

since $b_\varsigma(v,s)=(v,g)$ and ${}_cD_{0,t}^{\alpha}(s)=0$ (s being independent of t).

Definition 3.1.1 *We call the function $u(t)$ a weak solution of the problem (3.0.9)-(3.0.11) with*

$$u(0)=s\in M.\tag{3.1.37}$$

Chapter 4

Solvability for time fractional parabolic equation with an integral operator.

This chapter deals with the existence and uniqueness of a weak solution for two-dimensional time fractional with an integral operator.

4.1 Statement of the problem

The problem is as follows:

$$\begin{aligned}
&\partial_{0t}^{\alpha} u(x,t) - \Delta u = \int_0^t \mathcal{K}(x,s,u(x,s))\mathrm{d}s + f(x) \quad \text{ in } Q = \Omega \times (0,T), \\
&u(x,0) = 0, \\
&Au = 0 \quad \text{ on } \partial\Omega \times (0;T),
\end{aligned} \tag{4.1.1}$$

here, the kernel is supposed to satisfy:

$$\begin{aligned}
\|\mathcal{K}(x,t_2,u_2) - \mathcal{K}(x,t_1,u_1)\| \leqslant C \left(|t_2 - t_1| + \frac{1}{h^{\alpha-1}} \|u_2 - u_1\| \right), \\
u_1, u_2 \in L_2(\Omega), \quad t_1, t_2 \in I = [0,T].
\end{aligned} \tag{4.1.2}$$

$$u \in \vartheta \Rightarrow \mathcal{K} \in L_2(\Omega). \tag{4.1.3}$$

Where C is a constant and ϑ is the space defined by (1.1.3).

4.2 Motivation

Firstly, it is worth noting that the kernel can represent a variety of physical or chemical phenomena.

4.2.1 Motivation in nanomaterials

Applying equation (4.1.1) to the thermal properties of nanosystems could involve the following points:

Non-Fourier Conduction: At the nanoscale, heat conduction may not be instantly proportional to the local temperature gradient, which is contrary to Fourier's law. Fractional models make it possible to describe these differences by taking into account delayed interactions and long-range effects.

Confinement Effects: Nanomaterials, such as nanotubes or nanoparticles, have dimensions that are comparable to the mean free path of phonons (the heat carriers in materials). This can lead to abnormal phonon dispersion phenomena, affecting thermal conductivity.

Phonon-Phonon and Phonon-Defect Interactions: Interactions between phonons or between phonons and lattice defects can be strongly modified at the nanoscale, leading to complex heat transport processes that can be better described by equations fractional.

Materials with Thermal Memory: Some nanomaterials can exhibit thermal hysteresis or thermal path dependence, where the history of heating and cooling affects current thermal properties. Fractional derivatives are well suited to model such memory effects.

4.3 Weak formulation and α-Rothe sequence

Substituting (1.4.2) into (4.1.1) , we get

$$\frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{j-1} \left(y_{j-k}(x) - y_{j-k-1}(x)\right) a_k - \Delta y_j(x) = h \sum_{l=0}^{j-1} \mathcal{K}_l + f, \quad 0 < \alpha < 1, \tag{4.3.1}$$

here, $\mathcal{K}_j(x) = \mathcal{K}(x, jh, u_j(x)) \quad j = 1, \ldots, p$. Simplifying by eliminating x from $y_j(x)$, (4.3.1) can be expressed as follows:

$$-\Gamma(2-\alpha)\Delta y_j + \frac{y_j - y_{j-1}}{h^\alpha} = -\left(\sum_{k=1}^{j-1} \frac{y_{j-k} - y_{j-k-1})}{h^\alpha} a_k\right)$$
$$+\Gamma(2-\alpha)h\sum_{l=0}^{j-1}\mathcal{K}_l + \Gamma(2-\alpha)f, \quad 0<\alpha<1.$$

Hence, we obtain

$$\begin{aligned}
&\Gamma(2-\alpha)b_\zeta(v,y_j) + \frac{1}{h^\alpha}(v, y_j - y_{j-1})\\
&= \left(v, -\frac{1}{h^\alpha}\sum_{k=1}^{j-1}(y_{j-k} - y_{j-k-1})\, a_k + \Gamma(2-\alpha)h\sum_{l=0}^{j-1}\mathcal{K}_l + \Gamma(2-\alpha)f\right)\\
&\quad \forall v \in \vartheta, \quad 0<\alpha<1.
\end{aligned} \tag{4.3.2}$$

As in the previous chapters, let the function α-Rothe function $u_1(t) = u_1(x,t)$ defined by:

$$u_1(t) = y_{j-1} + \frac{y_j - y_{j-1}}{h^\alpha}(t - t_{j-1})^\alpha \quad 0<\alpha<1, \quad \textbf{in } I_j = [t_{j-1}, t_j], j = 1, \ldots, p.$$

Similarly, we have for any divisions $d_n, n = 2, 3\ldots,$ with

$$h_n = \frac{T}{2^{n-1}p}$$

the α-Rothe sequence:

$$\{u_n(t)\} \tag{4.3.3}$$

of functions:

$$u_n(t) = y_{j-1}^n + \frac{y_j^n(x) - y_{j-1}^n(x)}{h_n^\alpha}\left(t - t_{j-1}^n\right)^\alpha \quad 0<\alpha<1. \tag{4.3.4}$$

4.4 A priori estimates

Lemma 4.4.1 *There exist two constants C_1, C_2 independent of α and j such that*

$$\|y_j\| \leqslant C_1, \quad j = 1, \ldots, p, \tag{4.4.1}$$

$$\|Y_j\| \leqslant C_2, \quad j = 1, \ldots, p, \tag{4.4.2}$$

with

$$Y_j(x) = \frac{y_j(x) - y_{j-1}(x)}{h^\alpha}.$$

Proof. First, since u is Lipschitz and $y_0 = 0$, one has

$$\|y_j\| \leqslant \|y_1\| + \|y_2 - y_1\| + \ldots + \|y_j - y_{j-1}\| \leqslant \rho j h \leqslant \rho T = C_1,$$

where ρ is the constant of Lipschitz .

Next, let $v = Y_1$ in (4.3.2) for $j = 1$, we get

$$\Gamma(2-\alpha) b_\zeta(Y_1, y_1) + \frac{1}{h^\alpha}(Y_1, y_1) = \Gamma(2-\alpha)(Y_1, h\mathcal{K}_0 + f), \quad 0 < \alpha < 1.$$

Hence, since $b_\zeta(y_1, Y_1) = \dfrac{1}{h^\alpha} b_\zeta(y_1, y_1) \geqslant 0$ we obtain by S-C inequality that

$$\|Y_1\| \leqslant \Gamma(2-\alpha)(h\|\mathcal{K}_0\| + \|f\|). \tag{4.4.3}$$

Subtract (4.3.2) for $j = 1$ from their own for $j = 2$, hence we have

$$\Gamma(2-\alpha) b_\zeta(v, y_2 - y_1) + (v, Y_2 - Y_1) = (v, -a_1 Y_1 + \Gamma(2-\alpha) h\mathcal{K}_1 + f) \quad \forall v \in \vartheta,$$

from where, putting $v = Y_2$, then we can get

$$\|Y_2\| \leqslant (a_0 - a_1)\|Y_1\| + \Gamma(2-\alpha) h \|\mathcal{K}_1\|. \tag{4.4.4}$$

Similarly, we obtain

$$\|Y_3\| \leqslant (a_0 - a_1)\|Y_2\| + (a_1 - a_2)\|Y_1\| + \Gamma(2-\alpha) h \|\mathcal{K}_2\|.$$

Thus, similarly as above we can estimate

$$\|Y_j\| \leqslant \sum_{k=1}^{j-1} (a_{k-1} - a_k)\|Y_{j-k}\| + \Gamma(2-\alpha) h \|\mathcal{K}_{j-1}\|. \tag{4.4.5}$$

Adding up (4.4.3)–(4.4.5) yields

$$\|Y_j\| \leqslant \Gamma(2-\alpha)\psi_0 \|f\| + \Gamma(2-\alpha)h\left(\psi_0 \|\mathcal{K}_0\| + \psi_1 \|\mathcal{K}_1\| + \ldots + \psi_{j-1} \|\mathcal{K}_{j-1}\|\right) \tag{4.4.6}$$

with

$$\psi_l = (a_0 - a_1)^{j-(l+1)} + \sum_{k=1}^{j-(l+2)} (a_0 - a_1)^{j-k-(l+2)}(a_k - a_{k-1})(j-k-(l+1)),$$
$$\text{for all } \; l = 0, 1, \ldots, j-1.$$

Now, we can easily proof that $\psi_0 < \psi_1 < \ldots < \psi_{j-1} = 1$, so (4.4.6) yields

$$\|Y_j\| \leqslant \Gamma(2-\alpha)\left[\|f\| + h\left(\|\mathcal{K}_0\| + \|\mathcal{K}_1\| + \ldots + \|\mathcal{K}_{j-1}\|\right)\right]. \tag{4.4.7}$$

In the other hand, by using the fact that $y_0 = 0$, we have from (4.1.2)

$$\begin{aligned}
\|\mathcal{K}_1\| &\leqslant \|\mathcal{K}_0\| + \|\mathcal{K}_1 - \mathcal{K}_0\| \\
&\leqslant \|\mathcal{K}_0\| + C\left(h + \frac{\|y_1\|}{h^{\alpha-1}}\right) \\
&= \|\mathcal{K}_0\| + Ch\left(1 + \|Y_1\|\right),
\end{aligned}$$

then,

$$\begin{aligned}
\|\mathcal{K}_2\| &\leqslant \|\mathcal{K}_0\| + \|\mathcal{K}_2 - \mathcal{K}_0\| \\
&\leqslant \|\mathcal{K}_0\| + C\left(2h + \frac{\|y_2\|}{h^{\alpha-1}}\right) \\
&\leqslant \|\mathcal{K}_0\| + Ch\left(2 + \frac{\|y_2 - y_1\|}{h^{\alpha}} + \frac{\|y_1\|}{h^{\alpha}}\right) \\
&= \|\mathcal{K}_0\| + Ch\left(2 + \|Y_1\| + \|Y_2\|\right) \\
&= \|\mathcal{K}_0\| + Ch\left(2 + \|Y_1\| + \|Y_2\|\right),
\end{aligned}$$

hence, we can conclude from an analogy argument that

$$\begin{aligned}
\|\mathcal{K}_{j-1}\| &\leqslant \|\mathcal{K}_0\| + \|\mathcal{K}_{j-1} - \mathcal{K}_0\| \\
&\leqslant \|\mathcal{K}_0\| + Ch\left(j - 1 + \|Y_1\| + \|Y_2\| + \ldots + \|Y_{j-1}\|\right).
\end{aligned} \tag{4.4.8}$$

Adding up the above results into (4.4.7), we have

$$\|Y_j\| \leqslant \Gamma(2-\alpha)\left[\|f\| + jh\,\|\mathcal{K}_0\| + Ch^2\,(1+2+\ldots+j-1)\right]$$
$$+\,Ch^2\left[(j-1)\,\|Y_1\| + (j-2)\,\|Y_2\| + \ldots + \|Y_{j-1}\|\right].$$

Now,

$(j-1)\,h < ph = T$. So

$$\|Y_j\| \leqslant \Gamma(2-\alpha)\left[\|f\| + T\,\|\mathcal{K}_0\| + CT^2\,(1+2+\ldots+j-1)\right]$$
$$+\,CTh\,(\|Y_1\| + \|Y_2\| + \ldots + \|Y_{j-1}\|) \quad j = 1,2,\ldots p.$$

Using Lemma 1.1.2 , we get, finally

$$\|Y_j\| \leqslant \Gamma(2-\alpha)\left(\|f\| + T\,\|\mathcal{K}_0\| + CT^2\right)e^{CT(j-1)h}$$
$$\leqslant \Gamma(2-\alpha)\left(\|f\| + T\,\|\mathcal{K}_0\| + CT^2\right)e^{CT^2}, \quad j = 1,2,\ldots p.$$

Then (4.4.2) holds with $C_2 = \Gamma(2-\alpha)\left(\|f\| + T\,\|K_0\| + CT^2\right)e^{CT^2}$, and so our proof is complete. ■

Remark 4.4.1 *Since (4.4.1), (4.4.2) are independent of h they rest valid for all every division $d_n, n = 1,2,\ldots$. Consequently, we obtain*

$$\left\|y_j^n\right\| \leqslant c_1, \tag{4.4.9}$$

$$\left\|Y_j^n\right\| \leqslant c_2, \tag{4.4.10}$$

for all $n = 1,2,\ldots,$ and $1 \leqslant j \leqslant 2^{n-1}p$, where

$$Y_j^n(x) = \frac{y_j^n(x) - y_{j-1}(x)}{h_n^\alpha}. \tag{4.4.11}$$

By the definition (1.1.3) of the space ϑ, from (4.4.9) and (1.3.3), we arrive at the following:

Corollary 4.4.1 *There exists a constant c_3 such that*

$$\left\|y_j^n\right\|_\vartheta \leqslant c_3, \tag{4.4.12}$$

for all $n = 1, 2, \ldots,$ *and* $1 \leqslant j \leqslant 2^{n-1}p$.

Lemma 4.4.2 *There exists* c_4 *independent of* j *and* n *such that*

$$\left\|\mathcal{K}_j^n\right\| \leqslant c_4.$$

where $\mathcal{K}_j^n(x) = \mathcal{K}\left(x, jh_n, y_j^n(x)\right)$.

Proof. From (4.4.8) and for the division d_n, we know

$$\left\|\mathcal{K}_j^n\right\| \leqslant \|\mathcal{K}_0^n\| + Ch_n\left(j + \|Y_1^n\| + \|Y_2^n\| + \ldots + \left\|Y_j^n\right\|\right),$$

which gives from (4.4.10)

$$\left\|\mathcal{K}_j^n\right\| \leqslant \|\mathcal{K}_0^n\| + CT + CTc_2 = c_4. \tag{4.4.13}$$

■

4.5 Exixtence of weak solution

Let's first introduce:

Lemma 4.5.1 *The* α*-Rothe sequence (4.3.3) admits a subsequence* $\{u_{n_k}(t)\}$ *weakly convergent in* $L_2(I, \vartheta)$ *to a function* $u \in L_2(I, \vartheta)$*, we write*

$$u_{n_k} \rightharpoonup u \quad \text{in} \;\; L_2(I, \vartheta). \tag{4.5.1}$$

Proof. The proof is the same as that of Corollary 2.4.2. ■

Remark 4.5.1 *[36] Note that besides (4.5.1) it holds*

$$u_{n_k} \to u \quad \text{in} \quad C(I, L_2(\Omega)). \tag{4.5.2}$$

Corollary 4.5.1 *There exists a subsequence* $\left\{\mathbb{Z}_{n_k}^\alpha(t)\right\}$*and a function* $\mathbb{Z}^\alpha$ *such that*

$$\mathbb{Z}_{n_k}^\alpha \rightharpoonup \mathbb{Z}^\alpha \quad \text{in} \;\; L_2\left(I, L_2(\Omega)\right), \tag{4.5.3}$$

where

$$\mathbb{Z}_n^\alpha(t) : I \to L_2(\Omega), \quad n = 1, 2, \ldots$$

defined by

$$\mathbb{Z}_n^\alpha(t) = \begin{cases} \Gamma(\alpha+1)Y_1^{n,\alpha} \;\; for \quad t = 0, \\ \Gamma(\alpha+1)Y_j^{n,\alpha} \;\; for \quad t \in \widetilde{I}_j^n = (t_{j-1}^n, t_j^n], \;\; j = 1, \ldots, 2^{n-1}p. \end{cases} \tag{4.5.4}$$

taking into account (4.3.4), (4.5.4) and using the same reasoning as in the proof of Lemma 2.4.1, we arrived at the following Lemma:

Lemma 4.5.2

$$ {}_cD_{0,t}^\alpha u(t) = \mathbb{Z}^\alpha(t) \;\; in \;\; L_2\left(I, L_2(\Omega)\right) \quad for \; a. \; a. \; t \in I. \tag{4.5.5}$$

Corollary 4.5.2 *The function $u(t)$ verifies*

$$u(0) = 0 \quad in \;\; C\left(I, L_2(\Omega)\right), \tag{4.5.6}$$

$$u \in AC\left(I, L_2(\Omega)\right). \tag{4.5.7}$$

Proof. From (4.3.4), (4.5.4) we have by the Riemann-Liouville integral definition that

$$D_{0,t}^{-\alpha}\mathbb{Z}_{n_k}^\alpha(t) = u_{n_k}(t),$$

Now, using the same reasoning as in (4.5.5)'s proof, we obtain

$$D_{0,t}^{-\alpha}\mathbb{Z}^\alpha(t) = u(t),$$

finally, by Theorem 1.2.1, we obtain the desired result. ■

Lemma 4.5.3 *[36] If*

$$u_{n_k} \rightharpoonup u \quad in \;\; L_2(I, \vartheta),$$

then also

$$\widetilde{u}_{n_k} \rightharpoonup u \quad in \;\; L_2(I, \vartheta), \tag{4.5.8}$$

where $\{\widetilde{u}_n(t)\}$ *is the sequence defined by:*

$$\widetilde{u}_n(t) = \begin{cases} y_1^n \;\; for\; t = 0 \\ y_j^n \;\; in \;\; I_j^n = (t_{j-1}^n, t_j^n],\; j = 1, \ldots, p. \end{cases} \tag{4.5.9}$$

Let

$$e(t) = e(x,t) = \mathcal{K}(x,t,u(x,t)), \tag{4.5.10}$$

since $u \in \vartheta$ for a. a. $t \in I$ we have, by (4.1.3) $e \in L_2(\Omega)$ for a.a. $t \in I$, consequently we have the following corollary:

Corollary 4.5.3

$$\lim_{n_k \to \infty} \widetilde{e}_{n_k}(t) = e(t) \;\; in \;\; L_2(\Omega) \;\; for\; a.a.\; t \in I, \tag{4.5.11}$$

where

$$\widetilde{e}_n(t) : I \to L_2(\Omega), \quad n = 1, 2, \ldots$$

defined by

$$\widetilde{e}_n(t) = \begin{cases} \mathcal{K}_0^n \quad for \quad t = 0, \\ \mathcal{K}_{j-1}^n \quad in \quad \widetilde{I}_j^n = (t_{j-1}^n, t_j^n],\; j = 1, \ldots, 2^{n-1}p. \end{cases}$$

Proof. Let $\varepsilon > 0$, we show that

$$\|e(t) - \widetilde{e}_{n_k}(t)\| \leqslant \varepsilon \quad \text{for a.a. } t \in I \quad \text{if } n_k > n_0(\varepsilon). \tag{4.5.12}$$

By (4.1.2) we have for a fixed division d_{n_k} and for an arbitrary $t \in \widetilde{I}_{j+1}^{n_k}$

$$\begin{aligned} \|e(t) - \widetilde{e}_{n_k}(t)\| &= \left\|\mathcal{K}(x,t,u(t)) - \mathcal{K}\left(x, jh_{n_k}, y_j^{n_k}\right)\right\| \\ &\leqslant C\left(|t - jh_{n_k}| + \frac{1}{h_{n_k}^{\alpha-1}}\left\|u(t) - y_j^{n_k}\right\|\right). \\ &\leqslant C\left[h_{n_k} + \|u(t) - u\left(jh_{n_k}\right)\| + \left\|u\left(jh_{n_k}\right) - y_j^{n_k}\right\|\right]. \end{aligned}$$

Choosing $\eta = \dfrac{\varepsilon}{3C}$, hence, by (4.5.2) and (4.5.7), we get the desired result. ∎

Remark 4.5.2 *From* (4.5.11), (4.5.12) *we can immediately define*

$$\mathbb{E}(t) = \int_0^t e(s)ds, \tag{4.5.13}$$

$$\mathbb{E}_n(t) = \int_0^t \widetilde{e}_n(s)ds. \tag{4.5.14}$$

The demonstration of the following corollary is almost the same as in [36]. Conveniently, we provide the proof.

Corollary 4.5.4 *Define*

$$\widetilde{\mathbb{E}}_n(t) = \begin{cases} h_n \mathcal{K}_0^n \ \ for \quad t = 0, \\ h_n \left(\mathcal{K}_0^n + \ldots + \mathcal{K}_{j-1}^n\right) \ \ in \ \widetilde{I}_j^n, \end{cases} \tag{4.5.15}$$

then

$$\lim_{n_k \to \infty} \widetilde{\mathbb{E}}_{n_k} = \mathbb{E} \quad \ in \ \ L_2(I, L_2(\Omega)). \tag{4.5.16}$$

Proof. First, from (4.5.12) we have immediately

$$\lim_{n_k \to \infty} \mathbb{E}_{n_k} = \mathbb{E} \quad \text{in} \ \ L_2(I, L_2(\Omega)). \tag{4.5.17}$$

Now, from (4.5.14), (4.5.15) we get, for $t \in \widetilde{I}_j^n$

$$\begin{aligned} &\mathbb{E}_n(t) - \widetilde{\mathbb{E}}_n(t) = \int_0^{h_n} \widetilde{e}_n(s)\mathrm{d}s + \ldots + \int_{(j-2)h_n}^{(j-1)h_n} \widetilde{e}_n(s)\mathrm{d}s + \\ &+ \int_{(j-1)h_n}^{t} \widetilde{e}_n(s)\mathrm{d}s - \widetilde{\mathbb{E}}_n(t) = h_n \left(\mathcal{K}_0^n + \ldots + \mathcal{K}_{j-2}^n\right) + \\ &+ \left[t - (j-1)h_n\right] \mathcal{K}_{j-1}^n - h_n \left(\mathcal{K}_0^n + \ldots + \mathcal{K}_{j-1}^n\right) = (t - jh_n)\, \mathcal{K}_{j-1}^n. \end{aligned}$$

Hence, for $t \in \widetilde{I}_j^n$ it holds that

$$\left\| \mathbb{E}_n(t) - \widetilde{\mathbb{E}}_n(t) \right\| \leqslant h_n \left\| \mathcal{K}_{j-1}^n \right\|.$$

Combining (4.5.17), (4.4.13), thus we complete the proof. ■

Consequently, according to (4.4.11), and for the division d_n, (4.3.2) can be expressed as:

$$\begin{aligned} &\Gamma(2-\alpha) b_\zeta(v, y_j^n) + \left(v, \sum_{k=0}^{j-1} Y_{j-k}^n a_k\right) \\ &= \Gamma(2-\alpha)\left(v, h_n \left(\mathcal{K}_0^n + \mathcal{K}_1^n + \ldots \mathcal{K}_{j-1}^n\right) + \Gamma(2-\alpha) f\right) \quad \forall v \in \vartheta, \quad 0 < \alpha < 1. \end{aligned} \tag{4.5.18}$$

Let $v(t) \in L_2(I,\vartheta)$. Using (4.5.4), (4.5.9), and (4.5.15), and let $f(t)$ be a function from I into $L_2(\Omega)$ by $f(t) = f \quad \forall t \in I$, then (4.5.18) becomes

$$\Gamma(2-\alpha)b_\zeta(v,\widetilde{u}_n)+\frac{1}{\Gamma(\alpha+1)}\left(v,\sum_{l=0}^{j-1}a_l\mathbb{Z}_n^\alpha\right)=\Gamma(2-\alpha)\left(v,\widetilde{\mathbb{E}}_n\right)+\Gamma(2-\alpha)\left(v,f\right)\quad \text{for a.a. t} \in I. \tag{4.5.19}$$

From (4.5.1) take the indices n_k only and integrate (4.5.19), we know

$$\begin{aligned}\Gamma(2-\alpha)\int_0^T b_\zeta(v,\widetilde{u}_{n_k})dt+\frac{1}{\Gamma(\alpha+1)}\sum_{l=0}^{j-1}a_l\int_0^T\left(v,\mathbb{Z}_{n_k}^\alpha\right)dt\\ =\Gamma(2-\alpha)\int_0^T\left(v,\widetilde{\mathbb{E}}_{n_k}\right)dt+\Gamma(2-\alpha)\int_0^T\left(v,f\right)dt.\end{aligned} \tag{4.5.20}$$

Now,

$$v \in L_2(I,\vartheta) \Rightarrow v \in L_2(I, L_2(\Omega)).$$

Thus, (4.5.3), (4.5.5) imply

$$\lim_{n_k\to\infty}\frac{1}{\Gamma(\alpha+1)}\int_0^T\left(v,\sum_{l=0}^{j-1}a_l\mathbb{Z}_{n_k}^\alpha\right)dt=\frac{1}{\Gamma(\alpha+1)}\sum_{l=0}^{j-1}a_l\int_0^T\left(v,{}_cD_{0,t}^\alpha u\right)dt.$$

Similarly, by (4.5.16)

$$\lim_{n_k\to\infty}\Gamma(2-\alpha)\int_0^T\left(v,\widetilde{\mathbb{E}}_{n_k}\right)dt=\Gamma(2-\alpha)\int_0^T\left(v,\mathbb{E}\right)dt,$$

where $\mathbb{E}(t)=\int_0^t\mathcal{K}(x,s,u(x,s))ds$, by (4.5.10), (4.5.13). Finally, for $v \in L_2(I,\vartheta)$ fixed,

$$\int_0^T b_\zeta(v,u)dt$$

is a bounded functional in $L_2(I,\vartheta)$, thus

$$\lim_{n_k\to\infty}\Gamma(2-\alpha)\int_0^T b_\zeta(v,\widetilde{u}_{n_k})dt=\Gamma(2-\alpha)\int_0^T b_\zeta(v,u)dt$$

holds by (4.5.8). Consequently, for $n_k \to \infty$, (4.5.20) gives

$$\int_0^T b_\zeta(v,u)dt + \frac{1}{\Gamma(\alpha+1)} \sum_{l=0}^{j-1} a_l \int_0^T \left(v, {}_cD_{0,t}^\alpha u\right) dt = \Gamma(2-\alpha) \int_0^T (v, \mathbb{E})\, dt + \Gamma(2-\alpha) \int_0^T (v, f)\, dt. \tag{4.5.21}$$

The function $v \in L_2(I, \vartheta)$ having been chosen arbitrarily, (4.5.21) holds for every $v \in L_2(I, \vartheta)$. In this sense, (4.1.1) is fulfilled. hence, we have

$$u \in L_2(I, \vartheta) \cap AC\left(I, L_2(\Omega)\right), \tag{4.5.22}$$

$${}_cD_{0,t}^\alpha u(t) \in L_2\left(I, L_2(\Omega)\right), \tag{4.5.23}$$

$$u(0) = 0 \ \text{ in } \ C\left(I, L_2(\Omega)\right), \tag{4.5.24}$$

$$\begin{aligned} &\int_0^T b_\zeta(v,u)dt + \frac{1}{\Gamma(\alpha+1)} \sum_{l=0}^{j-1} a_l \int_0^T \left(v, {}_cD_{0,t}^\alpha u\right) dt \\ &= \Gamma(2-\alpha) \int_0^T (v, \mathbb{E})\, dt + \Gamma(2-\alpha) \int_0^T (v, f)\, dt \quad \forall v \in L_2(I, \vartheta). \end{aligned} \tag{4.5.25}$$

Definition 4.5.1 *A function $u(t)$ satsfying (4.5.22) - (4.5.25) is called a weak solution to our problem.*

4.6 Uniqueness

Let u_1 and u_2 be two weak solutions such that $u_1 \neq u_2$, let

$$u^* = u_1 - u_2.$$

We have to show that $u^* = 0$.

By (4.5.22) – (4.5.25), $u^*(t)$ satisfies

$$\begin{aligned} &u^* \in L_2(I, \vartheta), \\ &u^* \in AC\left(I, L_2(\Omega)\right), \\ &{}_cD_{0,t}^\alpha u^* \in L_2\left(I, L_2(\Omega)\right), \\ &u^*(0) = 0 \ \text{ in } \ C\left(I, L_2(\Omega)\right), \end{aligned}$$

$$\begin{aligned}&\Gamma(2-\alpha)\int_0^T b_\zeta(v,u^*)\mathrm{d}t+\int_0^T\left(v,\sum_{l=1}^{j-1}a_{lc}D_{0,t}^\alpha u^*\right)\mathrm{d}t\\&=\Gamma(2-\alpha)\int_0^T\left(v,\int_0^t\left[\mathcal{K}\left(x,s,u_2(s)\right)-\mathcal{K}\left(x,s,u_1(s)\right)\right]\mathrm{d}s\right)\mathrm{d}t\quad\forall v\in L_2(I,\vartheta).\end{aligned}\tag{4.6.1}$$

According to (4.1.2), we have, first

$$\begin{aligned}&\left\|\int_0^t\left[\mathcal{K}\left(x,s,u_2(s)\right)-\mathcal{K}\left(x,s,u_1(s)\right)\right]\right\|\mathrm{d}s\\&\leqslant\int_0^t\left\|\mathcal{K}\left(x,s,u_2(s)\right)-\mathcal{K}\left(x,s,u_1(s)\right)\right\|\mathrm{d}s\\&\leqslant\int_0^t\frac{C}{h^{\alpha-1}}\left\|u_2(s)-u_1(s)\right\|\mathrm{d}s=\frac{C}{h^{\alpha-1}}\int_0^t\|u^*(s)\|\mathrm{d}s\leqslant C\int_0^t\|u^*(s)\|\mathrm{d}s.\end{aligned}\tag{4.6.2}$$

Dividing I into subintervals of lengths l. Since $\|u(t)\|$ is continuous in $[0,l]$ $(u(t)\in C\left(I,L_2(\Omega)\right))$ there exists $t_1\in[0,l]$ such that

$$\max_{t\in[0,l]}\|u^*(t)\|=\left\|u^*\left(t_1\right)\right\|.\tag{4.6.3}$$

For $v(t)$ in (4.6.1), choose the function

$$v\left(t\right)=\begin{cases}u^*\left(t\right)\ \text{for}\ \ t\in[0,t_1],\\ \ 0\quad\ \text{for}\ \ t\in(t_1,T],\end{cases}$$

thus, we have

$$\begin{aligned}&\Gamma(2-\alpha)\int_0^{t_1}b_\zeta(u^*,u^*)\mathrm{d}t+\sum_{l=1}^{j-1}a_l\int_0^T\left(u^*,{}_cD_{0,t}^\alpha u^*\right)\mathrm{d}t\\&=\Gamma(2-\alpha)\int_0^{t_1}\left(u^*,\int_0^t\left[\mathcal{K}\left(x,s,u_2(s)\right)-\mathcal{K}\left(x,s,u_1(s)\right)\right]\mathrm{d}s\right)\mathrm{d}t.\end{aligned}\tag{4.6.4}$$

From the $\vartheta-$ellipticity of (1.3.1), we have first, that

$$\Gamma(2-\alpha)\int_0^{t_1}b_\zeta(u^*,u^*)\mathrm{d}t\geqslant 0.\tag{4.6.5}$$

Additionally, according to Lemma 1.2.1 one has

$$\int_0^{t_1}\left(u^*,\sum_{l=1}^{j-1} a_{l c}D^{\alpha}_{0,t}u\right)\mathrm{d}t \geqslant \frac{\sum_{l=1}^{j-1} a_l}{2}\int_0^{t_1} {}_cD^{\alpha}_{0,t}\left\|u\right\|^2 = c_{5c}D^{\alpha}_{0,t}\left\|y(t_1)\right\|^2 \tag{4.6.6}$$

with c_5 is a constant. Now, (4.6.2) yields

$$\begin{aligned}
&\left\|\Gamma(2-\alpha)\int_0^{t_1}\left(u^*,\int_0^t\left[\mathcal{K}\left(x,s,u_2(s)\right)-\mathcal{K}\left(x,s,u_1(s)\right)\right]\mathrm{d}s\right)\mathrm{d}t\right\| \\
&\leqslant \Gamma(2-\alpha)\int_0^{t_1}\left\|u^*(t_1)\right\|\left\|\int_0^t\left[\mathcal{K}\left(x,s,u_2(s)\right)-\mathcal{K}\left(x,s,u_1(s)\right)\right]\right\|\mathrm{d}s \\
&\leqslant \Gamma(2-\alpha)\int_0^l\left\|u^*(t_1)\right\|\left(C\int_0^l\|u^*(s)\|\mathrm{d}s.\right)\leqslant Cl^2\left\|u^*(t_1)\right\|^2.
\end{aligned} \tag{4.6.7}$$

Combining (4.6.4) – (4.6.7) , we can get

$$ {}_cD^{\alpha}_{0,t}\left\|u^*(t_1)\right\|^2 \leqslant c_6\left\|u^*(t_1)\right\|^2$$

where c_6 is a constant. Using Lemma 1.2.2, we get from where, with regard to (4.5.6),

$$\|u^*(t_1)\| = 0,$$

(4.6.3) then gives

$$u^*(t) = 0 \ \text{ in } L_2(\Omega) \quad \forall t\in[0,l]. \tag{4.6.8}$$

Applying the same procedure now in $[l, 2l]$ with

$$v\left(t\right) = \begin{cases} u^*\left(t\right) \ \text{for } t\in[0,t_2], \\ 0 \quad \text{for } t \ \in(t_2,T], \end{cases}$$

where t_2 is a point at which $\|u(t)\|$ attains its maximum in $[l, 2l]$, and using the just obtained result (4.6.8), we get

$$u^*(t) = 0 \quad \text{in } L_2(\Omega) \quad \forall t\in[l,2l].$$

Following a finite number of steps , we thus come to

$$u^*(t) = 0 \quad \text{in } L_2(\Omega) \quad \forall t\in I,$$

hence the uniqueness follows.

Conclusion and perspectives

Two objectives were successfully achieved in this study. Firstly, a new function was constructed using the discretization method for a time-fractional equation. Secondly, the proposed method was validated by demonstrating the convergence of the $\alpha-$ -Rothe sequence to the unique weak solution of several problems. As we have seen, TFDE is closely linked to various nanoscience phenomena. Consequently, the authors are optimistic that the obtained results can be extended to a wide range of problems across various scientific fields, such as mechanics, economics, and engineering. This work not only provides a robust mathematical framework but also opens new avenues for interdisciplinary research. Future studies could explore the application of this method to more complex systems, particularly those involving multi-scale phenomena or non-linear dynamics.

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