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NEW FIXED-POINT RESULTS FOR MULTI-VALUED
F-CONTRACTIONS OF HARDY-ROGERS-TYPE IN b -METRIC
SPACES WITH APPLICATIONS

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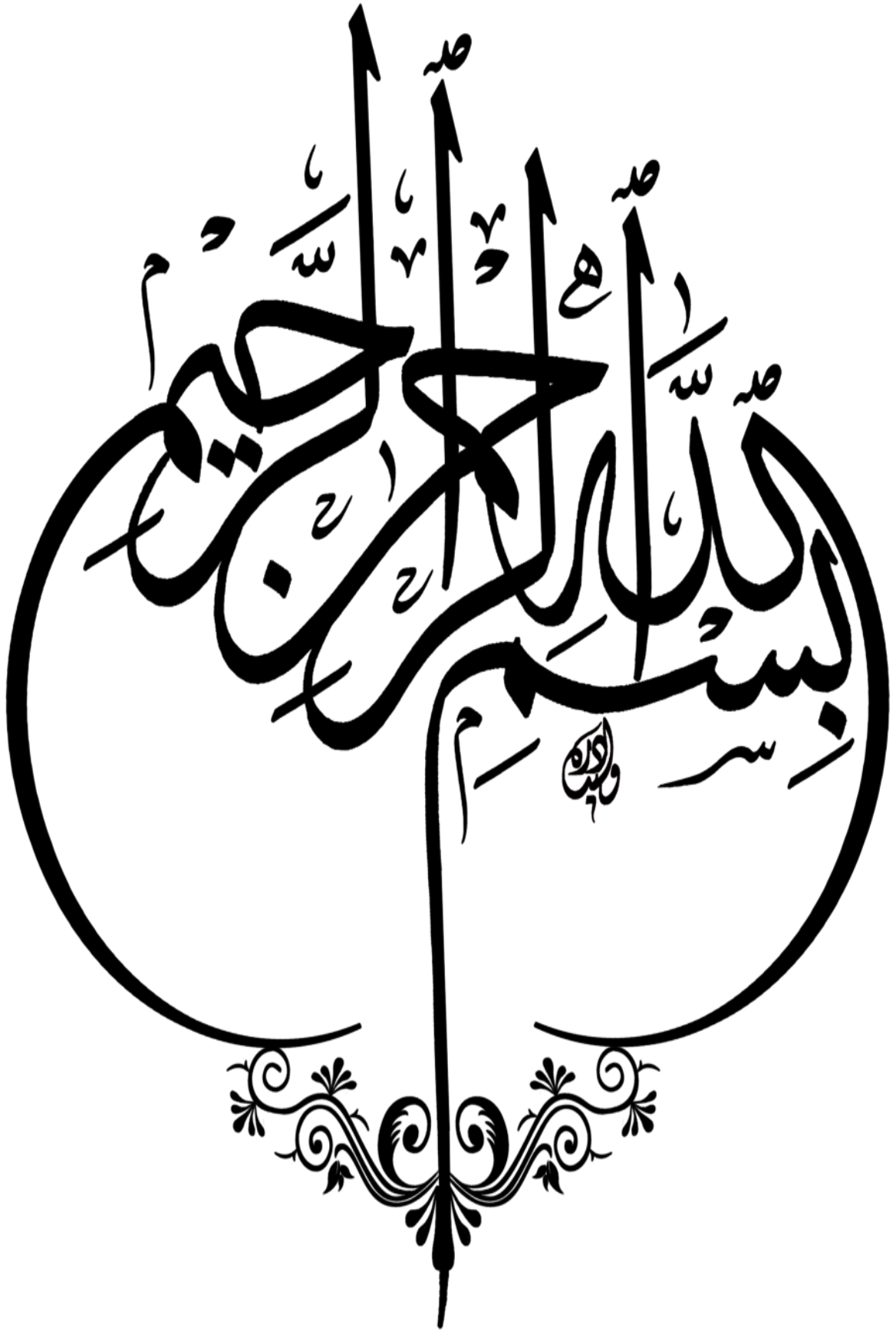
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أَلْحَمْدُ لِلَّهِ الَّذِي بِفَضْلِهِ تَتِمُّ الصَّالِحَاتُ وَ الصَّلَاةُ وَ السَّلَامُ عَلَى رَسُولِ اللَّهِ حَبِيبِنَا مُحَمَّدٍ
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*D*edication

*W*e dedicate this fruit of our long years of study first of all to :

*O*ur dear parents, who are the light of our life, who suffered and sacrificed so much to make us happy, for their advice, affection and encouragement.

*T*hank you for all your efforts for us, may God keep you, protect you, and bless your life.

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Abstract

The purpose of this thesis is to study the existence and the uniqueness of fixed point in some theorems for multi-valued mappings, which satisfy an F -contractions condition of Hardy-Rogers type with respect to the functional H , in the setting of complete b -metric spaces. An application, for the solution of integral inclusion, are given to illustrate the usability of the obtained results.

Such that " H " represent the Hausdorff metric which is a tool to measure distance between two sets of a metric space.

Keywords: multi-valued mappings, F -contractions mappings, Fixed point, b -metric spaces, Hausdorff metric.



المُلخَص

الغرض من هذه المذكرة هو دراسة وجود و وحدانية النقطة الثابتة على مستوى بعض تطبيقات فضاء مُتعدّد القيم تحت شروط تقلّصات هاردي روجر في فضاء b متري مُتكامل مع احترام خواص الدالة H .

حيث تُمثّل " H " المسافة بين مجموعتين من فضاء متري و تُدعى " $Hausdorff\ metric$ ".
الكلمات المفتاحية: فضاء مُتعدّد القيم، النقطة الثابتة، فضاء b متري، تقلّصات هاردي روجر.

*N*otation

- \mathbb{N} the sets of positive integers.
- \mathbb{R} the sets of real numbers.
- X a nonempty set.
- $K(X)$ the set of all nonempty compact subsets of X .
- $C(X)$ the set of all nonempty closed subsets of X .
- $C([0, r], \mathbb{R})$ the space of all continuous real-valued functions on $[0, r]$.
- $\mathcal{F}_c = \{F : (0, \infty) \rightarrow \mathbb{R}, F \text{ is nondecreasing continuous function}\}$.
- $\mathcal{S}_1 = \{\tau : (0, \infty) \rightarrow (0, \infty), \liminf_{t \rightarrow \eta^+} \tau(t) > 0, \text{ for all } \eta \geq 0\}$.
- $Fix(T)$ the fixed point set of T .
- $SFix(T)$ the strict fixed point set of T .

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*I*ntroduction

The Banach contraction mappings principle [1] is considered to be one of the most useful tools in fixed point theory. Since this principle has a lot of applications in different branches of mathematics. It has been extended, generalized and improved in many directions by considering different forms of mappings or various types of spaces. In 1973, Hardy and Rogers [2] gave a generalization of Reich fixed point theorem. In [3], an interesting generalization of Banach contraction principle is given by introducing the concept of F -contraction. After that, the notion of F -contraction of Hardy–Rogers is introduced in [4] as a generalization of F -contraction in complete metric spaces. One of the most prevalent generalization of the metric spaces was given in the article [5] through the notion of b -metric spaces. Henrikson has shown that if a metric space X is complete, then the collection of all nonempty closed bounded subsets of X is also complete with respect to the Hausdorff metric. Barich has proved that if a metric space X is complete, then the collection of all nonempty compact subsets of X is also complete with respect to the Hausdorff metric.

Nadler [6] was the first author who combined the notion of contraction with multi-valued mapping. Thus, the theory of fixed points for multi-valued mappings has been developed after the famous paper of him, it has an important role in various branches of pure and applied mathematics because of its many applications, for instance, in real and complex analysis as well as in optimal control problems. Over the years, this theory has increased its significance and hence in the literature there are many papers focusing on the discussion of abstract and practical problems involving multi-valued mappings. As a matter of fact, one of the most interesting various approaches utilized to develop this theory is based on methods of fixed point theory, often in view of the constructive character of fixed point theorems, especially in its metric branch (see, [7]).

Later on, many authors have been used different Hardy-Rogers contractive type conditions and gave their generalizations, extensions and applications ([6, 8, 9, 10,]).

CHAPTER

1

PRELIMINARIES

In this chapter, basing on [20], we recall some known definitions and results which will be used in the sequel. Throughout this paper, X will denote a nonempty set and the Picard sequence of a self-mapping $T : X \rightarrow X$ based on an arbitrary $x_0 \in X$ is given by $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$, where T^n denotes the n^{th} -iterates of T .

1.1 b-Metric Spaces

In 1989, Bakhtin [5] introduced the concept of b -metric spaces as a generalization of the metric spaces in the sense that the triangle inequality contains a suitable constant $s \geq 1$ (see also Czerwik [11]). Since then, several published papers have dealt with b -metric spaces and fixed point theory in the setting of b -metric spaces (see [12, 13, 14]). Note that the topological framework of a b -metric space with the topology induced by its convergence was studied.

We will first recall the definition of a b -metric space.

Definition 1.1.1 : Let X be a nonempty set and let $s \geq 1$ be a given real number. A mapping $\sigma : X \times X \rightarrow [0, \infty)$ is said to be a b -metric if, for all x, y, z in X , the following conditions hold:

- (b₁) $\sigma(x, y) = 0$ if and only if $x = y$,
- (b₂) $\sigma(x, y) = \sigma(y, x)$,
- (b₃) $\sigma(x, z) \leq s[\sigma(x, y) + \sigma(y, z)]$.

The pair (X, σ) is called a b -metric space with constant $s \geq 1$.

It is obvious from the above definition that the class of b -metric spaces is larger than that of metric spaces, since a b -metric space is a metric space when $s = 1$ but the converse is not true. The following classical examples illustrate this fact.

Example 1.1 : Let (X, d) be a metric space and let the mapping $\sigma_d : X \times X \rightarrow [0, \infty)$ be defined by

$$\sigma_d(x, y) = (d(x, y))^p, \text{ for all } x, y \in X,$$

where $p > 1$ is a fixed real number. Then (X, σ_d) is a b -metric space with $s = 2^{p-1}$.

In particular, if $X = \mathbb{R}$, $d(x, y) = |x - y|$ is the usual Euclidean metric and

$$\sigma_d(x, y) = (x - y)^2, \text{ for all } x, y \in \mathbb{R},$$

then (\mathbb{R}, σ_d) is a b -metric with $s = 2$. However, (\mathbb{R}, σ_d) is not a metric space on \mathbb{R} since (b₃) does not hold. Indeed,

$$\sigma_d(-2, 2) = 16 > 8 = 4 + 4 = \sigma_d(-2, 0) + \sigma_d(0, 2).$$

Example 1.2 : Let X be the set of Lebesgue measurable functions on $[0, 1]$ such that

$$\int_0^1 |f(x)|^2 dx < \infty.$$

Define $D : X \times X \rightarrow [0, \infty)$ by

$$D(f, g) = \int_0^1 |f(x) - g(x)|^2 dx,$$

then D satisfies the following properties

1. $D(f, g) = 0$ if and only if $f = g$,
2. $D(f, g) = D(g, f)$, for any $f, g \in X$,
3. $D(f, g) \leq 2(D(f, h) + D(h, g))$, for any points $f, g, h \in X$.

Clearly, (X, D) is a b -metric space with $s = 2$ but is not a metric space. For example, take $f(x) = 0, g(x) = 1$ and $h(x) = \frac{1}{2}$, for all $x \in [0, 1]$. Then

$$D(0, 1) = 1 > \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = D(0, \frac{1}{2}) + D(\frac{1}{2}, 1).$$

We present now the concepts of convergence, Cauchy sequence and completeness in b -metric spaces.

Definition 1.1.2 : Let (X, σ) be a b -metric space. Then a sequence $\{x_n\}$ in X is called:

- (a) convergent if and only if there exists $x \in X$ such that $\lim_{n \rightarrow \infty} \sigma(x_n, x) = 0$ and in this case we write $\lim_{n \rightarrow \infty} x_n = x$.
- (b) Cauchy if and only if $\lim_{n, m \rightarrow \infty} \sigma(x_n, x_m) = 0$.

Definition 1.1.3 : The b -metric space (X, σ) is said complete if every Cauchy sequence in X converges in X .

Remark 1.1.1 : In a b -metric space, the following assertions hold:

- (i) a convergent sequence has a unique limit,
- (ii) each convergent sequence is Cauchy.

Lemma 1.1.1 : Let (X, σ) be a b -metric space with constant $s \geq 1$ and $\{x_n\}, \{y_n\}$ two sequences such that $\lim_{n \rightarrow \infty} x_n = x, \lim_{n \rightarrow \infty} y_n = y$ in (X, σ) . Then $\lim_{n \rightarrow \infty} \sigma(x_n, y_n) = 0$ if and only if $x = y$.

It is worth recalling that a b -metric is generally not continuous. The following lemmas are very useful to manage this problem.

Lemma 1.1.2 : Let (X, σ) be a b -metric space with constant $s \geq 1$ and $\{x_n\}$ be a convergent sequence in X with $\lim x_n = x$. Then for each $y \in X$, we have

$$\frac{1}{s}\sigma(x, y) \leq \liminf_{n \rightarrow \infty} \sigma(x_n, y) \leq \limsup_{n \rightarrow \infty} \sigma(x_n, y) \leq s\sigma(x, y).$$

Lemma 1.1.3 : Let (X, σ) be a b -metric space with constant $s \geq 1$ and let $\{x_n\}$ be a sequence in X such that

$$\lim_{n \rightarrow \infty} \sigma(x_n, x_{n+1}) = 0. \quad (1.1)$$

If $\{x_n\}$ is not a Cauchy sequence in (X, σ) , then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that the following items hold:

$$\begin{aligned} \varepsilon &\leq \liminf_{k \rightarrow \infty} \sigma(x_{m(k)}, x_{n(k)}) \leq \limsup_{k \rightarrow \infty} \sigma(x_{m(k)}, x_{n(k)}) \leq s\varepsilon; \\ \frac{\varepsilon}{s} &\leq \liminf_{k \rightarrow \infty} \sigma(x_{m(k)}, x_{n(k)+1}) \leq \limsup_{k \rightarrow \infty} \sigma(x_{m(k)}, x_{n(k)+1}) \leq s^2\varepsilon; \\ \frac{\varepsilon}{s} &\leq \liminf_{k \rightarrow \infty} \sigma(x_{m(k)+1}, x_{n(k)}) \leq \limsup_{k \rightarrow \infty} \sigma(x_{m(k)+1}, x_{n(k)}) \leq s^2\varepsilon; \\ \frac{\varepsilon}{s^2} &\leq \liminf_{k \rightarrow \infty} \sigma(x_{m(k)+1}, x_{n(k)+1}) \leq \limsup_{k \rightarrow \infty} \sigma(x_{m(k)+1}, x_{n(k)+1}) \leq s^3\varepsilon. \end{aligned}$$

Inspired by the works in [16], we can state the following lemma.

Lemma 1.1.4 : Let all the conditions of Lemma 1.1.3 be satisfied. Then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that the following items hold:

$$\begin{aligned} \varepsilon^+ &\leq \liminf_{k \rightarrow \infty} \sigma(x_{m(k)}, x_{n(k)}) \leq \limsup_{k \rightarrow \infty} \sigma(x_{m(k)}, x_{n(k)}) \leq s\varepsilon^+; \\ \frac{\varepsilon}{s} &\leq \liminf_{k \rightarrow \infty} \sigma(x_{m(k)}, x_{n(k)+1}) \leq \limsup_{k \rightarrow \infty} \sigma(x_{m(k)}, x_{n(k)+1}) \leq s^2\varepsilon; \\ \frac{\varepsilon}{s} &\leq \liminf_{k \rightarrow \infty} \sigma(x_{m(k)+1}, x_{n(k)}) \leq \limsup_{k \rightarrow \infty} \sigma(x_{m(k)+1}, x_{n(k)}) \leq s^2\varepsilon; \\ \frac{\varepsilon}{s^2} &\leq \liminf_{k \rightarrow \infty} \sigma(x_{m(k)+1}, x_{n(k)+1}) \leq \limsup_{k \rightarrow \infty} \sigma(x_{m(k)+1}, x_{n(k)+1}) \leq s^3\varepsilon. \end{aligned}$$

Proof: If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and sequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of positive integers such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and $\sigma(x_{m(k)}, x_{n(k)}) > \varepsilon$, this implies that $\sigma(x_{m(k)}, x_{n(k)-1}) \leq \varepsilon$ for all $k \geq 1$. Using the relaxed triangle inequality (b_3), we have:

$$\begin{aligned} \sigma(x_{m(k)}, x_{n(k)}) &\leq s\sigma(x_{m(k)}, x_{n(k)-1}) + s\sigma(x_{n(k)-1}, x_{n(k)}) \\ &\leq s\varepsilon + s\sigma(x_{n(k)-1}, x_{n(k)}), \end{aligned}$$

this leads to

$$\frac{1}{s}\sigma(x_{m(k)}, x_{n(k)}) \leq \varepsilon + \sigma(x_{n(k)-1}, x_{n(k)}).$$

Since $\sigma(x_{n(k)-1}, x_{n(k)}) > 0$, then by taking limit superior as $k \rightarrow \infty$ with (1.1), we get

$$\frac{1}{s} \limsup_{k \rightarrow \infty} \sigma(x_{m(k)}, x_{n(k)}) \leq \varepsilon^+,$$

equivalently,

$$\limsup_{k \rightarrow \infty} \sigma(x_{m(k)}, x_{n(k)}) \leq s\varepsilon^+. \quad (1.2)$$

On the other hand, we have

$$\frac{1}{k} + \sigma(x_{m(k)}, x_{n(k)}) > \frac{1}{k} + \varepsilon, \text{ for all } k \geq 1.$$

Taking the limit inferior as $k \rightarrow \infty$, we have

$$\liminf_{k \rightarrow \infty} \sigma(x_{m(k)}, x_{n(k)}) \geq \varepsilon^+. \quad (1.3)$$

From (1.2) and (1.3), we obtain the first item of the above Lemme. Since the remaining items are the same as in Lemma (1.1.3), the proof is completed. ■

Remark 1.1.2 : Taking $s = 1$ (the case corresponding to a metric space (X, d)) in Lemma (1.1.3), we find Lemma (2.2) in [16]. More precisely, the above items become as follows:

$$\lim_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon^+,$$

and

$$\begin{aligned} \lim_{k \rightarrow \infty} d(x_{m(k)+1}, x_{n(k)}) &= \lim_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)+1}) \\ &= \lim_{k \rightarrow \infty} d(x_{m(k)+1}, x_{n(k)+1}) \\ &= \varepsilon. \end{aligned}$$

In 2015, An et al. [15] proved the following result:

Proposition 1.1.1 : Let (X, σ) be a b -metric space with constant $s \geq 1$. If σ is continuous with respect in one variable, then σ is continuous in other variable.

Obviously, we observe from the above result that if σ is not continuous with respect one variable, then σ is not continuous in each variable.

Example 3: Let $X = [0, \infty)$ and $\sigma : X \times X \rightarrow [0, \infty)$ be a mapping defined by

$$\sigma(x, y) = \begin{cases} d(x, y), & xy \neq 0, \\ 4d(x, y), & xy = 0, \end{cases}$$

where $d(x, y) = |x - y|$. Then the following hold:

- (1) (X, σ) is a complete b -metric space with constant $s = 4$,
- (2) σ is not a metric on X ,
- (3) σ is not continuous in each variable.

Proof: (1) We start to prove that (X, σ) is a b -metric space with constant $s = 4$. Clearly, (b_1) and (b_2) are satisfied. For (b_3) , we can easily observe that for any x, y in X ,

$$d(x, y) \leq \sigma(x, y) \leq 4d(x, y). \quad (1.4)$$

We consider then the following cases:

Case 1: Suppose that $xy \neq 0$. Then using (1.4), for any $z \in X$, we obtain:

$$\begin{aligned} \sigma(x, y) = d(x, y) &\leq d(x, z) + d(z, y) \leq \sigma(x, z) + \sigma(z, y) \\ &\leq 4(\sigma(x, z) + \sigma(z, y)). \end{aligned}$$

Case 2: Assume that $xy = 0$. Also, through (1.4), we have for any $z \in X$:

$$\sigma(x, y) = 4d(x, y) \leq 4d(x, z) + 4d(z, y) \leq 4(\sigma(x, z) + \sigma(z, y)).$$

Next, since (X, d) is a complete metric space, the completeness of (X, σ) follows immediately from (1.4).

(2) Indeed, σ is not a metric on X since we have

$$\sigma(0, 2) = 8 > 5 = 4 + 1 = \sigma(0, 1) + \sigma(1, 2).$$

(3) Let $x_n = \frac{1}{n}$ for each $n \in \mathbb{N}$. We have

$$\lim_{n \rightarrow \infty} \sigma\left(\frac{1}{n}, 0\right) = \lim_{n \rightarrow \infty} \frac{4}{n} = 0.$$

Then $\lim_{n \rightarrow \infty} x_n = 0$ in (X, σ) . On the other hand, we have

$$\lim_{n \rightarrow \infty} \sigma(x_n, 1) = 1 \neq 4 = \sigma(0, 1).$$

This, together with Proposition, proves that σ is not continuous in each variable. ■

1.2 F-contractions

Now, we review some results concerning F -contractions related to the existing literature. In 2012, Wardowski [3] introduced the notion of F -contraction as follows:

Definition 1.2.1 : Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be an F -contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$:

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \quad (1.5)$$

where \mathcal{F} is the family of all functions $F : (0, \infty) \rightarrow \mathbb{R}$ satisfying the following conditions:

(F₁) F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$, if $\alpha < \beta$, then $F(\alpha) < F(\beta)$.

(F₂) For each sequence $\{\alpha_n\}$ of positive numbers, the following holds:

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} F(\alpha_n) = -\infty.$$

(F₃) There exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

Remark 1.2.1 : Let $\alpha > 0$. Let the following functions $F_1(\alpha) = \ln(\alpha)$, $F_2 = \alpha + \ln(\alpha)$. Then, $F_1, F_2 \in \mathcal{F}$.

Remark 1.2.2 : Clearly, if F is an increasing function (not necessary strictly increasing), inequality (1.5) implies that T is a contractive mapping, i.e.,

$$d(Tx, Ty) < d(x, y), \quad \text{for all } x, y \in X, \quad x \neq y.$$

Hence, every F -contraction is a continuous mapping.

Wardowski's result is given as follows:

Theorem 1.2.1 : Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be an F -contraction. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to x^* .

Secelean showed that condition (F₂) can be replaced by an equivalent and more easier one (noted (F'₂) : $\inf F = -\infty$). Afterwards, Wardowski [3] proved a fixed point theorem concerning F -contractions when τ is taken as a function. In 2018, Lukács and Kajántó [17] extended Wardowski's theorem in the setting of b -metric spaces and omitted condition (F₂).

1.2.1 F-contraction of Hardy-Rogers type

We present here various types of F -contractions of Hardy-Rogers and some related works which will be needed for stating our results in the sequel.

In 2014, Wardowski and Dung proved the following result.

Theorem 1.2.2 : Let (X, d) be a complete metric space. Assume that there exists $\tau > 0$ and $F \in \mathcal{F}$ such that $T : X \rightarrow X$ satisfies

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(B_T^d(x, y)),$$

where

$$B_T^d(x, y) = ad(x, y) + bd(x, Tx) + cd(y, Ty) + e[d(x, Ty) + d(y, Tx)],$$

with $a, b, c, e \geq 0$ and $a + b + c + 2e < 1$. If T or F is continuous, then

(1) T has a unique fixed point $x^* \in X$.

(2) For all $x \in X$, the sequence $\{T^n x\}$ is convergent to x^* .

Afterwards, Cosentino and Vetro [4] introduced a new notion, namely the notion of F -contraction of Hardy-Rogers type given below.

Definition 1.2.2 : Let (X, d) be a metric space. A self-mapping T on X is called an F -contraction of Hardy-Rogers type if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(Q_T^d(x, y)),$$

where

$$Q_T^d(x, y) = (\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx)),$$

with $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$.

Later, Vetro [18] proved some new results about F -contraction of Hardy-Rogers type. Before enunciating these results. Let us note \mathbb{S} the family of all functions $\tau : (0, \infty) \rightarrow (0, \infty)$ satisfying the following property:

$$\liminf_{t \rightarrow \eta^+} \tau(t) > 0, \text{ for all } \eta \geq 0.$$

Let us consider also the following condition:

$$(F'_3) : F \text{ is continuous on } (0, \infty).$$

Henceforth, we denote by \mathfrak{F} the set of all functions $F : (0, \infty) \rightarrow \mathbb{R}$ satisfying the conditions (F_1) , (F_2) and (F'_3) and by \mathbb{F} the set of all functions $F : (0, \infty) \rightarrow \mathbb{R}$ satisfying the conditions (F_1) and (F_2) .

Vetro generalized the notion of F -contraction of Hardy-Rogers type as follows:

Definition 1.2.3 : Let (X, d) be a complete metric space. A self-mapping T on X is called an F -contraction of Hardy-Rogers type if there exist $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau(d(x, y)) + F(d(Tx, Ty)) \leq F(Q_T^d(x, y))$$

where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$, $\gamma \neq 1$ and $\alpha + \delta + L \leq 1$.

Next, the authors in [17] introduced the notion of F -weak contraction of Hardy-Rogers type in the setting of b -metric spaces as follows:

Definition 1.2.4 : Let (X, σ) be a b -metric space with constant $s \geq 1$, $a, b, c, e, f \geq 0$ real numbers and $T : X \rightarrow X$ an operator. If there exist $\tau > 0$ and $F \in \mathcal{F}_{s, \tau}$ such that for all $x, y \in X$ the inequality $\sigma(Tx, Ty) > 0$ implies

$$\tau + F(s\sigma(Tx, Ty)) \leq F(A_T^\sigma(x, y)),$$

where

$$A_T^\sigma(x, y) = a\sigma(x, y) + b\sigma(x, Tx) + c\sigma(y, Ty) + e\sigma(x, Ty) + f\sigma(y, Tx),$$

then T is called an **F -weak contraction of Hardy-Rogers type**.

1.2.2 F-contraction of Suzuki-Hardy-Rogers type

Vetro introduced the notion of F -contraction of Suzuki-Hardy-Rogers type given below:

Definition 1.2.5 : Let (X, d) be a complete metric space. A self-mapping T on X is called an F -contraction of Suzuki-Hardy-Rogers type if there exist $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that for all $x, y \in X$ with $Tx \neq Ty$

$$\frac{1}{2s}\sigma(x, Tx) < \sigma(x, y) \Rightarrow \tau(d(x, y)) + F(d(Tx, Ty)) \leq F(Q_T^d(x, y)),$$

where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$, $\gamma \neq 1$ and $\alpha + \delta + L \leq 1$.

Remark 1.2.3 : Vetro proved that if (F'_3) is weakened to the condition that F is upper semi-continuous on $(0, \infty)$, then Theorem holds for the strict inequality $\alpha + \delta + L < 1$.

Vetro established also the following corollaries.

Theorem 1.2.3 : Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be an F -contraction of Hardy-Rogers type. Then T has a fixed point. Moreover, if $\alpha + \delta + L \leq 1$, then the fixed point of T is unique.

The first Vetro's result is the following:

Theorem 1.2.4 : *Let (X, d) be a complete metric space. If T is an F -contraction of Hardy-Rogers type and F is continuous (i.e., $F \in \mathfrak{F}$), then T has a unique fixed point.*

The second Vetro's result (Suzuki-type version) is given as follows:

Theorem 1.2.5 : *Let (X, d) be a complete metric space. If T is an F -contraction of Suzuki-Hardy-Rogers type and F is continuous (i.e., $F \in \mathfrak{F}$), then T has a unique fixed point.*

Corollary 1.2.1 : *Let (X, d) be a complete metric space and let T be a self-mapping on X . Assume that there exist an upper semi-continuous $F \in \mathfrak{F}$ and $\tau \in \mathbb{S}$ such that for all $x, y \in X$ with $Tx \neq Ty$:*

$$\tau(d(x, y)) + F(d(Tx, Ty)) \leq F(\beta d(x, Tx) + \gamma d(y, Ty)),$$

where $\beta, \gamma \in [0, \infty)$ satisfying $\beta + \gamma = 1$ and $\gamma \neq 1$. Then T has a unique fixed point in X .

Corollary 1.2.2 : *Let (X, d) be a complete metric space and let T be a self-mapping on X . Assume that there exist a continuous $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that for all $x, y \in X$ with $Tx \neq Ty$:*

$$\tau(d(x, y)) + F(d(Tx, Ty)) \leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty)),$$

where $\alpha, \beta, \gamma \in [0, \infty)$ satisfying $\alpha + \beta + \gamma = 1$ and $\gamma \neq 1$. Then T has a unique fixed point in X .

In 2015, Cosentino et al. [19] introduced the following condition (noted (F_4) in [19], Definition 3.1):

Let $s \geq 1$. If $\{\alpha_n\} \subset (0, \infty)$ is a sequence such that $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$, for all $n \in \mathbb{N}$ and some $\tau > 0$, then $\tau + F(s^n \alpha_n) \leq F(s^{n-1} \alpha_{n-1})$, for all $n \in \mathbb{N}$.

In the same context, Lukács and Kajántó [17] defined a new class of functions (noted $\mathcal{F}_{s,\tau}$) satisfying an easier condition than (F_4) . Their definition is given below.

Definition 1.2.6 : *Let $s \geq 1$ and $\tau > 0$. We say that $F \in \mathbb{F}^*$ belongs to $\mathcal{F}_{s,\tau}$ if it is also satisfies:*

$(F_{s,\tau}) \inf F = -\infty$ and $x, y, z \in (0, \infty)$ are such that $\tau + F(sx) \leq F(y)$ and $\tau + F(sy) \leq F(z)$ then

$$\tau + F(s^2x) \leq F(sy),$$

where \mathbb{F}^* is the set of all functions $F : (0, \infty) \rightarrow \mathbb{R}$ satisfying the conditions (F_1) and (F_3) .

Lukács and Kajántó showed that if F is an increasing function, then $(F_{s,\tau})$ is equivalent to (F_4) and proved the fixed point result below :

Theorem 1.2.6 : *Suppose that (X, σ) is a b -metric space with constant $s \geq 1$ and $T : X \rightarrow X$ is an F -weak contraction of Hardy-Rogers type. If either $a + b + c + (s + 1)e < 1$ or $a + b + c + (s + 1)f < 1$ holds, then every $x_0 \in X$, the sequence $x_{n+1} = Tx_n$ converges to a fixed point of T . Moreover, if $a + e + f < s$ holds as well, then T has exactly one fixed point.*

Let $\omega \geq 1$ be a given real number. We denote by \mathcal{S}_ω the family of all functions $\tau : (0, \infty) \rightarrow (0, \infty)$ which satisfy the following condition:

$$\liminf_{t \rightarrow r} \tau(t) > 0, \text{ where } r \in [\eta^+, \eta^+\omega], \forall \eta \geq 0. \quad (A_\omega)$$

Remark 1.2.4 : *Obviously, if $\omega = 1$, condition (A_ω) becomes as follows:*

$$\liminf_{t \rightarrow \eta^+} \tau(t) > 0, \quad \forall \eta \geq 0. \quad (A_1)$$

Henceforth, we denote by \mathcal{S}_1 the set \mathcal{S}_ω when $\omega = 1$. Clearly, we have $\mathcal{S}_\omega \subseteq \mathcal{S}_1$.

Proposition 1.2.1 : *Let (X, σ) be a b -metric space with constant $s \geq 1$ and let λ be a given real number such that $1 \leq \lambda \leq s$. Let $T : X \rightarrow X$ be a mapping and $\{x_n\}$ the Picard sequence of T based on an arbitrary $x_0 \in X$. Assume that there exist a non-decreasing function F and $\tau \in \mathcal{S}_1$ such that for all $z \in X$ with $Tz \neq T^2z$,*

$$\tau(\sigma(z, Tz)) + F(\lambda\sigma(Tz, T^2z)) \leq F((d_1 + d_2)\sigma(z, Tz) + d_3\sigma(Tz, T^2z) + d_4\sigma(z, T^2z)), \quad (1.6)$$

where d_1, d_2, d_3, d_4 are non-negative real numbers satisfying:

$$d_1 + d_2 + d_3 + 2sd_4 = \frac{1}{s} \text{ and } d_3 \neq \frac{1}{s}. \quad (1.7)$$

Then $\lim_{n \rightarrow \infty} \sigma(x_n, x_{n+1}) = 0$.

Remark 1.2.5 : *Proposition (1.2.1) also furnishes that the sequence $\{\sigma_n\}$ is a strictly decreasing when $\sigma_n > 0$, for all $n \in \mathbb{N}_0$.*

Remark 1.2.6 : *Taking $s = 1$ (which yields $\lambda = 1$ as well) in Proposition (1.2.1) (it corresponds to the case of metric spaces). Moreover, for the function τ , we have used the condition that $\tau \in \mathcal{S}_1$ instead of the condition that $\tau \in \mathbb{S}$. This is a slightly weaker condition since $\mathbb{S} \subset \mathcal{S}_1$. In addition, we also change the condition that F is strictly increasing into the weaker condition that F is non-decreasing (i.e., the strictness of the monotonicity of F is not necessary).*

1.2.3 Extended F-contraction of Hardy–Rogers type

Let (X, σ) be a b -metric space with constant $s \geq 1$. We denote, for all $x, y \in X$,

$$Q_T^\sigma(x, y) = (\alpha\sigma(x, y) + \beta\sigma(x, Tx) + \gamma\sigma(y, Ty) + \delta\sigma(x, Ty) + L\sigma(y, Tx)),$$

with $\alpha, \beta, \gamma, \delta, L$ non-negative real numbers.

Definition 1.2.7 : Let (X, σ) be a b -metric space with constant $s \geq 1$. A mapping $T : X \rightarrow X$ is said to be an extended F -contraction of Hardy–Rogers type if there exist $F : (0, \infty) \rightarrow \mathbb{R}$ and $\tau \in \mathcal{S}_1$ such that for all $x, y \in X$,

$$\sigma(Tx, Ty) > 0 \Rightarrow \tau(\sigma(x, y)) + F(\sigma(Tx, Ty)) \leq F(Q_T^\sigma(x, y)). \quad (1.8)$$

Remark 1.2.7 : If F is non-decreasing, it is easy to see that every T which is an extended F -contraction of Hardy–Rogers type satisfies the following condition

$$\sigma(Tx, Ty) < Q_T^\sigma(x, y),$$

for all $x, y \in X$ with $Tx \neq Ty$.

Theorem 1.2.7 : Let (X, σ) be a complete b -metric space with constant $s \geq 1$, and $T : X \rightarrow X$ an extended F -contraction of Hardy–Rogers type with $F \in (F_c)$. Suppose that either (\mathcal{H}_s^1) or (\mathcal{H}_s^2) holds, where

$$(\mathcal{H}_s^1) \quad \alpha + \beta + \gamma + 2\delta s = \frac{1}{s} \quad \text{and} \quad \delta \neq \frac{1}{s},$$

$$(\mathcal{H}_s^2) \quad \alpha + \beta + \gamma + 2Ls = \frac{1}{s} \quad \text{and} \quad \beta \neq \frac{1}{s}.$$

Furthermore, we assume that $s^2\alpha + s^3(\delta + L) \leq 1$. Then T has a unique fixed point x^* for every $x_0 \in X$ the sequence $T^n x_0$ for all $n \in \mathbb{N}$ converges to x^* .

Example 1.3 : Let $X = [0, 7]$. Let $T : X \rightarrow X$ be a mapping given by

$$T(x) = \begin{cases} 7, & \text{if } x \in (0, 7], \\ 6, & \text{if } x = 0. \end{cases}$$

Let $\sigma : X \times X \rightarrow [0, \infty)$ be the mapping defined by

$$\sigma(x, y) = (x - y)^2, \quad \text{for all } x, y \in X.$$

By the Example 1.1, (X, σ) is a complete b -metric space with constant $s = 2$.

First, we observe that

$$\sigma(Tx, Ty) = 1 > 0 \Leftrightarrow [(x \in (0, 7] \wedge y = 0) \vee (y \in (0, 7] \wedge x = 0)].$$

Let $x, y \in X$ and denote

$$M(x, y) = \frac{1}{8}\sigma(x, y) + \frac{1}{8}\sigma(y, Ty) + \frac{1}{16}\sigma(x, Ty).$$

Now, we consider the following cases:

1. If $x \in (0, 7]$ and $y = 0$. In this case, we obtain

$$\begin{aligned} \frac{\sigma(x, 0)}{294} - \frac{1}{\sigma(Tx, T0) + 1} &\leq \frac{49}{294} - \frac{1}{2} = -\frac{1}{3} \\ &< -\frac{8}{\sigma(0, T0)} = -\frac{8}{36} \\ &< -\frac{1}{M(x, 0) + 1}. \end{aligned} \tag{1.9}$$

2. If $y \in (0, 7]$ and $x = 0$. In this case, we get

$$\begin{aligned} \frac{\sigma(0, y)}{294} - \frac{1}{\sigma(T0, Ty) + 1} &\leq \frac{49}{294} - \frac{1}{2} \\ &< -\frac{16}{\sigma(0, Ty)} = -\frac{16}{49} \\ &\leq -\frac{1}{M(0, y)} \\ &< -\frac{1}{M(0, y) + 1}. \end{aligned} \tag{1.10}$$

From (1.9) and (1.10), we arrive at a conclusion

$$\frac{\sigma(x, y)}{294} - \frac{1}{\sigma(Tx, Ty) + 1} \leq -\frac{1}{M(x, y) + 1},$$

for all $x, y \in X$ with $\sigma(Tx, Ty) = 1 > 0$.

Therefore, by choosing $F(t) = -\frac{1}{t+1}$ and $\tau(t) = \frac{t}{294}$, for all $t \in (0, \infty)$, we observe that

T is a multi-valued extended F -contraction of Hardy–Rogers type with $\alpha = \frac{1}{8} = \gamma, \delta = \frac{1}{16}, \beta = L = 0$ and all the conditions of Theorem 1.2.7 are satisfied with (\mathcal{H}_s^1) . Hence, T has a unique fixed point $x^* = 7$.

Theorem 1.2.8 : *Let (X, σ) be a complete b -metric space with constant $s \geq 1$, and $T : X \rightarrow X$ satisfying the contractive condition (1.8) with $F \in \mathcal{F}_c$ and $\tau \in \mathcal{S}_1$. Suppose that either (\mathcal{H}_s^1) or (\mathcal{H}_s^2) holds. Furthermore, we assume that $s^2\alpha + s^3(\delta + L) < 1$. Then T has a unique fixed point x^* for every $x_0 \in X$ the sequence $T^n x_0$ for all $n \in \mathbb{N}$ converges to x^* .*

1.2.4 Extended F-contraction of Suzuki-Hardy-Rogers type

Let (X, σ) be a b -metric space with constant $s \geq 1$. We denote, for all $x, y \in X$,

$$Q_T^\sigma(x, y) = (\alpha\sigma(x, y) + \beta\sigma(x, Tx) + \gamma\sigma(y, Ty) + \delta\sigma(x, Ty) + L\sigma(y, Tx)),$$

with $\alpha, \beta, \gamma, \delta, L$ non-negative real numbers.

Definition 1.2.8 : Let (X, σ) be a b -metric space with constant $s \geq 1$. A mapping $T : X \rightarrow X$ is said to be an extended F -contraction of Suzuki-Hardy-Rogers type if there exist $F : (0, \infty) \rightarrow \mathbb{R}$ and $\tau \in \mathcal{S}_1$ such that for all $x, y \in X$ with $Tx \neq Ty$,

$$\frac{1}{2s}\sigma(x, Tx) < \sigma(x, y) \Rightarrow \tau(\sigma(x, y)) + F(\sigma(Tx, Ty)) \leq F(Q_T^\sigma).$$

Remark 1.2.8 : If F is non decreasing, it is easy also to see from Definition (1.2.8) that every T which is an extended F -contraction of Suzuki-Hardy-Rogers type satisfies the following condition:

$$\sigma(Tx, Ty) < Q_T^\sigma(x, y),$$

for all $x, y \in X$ with $Tx \neq Ty$ and $\frac{1}{2s}\sigma(x, Tx) < \sigma(x, y)$.

Authors in [20] proved the following theorems:

Theorem 1.2.9 : Let (X, σ) be a completed b -metric space with constant $s \geq 1$ and $T : X \rightarrow X$ an extended F -contraction of Suzuki-Hardy-Rogers type with $F \in \mathcal{F}_c$. Suppose that either (\mathcal{H}_s^1) or (\mathcal{H}_s^2) holds. Furthermore, we assume that $s^2\alpha + s^3(\delta + L) \leq 1$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to x^* .

1.3 Compactness

In mathematics, property of some topological spaces that has its main use in the study of functions defined on such spaces. Formulation of this topological concept of compactness was motivated by the Heine-Borel theorem for Euclidean space, which states that compactness of a set is equivalent to the set's being closed and bounded.

Let (X, σ) be a b -metric space.

Definition 1.3.1 : We say that (X, σ) is a compact b -metric space if every sequence $\{x_n\}$ of elements in X has a sub-sequence converging to a point x of X .

Proposition 1.3.1 : Let (X, σ) be a b -metric space, we have :

- A compact subset of (X, σ) is at the both bounded and closed.
- A closed subset of a compact space is compact.
- If a subset of a metric space (X, σ) is compact then it is σ -bounded.

Proposition 1.3.2 : The direct image of a compact metric space by a continuous function is compact.

Definition 1.3.2 : Let (X, σ) be a b -metric space. For any $B \in K(X)$ there exists $y \in B$ such that

$$\sigma(x, B) = \sigma(x, y).$$

1.4 Multi-Valued Mappings

During the period 1941-45 several extensions of known fixed point theorems in which the transformation T takes each point of a compact metric space X into a closed subset of X were developed. These extensions first occurred in Von Neumann's work on the theory of games.

The following properties of some of the functionals defined above will be used throughout our main result.

Definition 1.4.1 : Let (X, σ) be a b -metric space. We denote the set of all nonempty compact subsets of X by $K(X)$. For $A, B \in K(x)$ except $A = B \Rightarrow \rho(A, B) = 0$, the mapping H is said to be a Pompeiu-Hausdorff metric induced by σ on $K(x)$ is given as:

$$H(A, B) = \begin{cases} \max\{\rho(A, B), \rho(B, A)\}, & \text{if the maximum exists,} \\ +\infty & \text{otherwise,} \end{cases}$$

Such that:

- **the excess functional** : $\rho : P(X) \times P(X) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ defined as

$$\rho(A, B) = \sup\{\sigma(x, B) / x \in A\},$$

- **the gap functional** : defined as

$$\sigma(x, B) = \inf_{y \in B} \sigma(x, y).$$

Lemma 1.4.1 : Let (X, σ) be a b -metric space with constant $s \geq 1$, and $A, B \in K(X)$. Then:

1. $\sigma(x, B) \leq \sigma(x, y)$, for any $y \in B$,
2. $\sigma(x, B) \leq H(A, B)$, for any $x \in A$,
3. $\sigma(x, A) \leq s[\sigma(x, y) + \sigma(y, A)]$, for all $x, y \in X$, $A \subset X$,
4. $\sigma(x_n, x_{n+p}) \leq s\sigma(x_n, x_{n+1}) + s^2\sigma(x_{n+1}, x_{n+2}) + \dots + s^{p-1}\sigma(x_{n+p-2}, x_{n+p-1})$
 $+ s^{p-1}\sigma(x_{n+p-1}, x_{n+p})$,

for any $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$,

5. $H(A, C) \leq s[H(A, B) + H(B, C)]$.

Proof: (the (2nd) inequality of Lemma (1.4.1)) when we take $A = \{x\}$ we get:

$$\begin{aligned} H(A, B) &= \max\left\{\sup_{z \in A} \sigma(z, B), \sup_{y \in B} \sigma(A, y)\right\} \\ &= \max\left\{\sigma(x, B), \sup_{y \in B} \sigma(x, y)\right\} \\ &= \max_{y \in B} \sigma(x, y), \end{aligned}$$

implies that $\sigma(x, y) = \sigma(A, y) \leq H(A, B)$. ■

Remark 1.4.1 Let $T : X \rightarrow K(X)$ be multi-valued operator. A point $x \in X$ is called fixed point for T if and only if $x \in T(x)$.

Theorem 1.4.1 : Let (X, σ) be a complete b -metric space with constant $s \geq 1$ and $T : X \rightarrow C(X)$ a multi-valued operator such that:

- (i) there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$, $\alpha + \beta + \gamma < \frac{s-1}{s^2}$ and $\beta + s\gamma < \frac{1}{s}$ such that

$$H(T(x), T(y)) \leq \alpha\sigma(x, y) + \beta[\sigma(x, T(x)) + \sigma(y, T(y))] + \gamma[\sigma(x, T(y)) + \sigma(y, T(x))],$$

for all $x, y \in X$.

- (ii) T is closed,

in these conditions $\text{Fix}(T) \neq \emptyset$.

Theorem 1.4.2 : *Let (X, σ) be a complete b -metric space with constant $s \geq 1$ and $T : X \rightarrow C(X)$ a multi-valued operator such that:*

(i) *there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$, $\alpha + \beta + \gamma < \frac{s-1}{s^2}$ and $\beta + s\gamma < \frac{1}{s}$ such that*

$$H(T(x), T(y)) \leq \alpha\sigma(x, y) + \beta[\sigma(x, T(x)) + \sigma(y, T(y))] + \gamma[\sigma(x, T(y)) + \sigma(y, T(x))],$$

for all $x, y \in X$.

(ii) *T is closed,*

If $SFix(T) \neq \emptyset$ then $SFix(T) = Fix(x) = \{x\}$.

Proof: Let $x \in SFix(T)$ and suppose that there exist $y \in Fix(T)$, $y \neq x$.

$$\begin{aligned} \sigma(x, y) = D(T(x), y) &\leq H(T(x), T(y)) \\ &\leq \alpha\sigma(x, y) + \beta[\sigma(x, T(x)) + \sigma(y, T(y))] + \gamma[\sigma(x, T(y)) + \sigma(y, T(x))] \\ &\leq \alpha\sigma(x, y) + 2\gamma\sigma(x, y). \end{aligned}$$

Hence $(1 - \alpha - 2\gamma)\sigma(x, y) \leq 0$.

Since $\alpha + 2\gamma < \alpha + \beta + 2s\gamma < \frac{s-1}{s^2} < 1$, we obtain that $\sigma(x, y) = 0$ which implies that $x = y$ and this is a contradiction.

In the sum up : $SFix(T) = Fix(T) = \{x\}$. ■

CHAPTER

2

FIXED POINT THEORY IN *B*-METRIC SPACES

The concept of a multi-valued contraction mapping was first introduced by Nadler Jr.[6] in 1967. He combined the ideas of multi-valued mappings and Lipschitz mappings, and proved some fixed point theorems in this respect. In this chapter it is our aim to add and to investigate a few new results.

Definition 2.0.1 : Let (X, σ) be a b -metric space with constant $s \geq 1$. A mapping $T : X \rightarrow K(X)$ is said to be a multi-valued extended F -contraction of Hardy-Rogers type if there exists $F : (0, \infty) \rightarrow \mathbb{R}$ and $\tau \in \mathcal{S}_1$ such that for all $x, y \in X$,

$$H(T(x), T(y)) > 0 \Rightarrow \tau(\sigma(x, y)) + F(H(T(x), T(y))) \leq F(Q_T^\sigma(x, y)). \quad (2.1)$$

Remark 2.0.1 : If F is non-decreasing, it is easy to see that every T which is a multi-valued extended F -contraction of Hardy-Rogers type satisfies the following condition

$$H(T(x), T(y)) < Q_T^\sigma(x, y), \quad (2.2)$$

for all $x, y \in X$ with $T(x) \cap T(y) = \emptyset$.

Theorem 2.0.1 : Let (X, σ) be a complete b -metric space with constant $s \geq 1$, and $T : X \rightarrow K(X)$ a multi-valued extended F -contraction of Hardy-Rogers type with $F \in (\mathcal{F}_c)$. Suppose that either (\mathcal{H}_s^1) or (\mathcal{H}_s^2) holds, where

$$(\mathcal{H}_s^1) \quad \alpha + \beta + \gamma + 2\delta s = 1/s \quad \text{and} \quad \gamma \neq \frac{1}{s},$$

$$(\mathcal{H}_s^2) \quad \alpha + \beta + \gamma + 2Ls = 1/s \quad \text{and} \quad \beta \neq \frac{1}{s}.$$

Furthermore, we assume that $\alpha + \beta + \gamma + 2\delta s + 2Ls \leq 1$ and $s^2\alpha + s^3(\delta + L) \leq 1$. Then T has a unique fixed point x and for every $x_0 \in X$ the sequence $T^n x_0$ for all $n \in \mathbb{N}$ converges to x .

Proof: First, we will prove the existence of a fixed point.

Let $x_0 \in X$. Since $\{x_0\}, T(x_0) \in K(X)$, there exists $x_1 \in T(x_0)$, where

$$\sigma(x_0, T(x_0)) = \sigma(x_0, x_1).$$

Now, as $T(x_0), T(x_1) \in K(X)$ there is a point $x_2 \in T(x_1)$, such that $\sigma(x_1, T(x_1)) = \sigma(x_1, x_2)$. Continuing in this manner, we create a sequence $\{x_n\}$, with $x_{n+1} \in T(x_n)$.

If there is a $n_0 \in \mathbb{N}$ such that $x_{n_0} \in T(x_{n_0})$, then The proof is complete because x_{n_0} is the fixed point of the multi-valued map T .

But, for any $n \in \mathbb{N}$, if $x_n \notin T(x_n)$, we have

$$\sigma(x_n, T(x_n)) = \sigma(x_n, x_{n+1}) > 0, \quad \text{for all } n \in \mathbb{N}, \quad (2.3)$$

and

$$H(T(x_n), T(x_{n+1})) > 0, \quad \text{for all } n \in \mathbb{N}.$$

Through the contractive inequality (2.1) with $x = x_{n-1}$ and $y = x_n$, to obtain for all $n \in \mathbb{N}$:

$$\begin{aligned} & \tau(\sigma(x_{n-1}, x_n)) + F(\sigma(x_n, T(x_n))) \\ & \leq \tau(\sigma(x_{n-1}, x_n)) + F(\max\{ \sup_{z \in T(x_{n-1})} \sigma(z, T(x_n)), \sup_{z \in T(x_n)} \sigma(T(x_{n-1}), z) \}) \\ & \leq F(\alpha\sigma(x_{n-1}, x_n) + \beta\sigma(x_{n-1}, T(x_{n-1})) + \gamma\sigma(x_n, T(x_n)) \\ & \quad + \delta\sigma(x_{n-1}, T(x_n)) + L\sigma(x_n, T(x_{n-1}))), \end{aligned}$$

according to (2.3) and the Definition (2.0.1), we get

$$\begin{aligned} & \tau(\sigma(x_{n-1}, x_n)) + F(\sigma(x_n, x_{n+1})) \\ & \leq F(\alpha\sigma(x_{n-1}, x_n) + \beta\sigma(x_{n-1}, x_n) + \gamma\sigma(x_n, x_{n+1}) \\ & \quad + \delta\sigma(x_{n-1}, T(x_n)) + L\sigma(x_n, T(x_{n-1}))) \\ & \leq F((\alpha + \beta)\sigma_{n-1} + \gamma\sigma_n + \delta\sigma(x_{n-1}, T(x_n)) + L\sigma(x_n, T(x_{n-1}))), \end{aligned}$$

using the (3rd) propriety of Lemma (1.4.1) and the monotonicity to obtain

$$\begin{aligned} \tau(\sigma_{n-1}) + F(\sigma_n) & \leq F((\alpha + \beta)\sigma_{n-1} + \gamma\sigma_n + \delta(s\sigma_{n-1} + s\sigma(x_n, T(x_n))) \\ & \quad + L(s\sigma_{n-1} + s\sigma(x_{n-1}, T(x_{n-1})))) \\ & \leq F((\alpha + \beta)\sigma_{n-1} + \gamma\sigma_n + s\delta\sigma_{n-1} + s\delta\sigma_n + 2sL\sigma_{n-1}) \\ & \leq F((\alpha + \beta + 2sL + \delta s)\sigma_{n-1} + (\gamma + \delta s)\sigma_n), \end{aligned} \tag{2.4}$$

with $\sigma(x_{n-1}, x_n) = \sigma_{n-1}$ and $\sigma(x_n, x_{n+1}) = \sigma_n$, that gives us

$$\sigma_n < \frac{(\alpha + \beta + 2sL + \delta s)}{1 - (\gamma + \delta s)} \sigma_{n-1}.$$

Using the fact :

$$\frac{\alpha + \beta + 2sL + \delta s}{1 - (\gamma + \delta s)} \leq 1,$$

notice that : $\sigma_n \leq \sigma_{n-1}$, i.e σ_n positive decreasing. That give for some $l \geq 0$

$$\lim_{n \rightarrow +\infty} \sigma_n = l.$$

If $l \neq 0$ through (2.4) we have

$$\tau(\sigma_{n-1}) + F(\sigma_n) \leq F((\alpha + \beta + 2sL + s\delta)\sigma_{n-1} + (\gamma + \delta s)\sigma_n).$$

Taking the upper limit we obtain:

$$\limsup_{n \rightarrow +\infty} \tau(\sigma_{n-1}) + F(l) \leq F((\alpha + \beta + \gamma + 2sL + 2s\delta)l),$$

Because $\alpha + \beta + \gamma + 2sL + 2s\delta \leq 1$ has been met, we've got

$$\limsup_{n \rightarrow +\infty} \tau(\sigma_{n-1}) \leq 0.$$

This implies that

$$\limsup_{n \rightarrow +\infty} \tau(\sigma_{n-1}) = 0,$$

which is contradiction.

That gives us

$$\lim_{n \rightarrow \infty} \sigma_n = 0, \tag{2.5}$$

through the (1st) propriety of Lemma (1.4.1) we obtain

$$\lim_{n \rightarrow \infty} \sigma(x_n, T(x_n)) = 0.$$

Now that we've established that $\{x_n\}$ is a Cauchy sequence, let's pretend that it isn't. Then there exist $\varepsilon > 0$ and two sequences $\{m_{(k)}\}$, $\{n_{(k)}\}$ of positive integers such that $n_{(k)}$ is the smallest index for which $n_{(k)} > m_{(k)} > k$ and $\sigma(x_{n_{(k)}}, x_{m_{(k)}}) > \varepsilon$. Due (2.5), this implies that $\sigma(x_{n_{(k)}-1}, x_{m_{(k)}}) \leq \varepsilon$ for all $k \geq 1$. Using the relaxed triangle inequality (b_3), we find:

$$\begin{aligned} \varepsilon &< \sigma(x_{m_{(k)}}, x_{n_{(k)}}) \\ &\leq s\sigma(x_{m_{(k)}}, x_{n_{(k)}-1}) + s\sigma(x_{n_{(k)}-1}, x_{n_{(k)}}) \\ &\leq s\varepsilon + s\sigma(x_{n_{(k)}-1}, x_{n_{(k)}}), \end{aligned}$$

this leads to

$$\frac{1}{s}\sigma(x_{m_{(k)}}, x_{n_{(k)}}) \leq \varepsilon + \sigma(x_{n_{(k)}-1}, x_{n_{(k)}}).$$

Since $\sigma(x_{n_{(k)}-1}, x_{n_{(k)}}) > 0$, then by taking limit superior as $k \rightarrow \infty$ with (2.5), we get

$$\limsup_{k \rightarrow \infty} \sigma(x_{m_{(k)}}, x_{n_{(k)}}) \leq s\varepsilon^+. \tag{2.6}$$

On the other hand, we have

$$\frac{1}{k} + \sigma(x_{m_{(k)}}, x_{n_{(k)}}) > \frac{1}{k} + \varepsilon, \text{ for all } k \geq 1,$$

taking the lower limit as $k \rightarrow \infty$, we have

$$\liminf_{k \rightarrow \infty} \sigma(x_{m_{(k)}}, x_{n_{(k)}}) \geq \varepsilon^+. \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$\varepsilon^+ \leq \liminf_{k \rightarrow \infty} \sigma(x_{m_{(k)}}, x_{n_{(k)}}) \leq \limsup_{k \rightarrow \infty} \sigma(x_{m_{(k)}}, x_{n_{(k)}}) \leq s\varepsilon^+.$$

Thus, we infer that there exists $k_0 \in \mathbb{N}$ such that $\{\sigma(x_{m(k)}, x_{n(k)})\}$ is bounded for all $k \geq k_0$ and thereby it has a convergent sub-sequence. It follows that there exists a real number l and a sub-sequence $\{k(p)\}_{p \geq k_0}$ such that:

$$\lim_{p \rightarrow \infty} \sigma(x_{m(k(p))}, x_{n(k(p))}) = l, \quad (2.8)$$

with

$$0 < \varepsilon^+ \leq \liminf_{k \rightarrow \infty} \sigma(x_{m(k(p))}, x_{n(k(p))}) \leq l \leq \limsup_{k \rightarrow \infty} \sigma(x_{m(k(p))}, x_{n(k(p))}) \leq s\varepsilon^+. \quad (2.9)$$

Through the relaxed triangle inequality, we find

$$\begin{aligned} \sigma_{m_p, n_p} = \sigma(x_{m(k(p))}, x_{n(k(p))}) &\leq s\sigma(x_{m(k(p))}, x_{m(k(p))+1}) + s\sigma(x_{m(k(p))+1}, x_{n(k(p))}) \\ &\leq s\sigma(x_{m(k(p))}, x_{m(k(p))+1}) + s^2\sigma(x_{m(k(p))+1}, x_{n(k(p))+1}) \\ &\quad + s^2\sigma(x_{n(k(p))+1}, x_{n(k(p))}), \end{aligned}$$

this gives

$$\begin{aligned} \sigma(x_{m(k(p))+1}, x_{n(k(p))+1}) &\geq \frac{1}{s^2}(\sigma_{m_p, n_p}) - s\sigma(x_{m(k(p))}, x_{m(k(p))+1}) \\ &\quad - s^2\sigma(x_{n(k(p))}, x_{n(k(p))+1}). \end{aligned} \quad (2.10)$$

For convenience, we set

$$H_{m_p, n_p} = H(T(x_{m(k(p))}), T(x_{n(k(p))})),$$

taking the upper limit as $p \rightarrow \infty$ we obtain:

$$\limsup_{p \rightarrow \infty} \sigma(x_{m(k(p))+1}, x_{n(k(p))+1}) \geq \frac{l}{s^2},$$

due (2nd) inequality of Lemma (1.4.1) and (2.3), this leads to

$$\limsup_{p \rightarrow \infty} H_{m_p, n_p} \geq \limsup_{p \rightarrow \infty} \sigma(x_{m(k(p))+1}, T(x_{n(k(p))})) \geq \frac{l}{s^2}. \quad (2.11)$$

Using (b₃), the monotonicity of F and $s^2\alpha + s^3(L + \delta) \leq 1$ to obtain:

$$\begin{aligned} &\tau(\sigma_{m_p, n_p}) + F(H_{m_p, n_p}) \\ &= \tau(\sigma_{m_p, n_p}) + F(\max\{\sup_{z \in T(x_{n(k(p))})} \sigma(z, T(x_{m(k(p))})), \sup_{z \in T(x_{m(k(p))})} \sigma(T(x_{n(k(p))}), z)\}) \\ &\leq F(\alpha\sigma_{m_p, n_p} + \beta\sigma(x_{m(k(p))}, T(x_{m(k(p))})) + \gamma\sigma(x_{n(k(p))}, T(x_{n(k(p))})) \\ &\quad + \delta\sigma(x_{m(k(p))}, T(x_{n(k(p))})) + L\sigma(x_{n(k(p))}, T(x_{m(k(p))}))), \end{aligned}$$

the above inequality leads to

$$\begin{aligned}
\tau(\sigma_{m_p, n_p}) + F(H_{m_p, n_p}) &\leq F(\alpha\sigma_{m_p, n_p} + \beta\sigma(x_{m_{(k(p))}}, x_{m_{(k(p))+1}) + \gamma\sigma(x_{n_{(k(p))}}, x_{n_{(k(p))+1}) \\
&\quad + \delta\sigma(x_{m_{(k(p))}}, T(x_{n_{(k(p))}))) + L\sigma(x_{n_{(k(p))}}, T(x_{m_{(k(p))}))) \\
&\leq F((\alpha + s(\delta + L))\sigma_{m_p, n_p} + (\beta + sL)\sigma_{m_{(k(p))}} + (\gamma + s\delta)\sigma_{n_{(k(p))}}) \\
&\leq F\left(\frac{1}{s^2}\sigma_{m_p, n_p} + (\beta + sL)\sigma_{m_{(k(p))}} + (\gamma + s\delta)\sigma_{n_{(k(p))}}\right),
\end{aligned}$$

with $\sigma_{m_{(k(p))}} = \sigma(x_{m_{(k(p))}}, x_{m_{(k(p))+1})$ and $\sigma_{n_{(k(p))}} = \sigma(x_{n_{(k(p))}}, x_{n_{(k(p))+1})$.

Now, we combine the above chain of inequalities with (2.8) and (2.11) through the fact that $F \in \mathcal{F}_c$ we obtain:

$$\begin{aligned}
\liminf_{p \rightarrow \infty} \tau(\sigma_{m_p, n_p}) + F\left(\frac{l}{s^2}\right) &\leq \limsup_{p \rightarrow \infty} \tau(\sigma_{m_p, n_p}) + F(\limsup_{p \rightarrow \infty} H_{m_p, n_p}) \\
&= \limsup_{p \rightarrow \infty} \tau(\sigma_{m_p, n_p}) + \limsup_{p \rightarrow \infty} F(H_{m_p, n_p}) \\
&= \limsup_{p \rightarrow \infty} [\tau(\sigma_{m_p, n_p}) + F(H_{m_p, n_p})] \\
&\leq \limsup_{p \rightarrow \infty} F\left(\frac{1}{s^2}\sigma_{m_p, n_p} + (\beta + sL)\sigma_{m_{(k(p))}} + (\gamma + s\delta)\sigma_{n_{(k(p))}}\right) \\
&\leq F\left(\frac{l}{s^2}\right).
\end{aligned}$$

As a result of this,

$$\liminf_{p \rightarrow \infty} \tau(\sigma_{m_{(k(p))}, n_{(k(p))}}) \leq 0,$$

contradiction with $\tau \in S_1$, it shows that $\{x_n\}$ is a Cauchy sequence converges to some point x in X , that is,

$$\lim_{n \rightarrow \infty} \sigma(x_n, x) = 0. \quad (2.12)$$

Following that, we show that $x \in T(x)$ that means x is a fixed point of T . Assume the opposite, i.e $x \notin T(x)$. When $\sigma(x, T(x)) = 0$, this provides that $x \in T(x)$; as a result of $T(x) \in K(X)$. The proof has been completed.

For this we will work on the other case, which is $0 < \sigma(x, T(x))$, then through (2.12) and the (2nd) and (3rd) inequalities of Lemma (1.4.1) we obtain

$$\begin{aligned}
\sigma(x, T(x)) &\leq s\sigma(x, x_{n+1}) + s\sigma(x_{n+1}, T(x)) \\
&\leq s\sigma(x, x_{n+1}) + sH(T(x_n), T(x)),
\end{aligned} \quad (2.13)$$

On the other side, there's the following contractive condition

$$\begin{aligned}
H(T(x_n), T(x)) &\leq \alpha\sigma(x_n, x) + \beta\sigma(x_n, T(x_n)) + \gamma\sigma(x, T(x)) + \delta\sigma(x_n, T(x)) + L\sigma(x, T(x_n)) \\
&\leq \alpha\sigma(x_n, x) + \beta\sigma(x_n, x_{n+1}) + \gamma\sigma(x, T(x)) + s\delta\sigma(x, T(x)) \\
&\quad + s\delta\sigma(x_n, x) + L\sigma(x, x_{n+1}),
\end{aligned} \quad (2.14)$$

implies that:

$$\begin{aligned}
\sigma(x, T(x)) &\leq s\sigma(x, x_{n+1}) + s\alpha\sigma(x_n, x) + s\beta\sigma(x_n, T(x_n)) + s\gamma\sigma(x, T(x)) + s\delta\sigma(x_n, T(x)) \\
&\leq s\sigma(x, x_{n+1}) + s\alpha\sigma(x_n, x) + s\beta\sigma(x_n, x_{n+1}) + s\gamma\sigma(x, T(x)) + s^2\delta\sigma(x_n, x) \\
&\quad + s^2\delta\sigma(x, T(x)) + sL\sigma(x, x_{n+1}).
\end{aligned} \tag{2.15}$$

The above inequality leads to

$$\begin{aligned}
\sigma(x, T(x)) &\leq (1 + L)s\sigma(x, x_{n+1}) + (s\alpha + s^2\delta)\sigma(x_n, x) \\
&\quad + s\beta\sigma(x_n, x_{n+1}) + (s\gamma + s^2\delta)\sigma(x, T(x)),
\end{aligned} \tag{2.16}$$

taking the limit superior as $n \rightarrow \infty$ in (2.16) to get

$$(1 - s\gamma - s^2\delta)\sigma(x, T(x)) \leq 0 \Rightarrow (1 - s\gamma)\sigma(x, T(x)) \leq s^2\delta\sigma(x, T(x)). \tag{2.17}$$

The contractive condition (2.2) and the symmetry condition can also be applied in a similar manner to obtain:

$$(1 - s\beta - Ls^2)\sigma(x, T(x)) \leq 0 \Rightarrow (1 - s\beta)\sigma(x, T(x)) \leq Ls^2\sigma(x, T(x)). \tag{2.18}$$

Again, according to (\mathcal{H}_s^1) and (\mathcal{H}_s^2) we consider the following cases:

1st case If (\mathcal{H}_s^1) hold, we have $(1 - \gamma) > 0$ and $\gamma + s\delta < \frac{1}{s}$.

Consequently, (2.17) implies that:

$$\sigma(x, T(x)) \leq \frac{s^2\delta}{(1 - s\gamma)}\sigma(x, T(x)) < \sigma(x, T(x)),$$

which is contradiction.

2nd case If (\mathcal{H}_s^2) hold, we have $(1 - \beta) > 0$ and $\beta + sL < \frac{1}{s}$.

Consequently, (2.18) implies that:

$$\sigma(x, T(x)) \leq \frac{Ls^2}{(1 - s\beta)}\sigma(x, T(x)) < \sigma(x, T(x)),$$

which is contradiction. Therefore, whether (\mathcal{H}_s^1) or (\mathcal{H}_s^2) holds, we obtain contradiction. So we have $x \in T(x) \Leftrightarrow Fix(t) \neq \emptyset$.

Finally, we will show that T has at most one fixed point.

Let x and y two distinct fixed points of T such that $x \neq y$, through the (2nd) inequality of Lemma (1.4.1) and the contractive inequality (2.1) we obtain:

$$\begin{aligned}
\sigma(T(x), y) &\leq H(T(x), T(y)) \\
&\leq \alpha\sigma(x, y) + \beta\sigma(x, T(x)) + \gamma\sigma(y, T(y)) + \delta\sigma(x, T(y)) + L\sigma(y, T(x)) \\
&= \alpha\sigma(x, y) + \delta\sigma(x, T(y)) + L\sigma(y, T(x)).
\end{aligned} \tag{2.19}$$

1. If $\alpha + \delta + L > 0$, note that $\alpha + \delta + L \leq 1$ we have:

$$\begin{aligned}\tau(\sigma(x, y)) + F(H(T(x), T(y))) &\leq F((\alpha + \delta + L)\sigma(x, y)) \\ &\leq F(\sigma(x, y)),\end{aligned}$$

as $\sigma(x, y) < H(T(x), T(y))$ we find

$$\tau(\sigma(x, y)) + F(H(T(x), T(y))) \leq F(H(T(x), T(y))),$$

this implies that

$$\tau(t) = 0,$$

which is contradiction.

2. if $\alpha + \delta + L = 0$, using (2.19) then:

$$H(T(x), T(y)) \leq (\alpha + \delta + L)H(T(x), T(y)) = 0,$$

gives that $H(T(x), T(y)) = 0$, which implies that $\sigma(x, y) = 0$. This gives that $y = x$. which is contradiction.

In the sum up : $Fix(T) = \{x\}$, then T has at most one fixed point. Which is the end of theorem proof. ■

Example 2.1 : Let $X = [0, 7]$. Let $\sigma : X \times X \rightarrow [0, \infty)$ be the mapping defined by

$$\sigma(x, y) = (x - y)^2, \text{ for all } x, y \in X.$$

Let $T : X \rightarrow K(X)$ be a mapping given by

$$T(x) = \begin{cases} \{0, 7\}, & \text{if } x \in (0, 7], \\ \{6\}, & \text{if } x = 0. \end{cases}$$

By the Example 1.1, (X, σ) is a complete b -metric space with constant $s = 2$.

To begin, we note that

$$\sup_{z \in T(x)} \sigma(z, T(y)) = \sup_{z \in T(y)} \sigma(T(x), z) = 1$$

That means $H(T(x), T(y)) = 1 > 0$.

Let $x, y \in X$ and denote

$$N(x, y) = \frac{1}{8}\sigma(x, y) + \frac{1}{8}\sigma(y, T(y)) + \frac{1}{16}\sigma(x, T(y)).$$

Now we'll look at the following cases:

1. If $x \in (0, 7]$ and $y = 0$. In this case, we obtain

$$\begin{aligned} \frac{\sigma(x, 0)}{294} - \frac{1}{H(T(x), T(0)) + 1} &\leq \frac{49}{294} - \frac{1}{2} \\ &< -\frac{8}{\sigma(0, T(0))} = -\frac{8}{36} \\ &< -\frac{1}{N(x, 0) + 1}. \end{aligned} \quad (2.20)$$

2. If $y \in (0, 7]$ and $x = 0$. In this case, we get

$$\begin{aligned} \frac{\sigma(0, y)}{294} - \frac{1}{H(T(0), T(y)) + 1} &\leq \frac{49}{294} - \frac{1}{2} \\ &< -\frac{16}{\sigma(0, T(y))} = -\frac{16}{49} \\ &\leq -\frac{1}{N(0, y)} \\ &< -\frac{1}{N(0, y) + 1}. \end{aligned} \quad (2.21)$$

From (2.20) and (2.21), we arrive at a conclusion

$$\frac{\sigma(x, y)}{294} - \frac{1}{H(T(x), T(y)) + 1} \leq -\frac{1}{N(x, y) + 1},$$

for all $x, y \in X$ with $H(T(x), T(y)) = 1 > 0$.

Therefore, by choosing $F(t) = -\frac{1}{t+1}$ and $\tau(t) = \frac{t}{294}$, for all $t \in (0, \infty)$, we observe that T is a multi-valued extended F -contraction of Hardy–Rogers type with $\alpha = \frac{1}{8} = \gamma, \delta = \frac{1}{16}, \beta = L = 0$ and all the conditions of Theorem 2.0.1 are contented with (\mathcal{H}_s^1) . As a result, T has a unique fixed point $x \in T(x)$ such that $x = 7$.

Theorem 2.0.2 : *Let (X, σ) be a complete b -metric space with constant $s \geq 1$, and $T : X \rightarrow K(X)$ a multi-valued extended F -contraction of Hardy–Rogers type with $F : (0, \infty) \rightarrow \mathbb{R}$ a non-decreasing function and $\tau \in \mathcal{S}_1$. Suppose that either (\mathcal{H}_s^1) or (\mathcal{H}_s^2) holds.*

We also assume that $\alpha + \beta + \gamma + 2\delta s + 2Ls < 1$ and $s^2\alpha + s^3(\delta + L) < 1$. Then, there's only one fixed point x in T and for every $x_0 \in X$ the sequence $T^n x_0$ for all $n \in \mathbb{N}$ converges to x .

Proof: First, we will prove the existence of a fixed point.

Let $x_0 \in X$. Since $\{x_0\}, T(x_0) \in K(X)$, there exists $x_1 \in T(x_0)$, where

$$\sigma(x_0, T(x_0)) = \sigma(x_0, x_1).$$

Now, as $T(x_0), T(x_1) \in K(X)$ there is a point $x_2 \in T(x_1)$, such that $\sigma(x_1, T(x_1)) = \sigma(x_1, x_2)$. Continuing in this manner, we create a sequence $\{x_n\}$, with $x_{n+1} \in T(x_n)$.

If there is $n_0 \in \mathbb{N}$ such that $x_{n_0} \in T(x_{n_0})$, then The proof is complete because x_{n_0} is the fixed point of the multi-valued map T .

But, if $x_n \notin T(x_n)$, for any $n \in \mathbb{N}$, we get

$$\sigma(x_n, T(x_n)) = \sigma(x_n, x_{n+1}) > 0, \text{ for all } n \in \mathbb{N}, \quad (2.22)$$

that provides

$$H(T(x_n), T(x_{n+1})) > 0, \text{ for all } n \in \mathbb{N}.$$

Through the (2nd) inequality of Lemma (1.4.1) and the contractive inequality (2.1) with $x = x_{n-1}$ and $y = x_n$, to obtain for all $n \in \mathbb{N}$:

$$\begin{aligned} \tau(\sigma(x_{n-1}, x_n)) + F(\sigma(x_n, T(x_n))) &\leq \tau(\sigma(x_{n-1}, x_n)) + F(H(T(x_{n-1}), T(x_n))) \\ &\leq F(\alpha\sigma(x_{n-1}, x_n) + \beta\sigma(x_{n-1}, T(x_{n-1})) + \gamma\sigma(x_n, T(x_n))) \\ &\quad + \delta\sigma(x_{n-1}, T(x_n)) + L\sigma(x_n, T(x_{n-1}))), \end{aligned}$$

in accordance with (2.22), for $\sigma_n = \sigma(x_n, x_{n+1})$, we find

$$\begin{aligned} \tau(\sigma(x_{n-1}, x_n)) + F(\sigma_n) &\leq F(\alpha\sigma(x_{n-1}, x_n) + \beta\sigma(x_{n-1}, x_n) + \gamma\sigma_n + \delta\sigma(x_{n-1}, T(x_n))) \\ &\quad + L\sigma(x_n, T(x_{n-1}))) \\ &\leq F((\alpha + \beta)\sigma(x_{n-1}, x_n) + \gamma\sigma_n + \delta\sigma(x_{n-1}, T(x_n)) + L\sigma(x_n, T(x_{n-1}))), \end{aligned}$$

using the (3rd) propriety of Lemma (1.4.1) and the monotonicity to obtain

$$\begin{aligned} \tau(\sigma(x_{n-1}, x_n)) + F(\sigma_n) &\leq F((\alpha + \beta)\sigma(x_{n-1}, x_n) + \gamma\sigma_n + \delta(s\sigma(x_{n-1}, x_n) + s\sigma(x_n, T(x_n)))) \\ &\quad + L(s\sigma(x_n, x_{n-1}) + s\sigma(x_{n-1}, T(x_{n-1})))) \\ &\leq F((\alpha + \beta)\sigma(x_{n-1}, x_n) + \gamma\sigma_n + s\delta\sigma(x_{n-1}, x_n) + s\delta\sigma_n + 2sL\sigma(x_{n-1}, x_n)) \\ &\leq F((\alpha + \beta + 2sL + \delta s)\sigma(x_{n-1}, x_n) + (\gamma + \delta s)\sigma_n), \end{aligned} \quad (2.23)$$

that gives us

$$\sigma_n < \frac{(\alpha + \beta + 2sL + \delta s)}{1 - (\gamma + \delta s)} \sigma(x_{n-1}, x_n).$$

Using the fact :

$$\frac{\alpha + \beta + 2sL + \delta s}{1 - (\gamma + \delta s)} < 1,$$

notice that : $\sigma_n \leq \sigma(x_{n-1}, x_n)$, i.e σ_n positive decreasing. That give for some $l \geq 0$

$$\lim_{n \rightarrow +\infty} \sigma_n = l.$$

If $l \neq 0$ through (2.23) we have

$$\tau(\sigma(x_{n-1}, x_n)) + F(\sigma_n) \leq F((\alpha + \beta + 2sL + s\delta)\sigma(x_{n-1}, x_n) + (\gamma + \delta s)\sigma_n).$$

Taking the upper limit we obtain:

$$\limsup_{n \rightarrow +\infty} \tau(\sigma(x_{n-1}, x_n)) + F(l) \leq F((\alpha + \beta + \gamma + 2sL + 2s\delta)l),$$

as we have this fact $\alpha + \beta + \gamma + 2sL + 2s\delta < 1$, we obtain

$$\limsup_{n \rightarrow +\infty} \tau(\sigma(x_{n-1}, x_n)) \leq 0.$$

According to the definition of the function τ , we get

$$\limsup_{n \rightarrow +\infty} \tau(\sigma(x_{n-1}, x_n)) = 0,$$

which is contradiction.

This provides us with

$$\lim_{n \rightarrow \infty} \sigma_n = 0, \tag{2.24}$$

through the (1st) propriety of Lemma (1.4.1) we obtain

$$\lim_{n \rightarrow \infty} \sigma(x_n, T(x_n)) = 0.$$

Now that we've established that $\{x_n\}$ is a Cauchy sequence, let's pretend that it isn't. Then there exist $\varepsilon > 0$ and two sequences $\{m_{(k)}\}$, $\{n_{(k)}\}$ of positive integers such that $n_{(k)}$ is the smallest index for which $n_{(k)} > m_{(k)} > k$ and $\sigma(x_{n_{(k)}}, x_{m_{(k)}}) > \varepsilon$. Due (2.24), this implies that $\sigma(x_{n_{(k)}-1}, x_{m_{(k)}}) \leq \varepsilon$ for all $k \geq 1$. Using the relaxed triangle inequality (b_3), we find:

$$\begin{aligned} \varepsilon &< \sigma(x_{m_{(k)}}, x_{n_{(k)}}) \\ &\leq s\sigma(x_{m_{(k)}}, x_{n_{(k)}-1}) + s\sigma(x_{n_{(k)}-1}, x_{n_{(k)}}) \\ &\leq s\varepsilon + s\sigma(x_{n_{(k)}-1}, x_{n_{(k)}}). \end{aligned}$$

Passing to the limit with respect to k in this latter inequality, by (2.24) we obtain

$$\varepsilon^+ \leq \liminf_{k \rightarrow \infty} \sigma(x_{m(k)}, x_{n(k)}) \leq \limsup_{k \rightarrow \infty} \sigma(x_{m(k)}, x_{n(k)}) \leq s\varepsilon^+.$$

We can deduce that there is $k_0 \in \mathbb{N}$ such that $\{\sigma(x_{m(k)}, x_{n(k)})\}$ is bounded for all $k \geq k_0$ and thereby it has a convergent sub-sequence. It follows that there exists a real number l and a sub-sequence $\{k(p)\}_{p \geq k_0}$ such that:

$$\lim_{p \rightarrow \infty} \sigma(x_{m(k(p))}, x_{n(k(p))}) = l, \quad (2.25)$$

with

$$0 < \varepsilon^+ \leq \liminf_{k \rightarrow \infty} \sigma(x_{m(k(p))}, x_{n(k(p))}) \leq l \leq \limsup_{k \rightarrow \infty} \sigma(x_{m(k(p))}, x_{n(k(p))}) \leq s\varepsilon^+. \quad (2.26)$$

Through the relaxed triangle inequality, we find

$$\begin{aligned} \sigma(x_{m(k(p))}, x_{n(k(p))}) &\leq s\sigma(x_{m(k(p))}, x_{m(k(p))+1}) + s\sigma(x_{m(k(p))+1}, x_{n(k(p))}) \\ &\leq s\sigma(x_{m(k(p))}, x_{m(k(p))+1}) + s^2\sigma(x_{m(k(p))+1}, x_{n(k(p))+1}) \\ &\quad + s^2\sigma(x_{n(k(p))+1}, x_{n(k(p))}), \end{aligned}$$

this gives

$$\begin{aligned} \sigma(x_{m(k(p))+1}, x_{n(k(p))+1}) &\geq \frac{1}{s^2}(\sigma(x_{m(k(p))}, x_{n(k(p))}) - s\sigma(x_{m(k(p))}, x_{m(k(p))+1}) \\ &\quad - s^2\sigma(x_{n(k(p))}, x_{n(k(p))+1})). \end{aligned} \quad (2.27)$$

We set this up, for convenience.

$$H_{m_p, n_p} = H(T(x_{m(k(p))}), T(x_{n(k(p))})),$$

taking the upper limit as $p \rightarrow \infty$ we obtain:

$$\limsup_{p \rightarrow \infty} \sigma(x_{m(k(p))+1}, x_{n(k(p))+1}) \geq \frac{l}{s^2},$$

due (2nd) inequality of Lemma (1.4.1) and (2.22), this leads to

$$\limsup_{p \rightarrow \infty} H_{m_p, n_p} \geq \limsup_{p \rightarrow \infty} \sigma(x_{m(k(p))+1}, T(x_{n(k(p))})) \geq \frac{l}{s^2}. \quad (2.28)$$

Let us put

$$\mu_s = \frac{l(1 - A_s)}{s^2 B_s}, \quad (2.29)$$

where

$$A_s = s^2\alpha + s^3(\delta + L) \quad (2.30)$$

and

$$B_s = 1 + \alpha + \beta + \gamma + 2s(\delta + L). \quad (2.31)$$

From the fact that $s^2\alpha + s^3(\delta + L) < 1$, we get $A_s < 1$ and $\mu_s > 0$. This implies, using (2.24), there exist $i_1, i_2 \geq k_0$ such that

$$\begin{aligned} \sigma_{m_{(k(p))}} &= \sigma(x_{m_{(k(p))}}, x_{m_{(k(p))}+1}) \leq \mu_s, \quad \text{for all } p \geq i_1, \\ \sigma_{n_{(k(p))}} &= \sigma(x_{n_{(k(p))}}, x_{n_{(k(p))}+1}) \leq \mu_s, \quad \text{for all } p \geq i_2. \end{aligned} \quad (2.32)$$

On the other hand, by virtue of (2.25) and $\mu_s > 0$, it follows that there exists $i_3 \geq k_0$ such that

$$\sigma_{m_p, n_p} = \sigma(x_{m_{(k(p))}}, x_{n_{(k(p))}}) \leq l + \mu_s, \quad \text{for all } p \geq i_3. \quad (2.33)$$

If $B_s = 1$, we get $\alpha = \beta = \gamma = \delta = L = 0$, which is contradiction with \mathcal{H}_s^1 or \mathcal{H}_s^2 .

If $B_s > 1$, we have $\mu_s < \frac{l}{s^2}$. Then due (2.28) and $\mu_s > 0$, there exist $i_4 \geq k_0$ such that

$$\sigma(x_{m_{(k(p))}+1}, x_{n_{(k(p))}+1}) > \frac{l - s^2\mu_s}{s^2} > 0, \quad \text{for all } p \geq i_4. \quad (2.34)$$

Using (b_3) , the contractive inequality (2.1), the monotonicity of F and (2.34) to obtain for all $p \geq i_4$:

$$\begin{aligned} \tau(\sigma_{m_p, n_p}) + F(H_{m_p, n_p}) &\leq F(\alpha\sigma_{m_p, n_p} + \beta\sigma(x_{m_{(k(p))}}, T(x_{m_{(k(p))}})) + \gamma\sigma(x_{n_{(k(p))}}, T(x_{n_{(k(p))}})) \\ &\quad + \delta\sigma(x_{m_{(k(p))}}, T(x_{n_{(k(p))}})) + L\sigma(x_{n_{(k(p))}}, T(x_{m_{(k(p))}}))), \end{aligned}$$

the above inequality leads to

$$\tau(\sigma_{m_p, n_p}) + F(H_{m_p, n_p}) \leq F((\alpha + s(\delta + L))\sigma_{m_p, n_p} + (\beta + sL)\sigma_{m_{(k(p))}} + (\gamma + s\delta)\sigma_{n_{(k(p))}}). \quad (2.35)$$

Setting $i = \max\{i_1, i_2, i_3, i_4\}$, we combine the above inequalities with (2.29)-(2.35) for all $p \geq i$ we arrive at :

$$\begin{aligned} \liminf_{p \rightarrow \infty} \tau(\sigma_{m_p, n_p}) + F\left(\frac{l - s^2\mu_s}{s^2}\right) &\leq \limsup_{p \rightarrow \infty} [\tau(\sigma_{m_p, n_p}) + F(H_{m_p, n_p})] \\ &\leq \limsup_{p \rightarrow \infty} F\left(\frac{A_s}{s^2}\sigma_{m_p, n_p} + (\beta + sL)\sigma_{m_{(k(p))}} \right. \\ &\quad \left. + (\gamma + s\delta)\sigma_{n_{(k(p))}}\right) \\ &\leq \limsup_{p \rightarrow \infty} F\left(\frac{A_s}{s^2}(l + \mu_s) + (\beta + sL)\mu_s + (\gamma + s\delta)\mu_s\right) \\ &\leq F\left(\frac{l - s^2\mu_s}{s^2}\right). \end{aligned}$$

implies that

$$\liminf_{p \rightarrow \infty} \tau(\sigma_{m_p, n_p}) \leq 0,$$

for all $p \geq i$, which is contradiction with $\tau \in S_1$, it shows that $\{x_n\}$ is a Cauchy sequence converges to some point x in X , that is,

$$\lim_{n \rightarrow \infty} \sigma(x_n, x) = 0. \quad (2.36)$$

Then, we demonstrate that $x \in T(x)$ i.e x is a fixed point of T . Assume the opposite $x \notin T(x)$ is true. Since $T(x)$ is compact, this means that the proof is complete when $\sigma(x, T(x)) = 0$.

Then through (2.36) and the (2nd) and the (3rd) inequalities of Lemma (1.4.1) we'll start with the other case, which is $\sigma(x, T(x)) > 0$ to obtain

$$\begin{aligned} \sigma(x, T(x)) &\leq s\sigma(x, x_{n+1}) + s\sigma(x_{n+1}, T(x)) \\ &\leq s\sigma(x, x_{n+1}) + sH(T(x_n), T(x)). \end{aligned} \quad (2.37)$$

On the other hand we have:

$$\begin{aligned} H(T(x_n), T(x)) &\leq \alpha\sigma(x_n, x) + \beta\sigma(x_n, T(x_n)) + \gamma\sigma(x, T(x)) + \delta\sigma(x_n, T(x)) + L\sigma(x, T(x_n)) \\ &\leq \alpha\sigma(x_n, x) + \beta\sigma(x_n, x_{n+1}) + \gamma\sigma(x, T(x)) + s\delta\sigma(x, T(x)) \\ &\quad + s\delta\sigma(x_n, x) + L\sigma(x, x_{n+1}), \end{aligned} \quad (2.38)$$

implies that:

$$\begin{aligned} \sigma(x, T(x)) &\leq s\sigma(x, x_{n+1}) + s\alpha\sigma(x_n, x) + s\beta\sigma(x_n, T(x_n)) + s\gamma\sigma(x, T(x)) + s\delta\sigma(x_n, T(x)) \\ &\leq s\sigma(x, x_{n+1}) + s\alpha\sigma(x_n, x) + s\beta\sigma(x_n, x_{n+1}) + s\gamma\sigma(x, T(x)) + s^2\delta\sigma(x_n, x) \\ &\quad + s^2\delta\sigma(x, T(x)) + sL\sigma(x, x_{n+1}). \end{aligned} \quad (2.39)$$

The above inequality leads to

$$\begin{aligned} \sigma(x, T(x)) &\leq (1 + L)s\sigma(x, x_{n+1}) + (s\alpha + s^2\delta)\sigma(x_n, x) \\ &\quad + s\beta\sigma(x_n, x_{n+1}) + (s\gamma + s^2\delta)\sigma(x, T(x)). \end{aligned} \quad (2.40)$$

Taking the limit superior as $n \rightarrow \infty$ in (2.40)

$$(1 - s\gamma - s^2\delta)\sigma(x, T(x)) \leq 0 \Rightarrow (1 - s\gamma)\sigma(x, T(x)) \leq s^2\delta\sigma(x, T(x)). \quad (2.41)$$

In a similar way, we can also apply the contractive condition (2.2) with the symmetry condition to obtain:

$$(1 - s\beta - Ls^2)\sigma(x, T(x)) \leq 0 \Rightarrow (1 - s\beta)\sigma(x, T(x)) \leq Ls^2\sigma(x, T(x)). \quad (2.42)$$

Again, according to (\mathcal{H}_s^1) and (\mathcal{H}_s^2) we consider the following cases:

1st case If (\mathcal{H}_s^1) hold, we have $(1 - \gamma) > 0$ and $\gamma + s\delta < \frac{1}{s}$.

Consequently, (2.41) implies that:

$$\sigma(x, T(x)) \leq \frac{s^2\delta}{(1 - s\gamma)}\sigma(x, T(x)) < \sigma(x, T(x)),$$

which is contradiction.

2nd case If (\mathcal{H}_s^2) hold, we have $(1 - \beta) > 0$ and $\beta + sL < \frac{1}{s}$.

Consequently, (2.42) implies that:

$$\sigma(x, T(x)) \leq \frac{Ls^2}{(1 - s\beta)}\sigma(x, T(x)) < \sigma(x, T(x)),$$

which is contradiction. Therefore, whether (\mathcal{H}_s^1) or (\mathcal{H}_s^2) holds, we obtain contradiction.

So we have $x \in T(x) \Leftrightarrow Fix(t) \neq \emptyset$.

Finally, we'll demonstrate that T has only one fixed point.

Let x and y be two distinct fixed points of T such that $x \neq y$, through the (2nd) inequality of Lemma (1.4.1) and the contractive inequality we obtain:

$$\begin{aligned} \sigma(T(x), y) &\leq H(T(x), T(y)) \\ &\leq \alpha\sigma(x, y) + \beta\sigma(x, T(x)) + \gamma\sigma(y, T(y)) + \delta\sigma(x, T(y)) + L\sigma(y, T(x)) \quad (2.43) \\ &= \alpha\sigma(x, y) + \delta\sigma(x, T(y)) + L\sigma(y, T(x)). \end{aligned}$$

1. If $\alpha + \delta + L > 0$, note that $\alpha + \delta + L \leq 1$ we have:

$$\begin{aligned} \tau(\sigma(x, y)) + F(H(T(x), T(y))) &\leq F((\alpha + \delta + L)\sigma(x, y)) \\ &\leq F(\sigma(x, y)), \end{aligned}$$

as $\sigma(x, y) < H(T(x), T(y))$ we find

$$\tau(\sigma(x, y)) + F(H(T(x), T(y))) \leq F(H(T(x), T(y))),$$

implies that

$$\tau(\sigma(x, y)) = 0,$$

which is contradiction with $\tau \in \mathcal{S}_1$.

2. if $\alpha + \delta + L = 0$, using (2.43) then:

$$H(T(x), T(y)) \leq (\alpha + \delta + L)H(T(x), T(y)) = 0,$$

gives that $H(T(x), T(y)) = 0$, which implies

$$\sigma(x, y) = 0.$$

This gives $y = x$.

which is contradiction.

In conclusion, we find that only one fixed point exists in T . Then, the Theorem proof has come to an end. ■

CHAPTER

3

APPLICATION TO INTEGRAL INCLUSION EQUATION OF VOLTERRA-TYPE

In this chapter, we will apply the theorem (2.0.2) to study the existence and the uniqueness of a solution for the following integral inclusion equation of Volterra-type as

$$x(t) \in \int_0^t M(t, s, x(s))ds + g(t), \quad t \in [0, r], \quad (3.1)$$

where $r > 0$ and $M : [0, r] \times [0, r] \times \mathbb{R} \rightarrow K(\mathbb{R})$, for each $x \in C([0, r], \mathbb{R})$, the operator $M(\cdot, \cdot, x)$ is continuous. Further the function $g : [0, r] \rightarrow \mathbb{R}$ is continuous.

Let $X = C([0, r], \mathbb{R})$ be the space of all continuous real-valued functions on $[0, r]$. Note that X is complete b -metric space by considering the following distance

$$\sigma(x, y) = \sup_{t \in [0, r]} e^{-2t}(x(t) - y(t))^2, \quad \text{for all } x, y \in X, \quad (3.2)$$

with $s = 2$.

For the integral inclusion given above, we can define a multi-valued operator $T : C([0, r], \mathbb{R}) \rightarrow$

$K(C([0, r], \mathbb{R}))$ as follows:

$$Tx(t) = \left\{ u \in C([0, r], \mathbb{R}) : u \in \int_0^t M(t, s, x(s))ds + g(t), t \in [0, r] \right\},$$

there exists $m : [0, r] \times [0, r] \times \mathbb{R} \rightarrow \mathbb{R}$ a continuous operator with $m(t, s, x(s)) \in M(t, s, x(s))$. This shows that $\int_0^t m(t, s, x(s))ds + g(t) \in Tx(t)$. Thus, the operator Tx is nonempty.

Theorem 3.0.1 : *Suppose that the following hypothesis hold:*

For all $s, t \in [0, r]$ we have

$$\sup \left\{ |m(t, s, x(s)) - n(t, s, y(s))|^2; (m, n) \in M_x \times M_y \right\} \leq \frac{2(x(s) - y(s))^2}{(x(s) - y(s))^2 + 16}.$$

Then the integral inclusion (3.1) has a unique solution.

Proof: We have to show that the operator T satisfies all conditions of Theorem (2.0.2).

Let $u \in T(x)$ and $v \in T(y)$. Using the hypothesis in the theorem above, we obtain

$$\begin{aligned} |u(t) - v(t)|^2 &\leq \left(\int_0^t |m_u(t, s, x(s)) - m_v(t, s, y(s))|^2 ds \right) \\ &\leq \int_0^t \frac{2(x(s) - y(s))^2}{(x(s) - y(s))^2 + 16} ds \\ &\leq \int_0^t \frac{2(x(s) - y(s))^2 e^{-2s} e^{2s}}{(x(s) - y(s))^2 e^{-2s} + 16} ds. \end{aligned}$$

This leads to

$$\begin{aligned} |u(t) - v(t)|^2 &\leq \frac{2\sigma(x, y)}{\sigma(x, y) + 16} \int_0^t e^{2s} ds \\ &\leq \frac{\sigma(x, y)e^{2t}}{\sigma(x, y) + 16}, \end{aligned}$$

which implies that

$$|u(t) - v(t)|^2 e^{-2t} \leq \frac{\sigma(x, y)}{\sigma(x, y) + 16}.$$

Taking the supremum with respect to $t \in [0, r]$ in the above inequality, we have

$$\sigma(u, v) \leq \frac{\sigma(x, y)}{\sigma(x, y) + 16} \leq \sigma(x, y). \quad (3.3)$$

So

$$\ln(\sigma(u, v)) \leq \ln \left(\frac{\sigma(x, y)}{\sigma(x, y) + 16} \right) < \ln \left(\frac{\sigma(x, y)}{16} \right),$$

this leads to

$$\ln \left(\frac{\sigma(x, y)}{\sigma(x, y) + 16} \right) + \ln(\sigma(u, v)) \leq \frac{\sigma(x, y)}{8},$$

functions u and v are arbitrary in $T(x)$ and $T(y)$, resp., therefore

$$\ln\left(\frac{\sigma(x, y)}{\sigma(x, y) + 16}\right) + \ln(H(T(x), T(y))) \leq \ln\left(\frac{1}{8}\sigma(x, y)\right), \quad (3.4)$$

By choosing $F(t) = \ln(t)$ and $\tau(t) = \ln\left(\frac{t}{t + 16}\right)$, for all $t \in (0, \infty)$, inequality (3.4) can be equivalently written as

$$\tau(\sigma(x, y)) + F(H(T(x), T(y))) \leq F\left(\frac{1}{8}\sigma(x, y)\right).$$

Therefore, all the conditions of Theorem (2.0.2) are satisfied with $\alpha = \frac{1}{8}, \gamma = \beta = L = \delta = 0$. Therefore, T has a unique fixed point, that is, the Volterra-type integral inclusion (3.1) has a unique solution. ■

Conclusion

This dissertation is divided into three chapters.

- In Chapter 1, we included some primly notions and definitions of various types contractions, methods and examples both on the fixed point as well as on the fixed point theorems.
- In Chapter 2, we defined the Hausdorff distance function H and we investigate a few new results on fixed point for $T : X \rightarrow K(X)$ by using the contractive condition of Hardy-Rogers type .
- In Chapter 3, an application, for the solution of integral inclusion, are given to illustrate the usability of the obtained results.

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