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Dedication

To those for whom Allah has ordained kindness and excellence:

"And be good to parents."

And to those whose rights are accompanied by the supplication of the Most Merciful:

"My Lord, have mercy upon them as they brought me up when I was small."

And to those whose companionship has been commanded in every circumstance:

"And accompany them in this world with kindness."

To my beloved parents, who have always been my greatest support through their endless love, sincere prayers, guidance, and sacrifices. May Allah bless them abundantly and grant them the highest ranks in Paradise.

To all my brothers and sisters, their spouses, and their children.

To everyone who paved the path of knowledge with their effort, patience, and time.

To every seeker of knowledge and inheritor of the Prophetic legacy.

To all my dear friends who stood beside me throughout this journey.

To everyone remembered in my heart, even if my pen fails to mention them.

I dedicate this humble work, asking Allah Almighty to make it sincerely for His Noble Face, to make me among the seekers of beneficial knowledge, to teach me what benefits me, to benefit me with what He has taught me, and to increase me in knowledge. Indeed, He is the All-Hearing, the All-Knowing.

May the peace and blessings of Allah be upon our Prophet Muhammad, his family, his companions, and all those who follow them in righteousness until the Day of Judgment.

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Abstract

This thesis presents an analytical study of nonlinear fractional boundary value problems involving Caputo and Riemann–Liouville fractional derivatives. The primary objective is to establish existence and uniqueness results for solutions of such problems defined on finite intervals under various boundary conditions. The analysis is developed within a rigorous mathematical framework that combines tools from fractional calculus, functional analysis, and fixed point theory. Two main classes of problems are investigated. The first class concerns Caputo fractional boundary value problems, including a nonlocal problem and a two-point problem with a convection term, for which existence and uniqueness results are obtained using the Banach contraction principle and the Schauder fixed point theorem, respectively. The second class addresses a Riemann–Liouville fractional boundary value problem with mixed boundary conditions, where the problem is reformulated as an equivalent integral equation via the associated Green’s function. A new sufficient condition for the existence of a unique solution is then derived by applying Rus’s fixed point theorem within a two-metric framework. The obtained criterion is shown to be less restrictive than existing results in the literature, thereby extending the applicability of the theory to a wider class of problems.

Keywords: Fractional boundary value problems, Caputo derivative, Riemann–Liouville derivative, existence, uniqueness .

Résumé

Cette thèse présente une étude analytique de problèmes aux limites fractionnaires non linéaires faisant intervenir les dérivées fractionnaires de Caputo et de Riemann–Liouville. L’objectif principal est d’établir des résultats d’existence et d’unicité pour les solutions de tels problèmes définis sur des intervalles finis sous diverses conditions aux limites. L’analyse est développée dans un cadre mathématique rigoureux combinant des outils du calcul fractionnaire, de l’analyse fonctionnelle et de la théorie du point fixe. Deux grandes classes de problèmes sont étudiées. La première concerne des problèmes aux limites fractionnaires de type Caputo, comprenant un problème non local et un problème à deux points avec un terme de convection, pour lesquels des résultats d’existence et d’unicité sont établis respectivement à l’aide du principe de contraction de Banach et du théorème du point fixe de Schauder. La seconde classe porte sur un problème aux limites fractionnaire de Riemann–Liouville avec des conditions aux limites mixtes, où le problème est reformulé en une équation intégrale équivalente via la fonction de Green associée. Une nouvelle condition suffisante pour l’existence d’une solution unique est ensuite obtenue en appliquant le théorème du point fixe de Rus dans un cadre à deux métriques. Le critère obtenu est démontré moins restrictif que les résultats existants dans la littérature, élargissant ainsi le champ d’applicabilité de la théorie.

Mots-clés: Problèmes aux limites fractionnaires, dérivée de Caputo, dérivée de Riemann–Liouville, existence, unicité.

الملخص

تقدم هذه الأطروحة دراسة تحليلية لمسائل القيم الحدية الكسرية غير الخطية التي تتضمن مشتقات كابوتو وريمان-ليوفيل الكسرية، وتهدف إلى إثبات نتائج الوجود والوحدانية لحلول هذه المسائل المعرفة على فترات منتهية تحت شروط حدية مختلفة. يُبنى التحليل ضمن إطار رياضي صارم يجمع بين أدوات الحساب التفاضلي والتكاملي الكسري والتحليل الدالي ونظرية النقطة الثابتة.

تُدرس صنفان رئيسيان من المسائل؛ يتعلق الصنف الأول بمسائل القيم الحدية الكسرية من نوع كابوتو، وتشمل مسألة غير محلية ومسألة ثنائية النقطة تحتوي على حد الحمل، وقد أُثبتت نتائج الوجود والوحدانية لكل منهما باستخدام مبدأ الانكماش لباناخ ونظرية النقطة الثابتة لشاوادر على التوالي.

أما الصنف الثاني فيتناول مسألة قيم حدية كسرية من نوع ريمان-ليوفيل ذات شروط حدية مختلطة، إذ تُعاد صياغة المسألة في صورة معادلة تكاملية مكافئة بالاستعانة بدالة غرين المرتبطة بها، ثم تُشتق شرط كافٍ جديد لوجود حل وحيد عبر تطبيق نظرية النقطة الثابتة لروس في إطار ذي مقياسين. ويُنَّ أن المعيار المتحصل عليه أقل تقييداً من النتائج الموجودة في الأدبيات، مما يوسع نطاق تطبيق النظرية على فئة أشمل من المسائل.

الكلمات المفتاحية: مسائل القيم الحدية الكسرية، مشتقة كابوتو، مشتقة ريمان-ليوفيل، الوجود، الوحدانية.

Notations

This section summarizes the main notations used throughout this work in order to enhance clarity and readability.

$\mathbb{N}$	set of natural numbers.
$\mathbb{N}^* = \mathbb{N} \setminus \{0\}$	set of nonzero natural numbers.
$\mathbb{R}$	set of real numbers.
$\mathbb{R}_+$	set of positive real numbers.
$\mathbb{C}$	set of complex numbers.
$L^p(\mathbb{R})$	space of measurable functions whose p -th power is integrable on $\mathbb{R}$, where $p \in [1, +\infty[$.
$L^\infty(\mathbb{R})$	space of essentially bounded measurable functions on $\mathbb{R}$.
$C(\mathbb{R})$	The space of continuous functions defined on $\mathbb{R}$.
$C^n(\Omega)$	The space of n -times continuously differentiable functions $f : \Omega \rightarrow \mathbb{R}$.
$AC([a, b])$	Space of absolutely continuous functions on $[a, b]$.
$AC^n([a, b])$	Space of functions whose $(n - 1)$ -th derivative is absolutely continuous on $[a, b]$.
$\Gamma(\cdot)$	Gamma function.
$B(\cdot, \cdot)$	Beta function.
$E_{\alpha, \beta}(\cdot)$	Mittag–Leffler function with two parameters.
$\operatorname{Re}(z)$	real part of a complex number z .
$\operatorname{Im}(z)$	imaginary part of z .
$[x]$	integer part (floor) of a real number x .
$\mathcal{I}_{a+}^\alpha$	left-sided Riemann–Liouville fractional derivative of order α .
$\mathcal{I}_{b-}^\alpha$	right-sided Riemann–Liouville fractional derivative of order α .
$\mathcal{D}_{a+}^\alpha$	the Riemann–Liouville fractional derivative defined on the left side of order α .
$\mathcal{D}_{b-}^\alpha$	the Riemann–Liouville fractional derivative defined on the right side of order α .
${}^C\mathcal{D}_{a+}^\alpha$	the Caputo fractional derivative of order α taken from the left side.
${}^C\mathcal{D}_{b-}^\alpha$	the Caputo fractional derivative of order α taken from the right side.
${}_2F_1$	Gauss hypergeometric function.
$\ \cdot\ $	norm in a normed space.

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General Introduction

Fractional calculus is a generalization of classical calculus in which the concepts of differentiation and integration are extended to arbitrary (non-integer) orders. This extension provides a powerful mathematical framework for modeling systems with memory and hereditary properties that cannot be accurately described by classical integer-order models. Due to its ability to capture complex dynamical behaviors, fractional calculus has attracted considerable attention and has been widely applied in physics, engineering, and applied sciences, including viscoelasticity, signal processing, control theory, and anomalous diffusion phenomena.

Various definitions of fractional derivatives have been introduced in the literature, among which the most commonly used are the Riemann–Liouville and Caputo derivatives. The Riemann–Liouville derivative is one of the earliest formulations and plays a fundamental role in theoretical analysis, particularly in the development of analytical results. On the other hand, the Caputo derivative is often preferred in practical applications due to its compatibility with classical initial and boundary conditions, which makes it more suitable for modeling real-world problems ([4],[14],[19],[24], [26]).

Boundary value problems (BVPs) for fractional differential equations have gained significant importance due to their ability to model a wide range of real-world phenomena more accurately than classical models ([16],[19],[26]). Such problems arise naturally in various fields, including viscoelastic materials, electrical circuits, fluid mechanics, and diffusion processes with memory effects ([4],[5],[10],[28]). The study of these problems is mainly concerned with establishing the existence, uniqueness, and qualitative properties of solutions, as well as their dependence on given data ([2], [3],[21],[36],[37]).

To analyze fractional boundary value problems, several analytical methods have been developed. Among these, fixed point theory plays a central role and has proven to be an effective and widely used approach. In this work, particular emphasis is placed on fixed point techniques. Specifically, the Banach contraction principle and Schauder fixed point theorem are employed as the main tools to establish the existence of solutions under appropriate conditions. In addition, Rus fixed point theorem is utilized to treat certain classes of nonlinear operators and to obtain existence results in more general settings. These methods provide a flexible and powerful framework for the analysis of nonlinear fractional boundary value problems ([2],[7],[15],[40]).

Another fundamental tool in the study of boundary value problems is the use of Green's functions. This approach allows the transformation of the differential equation into an equivalent integral equation, which is often easier to analyze. The Green's function representation plays a crucial role in deriving integral formulations of the problem and facilitates the application of fixed point theorems, including the Banach, Schauder, and Rus techniques, to establish the existence of solutions ([19],[22],[28]).

Thesis Structure

To provide a comprehensive study of fractional boundary value problems, this work is organized into three main parts:

- In the first chapter, we assembled the mathematical prerequisites necessary for the subsequent analysis, including functional analysis tools, fixed point theorems (Banach, Rus, and Schauder), classical special functions (Gamma, Beta, and Gauss hypergeometric functions), and the main operators of fractional calculus.
- The second chapter, we studied two classes of Caputo fractional boundary value problems. For the nonlocal problem, a unique solution was established via the Banach contraction principle under a Lipschitz condition on the nonlinearity and the nonlocal functional. For the two-point problem with a convection term, existence of at least one solution was obtained via the Schauder fixed point theorem, supported by a detailed analysis of the associated Green's function and its nonnegativity properties.
- In the third chapter, we investigated a Riemann–Liouville fractional boundary value problem of order $2 < \alpha \leq 3$. An equivalent integral formulation was derived via the associated Green's function. Using an L^2 -norm framework and Rus's fixed point theorem in a two-metric setting, we established a new sufficient condition for existence and uniqueness. For the special case $\alpha = 3$, the condition reduces to

$$K \frac{(d-c)^3}{\sqrt{315}} < 1,$$

which was shown to be less restrictive than the classical criterion of Laadjal, since $\sqrt{315} \approx 17.748 > 12$.

Study Objective

The objective of this work is to study fractional boundary value problems involving different types of fractional derivatives, with a particular focus on establishing existence results and analyzing the properties of solutions by using Green's function techniques, integral representations, and fixed point methods, within the framework of nonlinear, nonlocal, and nonstandard boundary conditions.

Fundamentals of Functional Analysis and Fractional Calculus

This chapter presents the basic preliminaries required for the subsequent analysis, including functional analysis tools, fixed point results, special functions, and fractional calculus operators.

I Foundations of Functional Analysis

This section is devoted to the basic foundations of functional analysis that are essential for the development of the subsequent results. In particular, we recall fundamental notions concerning metric spaces, normed spaces, Banach spaces, and different types of continuity.

Definition 1.1: [32]

A **metric space** is a pair (E, d) , where E is a nonempty set and $d : E \times E \rightarrow \mathbb{R}$ is a function, called a **metric** or **distance function**, satisfying the following properties for all $x, y, z \in E$:

- $d(x, y) \geq 0$;
- $d(x, y) = 0$ if and only if $x = y$;
- $d(x, y) = d(y, x)$;
- $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 1.2: [32]

A **norm** on a vector space E is a mapping $\|\cdot\| : E \rightarrow \mathbb{R}^+$ such that, for all $x, y \in E$ and all $\lambda \in \mathbb{K}$, the following properties hold:

- $\|x\| \geq 0$ with $\|x\| = 0$ if and only if $x = 0$;
- $\|\alpha x\| = |\alpha| \|x\|$;

- $\|x + y\| \leq \|x\| + \|y\|.$

The pair $(E, \|\cdot\|)$ is called a **normed vector space**.

Remark 1.1 Let $(E, \|\cdot\|)$ be a normed space. Then the mapping $d : E \times E \rightarrow \mathbb{R}$ defined by

$$d(x, y) = \|x - y\|$$

is a metric on E . Therefore, every normed space induces a natural metric space structure.

Definition 1.3: [31]

Let $(E, \|\cdot\|)$ be a normed space. A sequence $(x_n) \subset E$ is said to converge to a point $x \in E$ if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|x_n - x\| < \varepsilon \quad \text{for all } n \geq N.$$

In this case, we write $x_n \rightarrow x$ as $n \rightarrow \infty$.

Definition 1.4: [31]

Let $(E, \|\cdot\|)$ be a normed space. A sequence $(x_n) \subset E$ is said to be a **Cauchy sequence** if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \varepsilon \quad \text{for all } n, m \geq N.$$

Definition 1.5: [32]

A metric space (E, d) is said to be **complete** if every Cauchy sequence in E converges to a limit in E .

Definition 1.6: [32]

A Banach space is a normed vector space that is complete.

Example 1 [32] The following are important examples of Banach spaces:

- The space $C([a, b])$ of all continuous real-valued functions on $[a, b]$, equipped with the supremum norm

$$\|f\|_\infty = \sup_{t \in [a, b]} |f(t)|,$$

is a Banach space.

- The space $L^p([a, b])$, $1 \leq p \leq \infty$, endowed with the norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

and

$$\|f\|_\infty = \operatorname{ess\,sup}_{t \in [a, b]} |f(t)|,$$

is also a Banach space.

Definition 1.7: ([30], [35])

Let $T : E \rightarrow F$ be a mapping between normed spaces.

- The mapping T is said to be continuous at a point $x \in E$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in E$,

$$\|y - x\|_E < \delta \Rightarrow \|T(y) - T(x)\|_F < \varepsilon.$$

- T is said to be uniformly continuous on a subset $A \subset E$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in A$,

$$\|y - x\|_E < \delta \Rightarrow \|T(y) - T(x)\|_F < \varepsilon.$$

- A family $\{T_n\}_{n \in \mathbb{N}}$ of mappings $T_n : E \rightarrow F$ is said to be **equicontinuous** on a subset $A \subset E$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in A$ and all $n \in \mathbb{N}$,

$$\|y - x\|_E < \delta \Rightarrow \|T_n(y) - T_n(x)\|_F < \varepsilon.$$

II

Main Results in Fixed Point Theory

This section is devoted to several classical results in fixed point theory. We begin by introducing the Banach fixed point theorem for contraction mappings, which is a central tool in nonlinear analysis. We then discuss the Rus fixed point theorem, and finally present the Schauder fixed point theorem. ([9], [10], [39])

Definition 1.8

Let $T : E \rightarrow E$ be a mapping. An element $x \in E$ is called a **fixed point** of T if

$$T(x) = x.$$

Definition 1.9

Let (E, d) be a metric space. A mapping $T : E \rightarrow E$ is called a **contraction** if there exists a constant $0 < k < 1$ such that

$$d(T(x), T(y)) \leq k d(x, y), \quad \forall x, y \in E.$$

Theorem 1.1: [Banach Fixed Point Theorem]

Let E be a Banach space and let $T : E \rightarrow E$ be a contraction mapping. Then T has a unique fixed point in E , i.e., there exists a unique $x \in E$ such that $T(x) = x$.

Theorem 1.2: [29][Rus Fixed Point Theorem]

Let E be a nonempty set equipped with two metrics ρ and σ , and assume that (E, ρ) is a complete metric space. Let $T : E \rightarrow E$ be an operator satisfying the following conditions:

1. T is continuous with respect to the metric ρ ;
2. there exists a constant $\delta > 0$ such that

$$\rho(Tu, Tv) \leq \delta \sigma(u, v), \quad \forall u, v \in E;$$

3. there exists a constant $0 < \theta < 1$ such that

$$\sigma(Tu, Tv) \leq \theta \sigma(u, v), \quad \forall u, v \in E.$$

Then T possesses a unique fixed point $w \in E$, i.e.,

$$\exists! w \in E \text{ such that } T(w) = w.$$

Theorem 1.3: [Schauder Fixed Point Theorem]

Let (E, d) be a complete metric space and let K be a nonempty closed and convex subset of E . Assume that $T : K \rightarrow K$ is a continuous mapping such that the image $T(K)$ is relatively compact in E . Then T admits at least one fixed point in K .

Theorem 1.4: [Arzelà-Ascoli Theorem]

Let A be a subset of $C(J, E)$. Then A is relatively compact in $C(J, E)$ if and only if the following conditions are satisfied:

1. The set A is uniformly bounded; that is, there exists a constant $M \geq 0$ such that

$$\|f(x)\| \leq M, \quad \forall x \in J, \forall f \in A.$$

2. The family A is equicontinuous on J , i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|t_1 - t_2| < \delta \Rightarrow \|f(t_1) - f(t_2)\| < \varepsilon,$$

for all $t_1, t_2 \in J$ and all $f \in A$.

3. For each $x \in J$, the set $\{f(x) : f \in A\}$ is relatively compact in E .

III**Special Functions and Their Properties**

This section is devoted to the presentation of some classical special functions and their fundamental properties, which play an important role in fractional calculus.

1**Analytical Properties of the Gamma Function**

We recall some basic analytical properties of the Gamma function ([5],[18],[19],[24],[26]).

Definition 1.10

For every complex number α with $\operatorname{Re}(\alpha) > 0$, the **Gamma function** is defined by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt. \quad (1.1)$$

1.1 Properties of the Gamma Function

- **Recursive (functional) property:**

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha), \quad \alpha > 0. \quad (1.2)$$

The Gamma function satisfies the fundamental recurrence relation extending the factorial.

$$\begin{aligned}
 \Gamma(\alpha + 1) &= \int_0^{+\infty} t^\alpha e^{-t} dt \\
 &= \int_0^{+\infty} t^\alpha d(-e^{-t}) \\
 &= \left[-t^\alpha e^{-t} \right]_0^{+\infty} + \alpha \int_0^{+\infty} t^{\alpha-1} e^{-t} dt \\
 &= \alpha \Gamma(\alpha), \quad \alpha > 0.
 \end{aligned}$$

- **Extension of the Gamma function to the half-plane $\operatorname{Re}(\alpha) \leq 0$:**

$$\Gamma(\alpha) = \frac{\Gamma(\alpha + k)}{(\alpha)_k}, \quad \operatorname{Re}(\alpha) > -k, \quad k \in \mathbb{N}^*, \quad \alpha \notin \mathbb{Z}_0^-, \quad (1.3)$$

where $(\alpha)_k$ denotes the Pochhammer symbol defined by

$$(\alpha)_0 = 1, \quad (\alpha)_k = \alpha(\alpha + 1) \cdots (\alpha + k - 1), \quad k \in \mathbb{N}.$$

$$\begin{aligned}
 \Gamma(\alpha + 1) &= \alpha \Gamma(\alpha), \\
 \Gamma(\alpha + 2) &= (\alpha + 1) \Gamma(\alpha + 1) = (\alpha + 1) \alpha \Gamma(\alpha), \\
 \Gamma(\alpha + 3) &= (\alpha + 2) \Gamma(\alpha + 2) = (\alpha + 2)(\alpha + 1) \alpha \Gamma(\alpha), \\
 &\vdots \\
 \Gamma(\alpha + k) &= (\alpha + k - 1)(\alpha + k - 2) \cdots (\alpha + 1) \alpha \Gamma(\alpha).
 \end{aligned} \quad (1.4)$$

Hence, we obtain

$$\Gamma(\alpha + k) = (\alpha)_k \Gamma(\alpha). \quad (1.5)$$

- **Consistency with the factorial:**

$$\Gamma(n + 1) = n!, \quad n \in \mathbb{N}. \quad (1.6)$$

The Gamma function extends the factorial by repeated use of its recurrence

relation.

$$\begin{aligned}
 \Gamma(n+1) &= n \Gamma(n) \\
 &= n(n-1) \Gamma(n-1) \\
 &= n(n-1)(n-2) \Gamma(n-2) \\
 &\vdots \\
 &= n \cdot (n-1) \cdots 2 \cdot \Gamma(1) \\
 &= n!, \quad n \in \mathbb{N}.
 \end{aligned}$$

• **Special value:**

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \quad (1.7)$$

The value of $\Gamma(\frac{1}{2})$ is obtained using the Gaussian integral.

$$\begin{aligned}
 \Gamma\left(\frac{1}{2}\right) &= \int_0^{+\infty} t^{-1/2} e^{-t} dt \\
 &= \int_0^{+\infty} x^{-1} e^{-x^2} \cdot 2x dx \quad (t = x^2) \\
 &= 2 \int_0^{+\infty} e^{-x^2} dx \\
 &= \left(\int_{-\infty}^{+\infty} e^{-x^2} dx \right) \\
 &= \sqrt{\pi}.
 \end{aligned}$$

Theorem 1.5: [8][Weierstrass product for the Gamma function]

For $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, we have

$$\frac{1}{\Gamma(\alpha)} = \alpha e^{\gamma\alpha} \prod_{k=1}^{\infty} \left(1 + \frac{\alpha}{k}\right) e^{-\alpha/k},$$

where γ is the Euler–Mascheroni constant.

Proposition 1 [*Limit representation of the Gamma function*] Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$. Then

$$\Gamma(\alpha) = \lim_{n \rightarrow \infty} \frac{n! n^\alpha}{(\alpha+1)(\alpha+2) \cdots (\alpha+n)}.$$

Proof. From the Weierstrass product formula,

$$\begin{aligned}\frac{1}{\Gamma(\alpha)} &= \alpha e^{\gamma\alpha} \prod_{k=1}^{\infty} \left(1 + \frac{\alpha}{k}\right) e^{-\alpha/k} \\ &= \alpha e^{\gamma\alpha} \prod_{k=1}^n \left(1 + \frac{\alpha}{k}\right) e^{-\alpha/k} \prod_{k=n+1}^{\infty} \left(1 + \frac{\alpha}{k}\right) e^{-\alpha/k}.\end{aligned}$$

Since

$$\prod_{k=1}^n \left(1 + \frac{\alpha}{k}\right) = \frac{(\alpha+1) \cdots (\alpha+n)}{n!},$$

we rewrite

$$\begin{aligned}\frac{n!}{(\alpha+1) \cdots (\alpha+n)} &= \prod_{k=1}^n \left(1 + \frac{\alpha}{k}\right)^{-1} \\ &= \exp\left(-\sum_{k=1}^n \log\left(1 + \frac{\alpha}{k}\right)\right).\end{aligned}$$

Using the expansion

$$\log\left(1 + \frac{\alpha}{k}\right) = \frac{\alpha}{k} + O\left(\frac{1}{k^2}\right),$$

we obtain

$$\begin{aligned}\sum_{k=1}^n \log\left(1 + \frac{\alpha}{k}\right) &= \alpha \sum_{k=1}^n \frac{1}{k} + O(1) \\ &= \alpha(\log n + \gamma) + O(1).\end{aligned}$$

This gives

$$\frac{n!}{(\alpha+1) \cdots (\alpha+n)} = n^{-\alpha} e^{-\gamma\alpha} (1 + o(1)),$$

and therefore

$$\frac{n! n^{\alpha}}{(\alpha+1) \cdots (\alpha+n)} \longrightarrow \Gamma(\alpha).$$

■

2 The Beta Function and Its Relation to the Gamma Function

The Beta function is closely connected to the Gamma function and plays an important role in integral calculus and fractional calculus.

Definition 1.11: [18]

. Let $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re}(\alpha) > 0$ and $\operatorname{Re}(\beta) > 0$. The Beta function is defined by

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt. \quad (1.8)$$

Theorem 1.6

Let $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$ and $\operatorname{Re}(\beta) > 0$. Then the Beta and Gamma functions satisfy

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

Proof. We start from the product of two Gamma functions:

$$\begin{aligned} \Gamma(\alpha)\Gamma(\beta) &= \left(\int_0^{+\infty} t^{\alpha-1} e^{-t} dt \right) \left(\int_0^{+\infty} s^{\beta-1} e^{-s} ds \right) \\ &= \iint_{(0,+\infty)^2} t^{\alpha-1} s^{\beta-1} e^{-(t+s)} dt ds. \end{aligned}$$

We apply a change of variables.

$$u = t + s, \quad v = \frac{t}{t + s},$$

which gives $t = uv$ and $s = u(1 - v)$. The Jacobian determinant of this change of variables is

$$\left| \frac{\partial(t, s)}{\partial(u, v)} \right| = u,$$

and the domain becomes $(0, +\infty) \times (0, 1)$.

Hence,

$$\begin{aligned} \Gamma(\alpha)\Gamma(\beta) &= \int_0^{+\infty} \int_0^1 (uv)^{\alpha-1} (u(1-v))^{\beta-1} e^{-u} u dv du \\ &= \int_0^{+\infty} u^{\alpha+\beta-1} e^{-u} du \int_0^1 v^{\alpha-1} (1-v)^{\beta-1} dv \\ &= \Gamma(\alpha + \beta) B(\alpha, \beta). \end{aligned}$$

Therefore, we conclude that

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

■

2.1 Properties of the Beta Function

- **Symmetry property:** Let $\alpha, \beta > 0$.

$$B(\alpha, \beta) = B(\beta, \alpha).$$

For $\alpha, \beta > 0$, the Beta function is symmetric in its arguments.

$$\begin{aligned} B(\alpha, \beta) &= \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt \\ &= \int_0^1 (1-u)^{\alpha-1} u^{\beta-1} du \quad (u = 1-t) \\ &= \int_0^1 u^{\beta-1} (1-u)^{\alpha-1} du \\ &= B(\beta, \alpha). \end{aligned}$$

- **Functional identity:** Assume that $\alpha, \beta > 0$. Then the Beta function satisfies:

$$\alpha B(\alpha, \beta + 1) = \beta B(\alpha + 1, \beta).$$

We start from the definition of the Beta function:

$$\begin{aligned} \alpha B(\alpha, \beta + 1) &= \alpha \int_0^1 t^{\alpha-1} (1-t)^{\beta} dt \\ &= \int_0^1 (1-t)^{\beta} d(t^{\alpha}) \\ &= \left[(1-t)^{\beta} t^{\alpha} \right]_0^1 - \int_0^1 t^{\alpha} d((1-t)^{\beta}). \end{aligned}$$

Since

$$\left[(1-t)^{\beta} t^{\alpha} \right]_0^1 = 0, \quad d((1-t)^{\beta}) = -\beta(1-t)^{\beta-1} dt,$$

we obtain

$$\begin{aligned} \alpha B(\alpha, \beta + 1) &= \beta \int_0^1 t^{\alpha} (1-t)^{\beta-1} dt \\ &= \beta B(\alpha + 1, \beta). \end{aligned}$$

- **Recurrence relation:** For $\alpha, \beta > 0$,

$$B(\alpha + 1, \beta) = \frac{\alpha}{\alpha + \beta} B(\alpha, \beta).$$

This relation is obtained by integration by parts in the Beta function representation.

$$\begin{aligned}
 B(\alpha + 1, \beta) &= \int_0^1 t^\alpha (1-t)^{\beta-1} dt \\
 &= \left[-\frac{t^\alpha (1-t)^\beta}{\beta} \right]_0^1 + \frac{\alpha}{\beta} \int_0^1 t^{\alpha-1} (1-t)^\beta dt \\
 &= \frac{\alpha}{\beta} \int_0^1 t^{\alpha-1} (1-t)^\beta dt \\
 &= \frac{\alpha}{\beta} B(\alpha, \beta + 1) \\
 &= \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha + \beta} B(\alpha, \beta) \\
 &= \frac{\alpha}{\alpha + \beta} B(\alpha, \beta).
 \end{aligned}$$

• **Special case:**

$$B(1, \beta) = \frac{1}{\beta}.$$

From the definition,

$$\begin{aligned}
 B(1, \beta) &= \int_0^1 (1-t)^{\beta-1} dt \\
 &= \int_0^1 u^{\beta-1} du \quad (u = 1-t) \\
 &= \frac{1}{\beta}.
 \end{aligned}$$

• **Derivative with respect to α :** Let $\alpha, \beta > 0$. Then

$$\frac{\partial}{\partial \alpha} B(\alpha, \beta) = B(\alpha, \beta) \left(\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{\Gamma'(\alpha + \beta)}{\Gamma(\alpha + \beta)} \right).$$

Consider the expression of $B(\alpha, \beta)$ written in terms of Gamma functions. Differentiating with respect to α , we obtain:

$$\begin{aligned}
 \frac{\partial}{\partial \alpha} B(\alpha, \beta) &= \frac{\Gamma'(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} - \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma'(\alpha + \beta)}{\Gamma(\alpha + \beta)^2} \\
 &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \left(\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{\Gamma'(\alpha + \beta)}{\Gamma(\alpha + \beta)} \right).
 \end{aligned}$$

Recognizing that the factor $\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is precisely $B(\alpha, \beta)$, we conclude that

$$\frac{\partial}{\partial \alpha} B(\alpha, \beta) = B(\alpha, \beta) \left(\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{\Gamma'(\alpha + \beta)}{\Gamma(\alpha + \beta)} \right).$$

3 Basic Properties of the Gauss Hypergeometric Function

The Gauss hypergeometric function is one of the most important special functions in mathematical analysis and appears frequently in the study of differential equations and fractional calculus. It generalizes many classical functions through a single power series representation.

Definition 1.12: [19]

Let $p_1, p_2 \in \mathbb{C}$ and $p_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$. The Gauss hypergeometric function ${}_2F_1$ is defined for $|z| < 1$ by

$${}_2F_1(p_1, p_2; p_3; z) = \sum_{k=0}^{\infty} \frac{(p_1)_k (p_2)_k}{(p_3)_k} \frac{z^k}{k!},$$

where the Pochhammer symbol is defined by

$$(a)_0 = 1, \quad (a)_k = a(a+1) \cdots (a+k-1), \quad k \geq 1.$$

The series converges for $|z| < 1$, and in particular under the condition

$$\operatorname{Re}(p_3 - p_1 - p_2) > 0.$$

Moreover, if $0 < \operatorname{Re}(p_2) < \operatorname{Re}(p_3)$ and $|\arg(1-z)| < \pi$, then ${}_2F_1$ admits the Euler integral representation

$${}_2F_1(p_1, p_2; p_3; z) = \frac{\Gamma(p_3)}{\Gamma(p_2)\Gamma(p_3-p_2)} \int_0^1 x^{p_2-1} (1-x)^{p_3-p_2-1} (1-zx)^{-p_1} dx.$$

which guarantees absolute convergence at $z = 1$. In this case, the function admits the closed-form expression

$${}_2F_1(p_1, p_2; p_3; 1) = \frac{\Gamma(p_3) \Gamma(p_3 - p_1 - p_2)}{\Gamma(p_3 - p_1) \Gamma(p_3 - p_2)}.$$

Polynomial reduction. ([19],[34]) When one of the upper parameters, for instance p_1 , is a non-positive integer such that $p_1 = -m$ with $m \in \mathbb{N}$, the hypergeometric expansion

contains only finitely many nonzero terms. Consequently, the Gauss hypergeometric function becomes a polynomial of degree m given by

$${}_2F_1(-m, p_2; p_3; z) = \sum_{k=0}^m \frac{(-m)_k (p_2)_k}{(p_3)_k} \frac{z^k}{k!}.$$

Indeed, this follows from the fact that the Pochhammer symbol satisfies $(-m)_k = 0$ whenever $k > m$, and therefore all subsequent terms in the infinite series vanish.

Proposition 2 (Differentiation property) ([19],[34]) *The Gauss hypergeometric function satisfies the following differentiation rule:*

$$\frac{d}{dz} {}_2F_1(p_1, p_2; p_3; z) = \frac{p_1 p_2}{p_3} {}_2F_1(p_1 + 1, p_2 + 1; p_3 + 1; z).$$

Proof. Starting from the series representation,

$${}_2F_1(p_1, p_2; p_3; z) = \sum_{k=0}^{\infty} \frac{(p_1)_k (p_2)_k}{(p_3)_k} \frac{z^k}{k!},$$

we differentiate term-by-term:

$$\begin{aligned} \frac{d}{dz} {}_2F_1(p_1, p_2; p_3; z) &= \sum_{k=1}^{\infty} \frac{(p_1)_k (p_2)_k}{(p_3)_k} \frac{k z^{k-1}}{k!} \\ &= \sum_{k=1}^{\infty} \frac{(p_1)_k (p_2)_k}{(p_3)_k} \frac{z^{k-1}}{(k-1)!}. \end{aligned}$$

Setting $n = k - 1$ gives

$$\frac{d}{dz} {}_2F_1(p_1, p_2; p_3; z) = \sum_{n=0}^{\infty} \frac{(p_1)_{n+1} (p_2)_{n+1}}{(p_3)_{n+1}} \frac{z^n}{n!}.$$

Using $(p)_{n+1} = p(p+1)_n$, we obtain

$$\begin{aligned} \frac{d}{dz} {}_2F_1(p_1, p_2; p_3; z) &= \sum_{n=0}^{\infty} \frac{p_1 (p_1 + 1)_n p_2 (p_2 + 1)_n}{p_3 (p_3 + 1)_n} \frac{z^n}{n!} \\ &= \frac{p_1 p_2}{p_3} \sum_{n=0}^{\infty} \frac{(p_1 + 1)_n (p_2 + 1)_n}{(p_3 + 1)_n} \frac{z^n}{n!}. \end{aligned}$$

Recognizing the series again,

$$\frac{d}{dz} {}_2F_1(p_1, p_2; p_3; z) = \frac{p_1 p_2}{p_3} {}_2F_1(p_1 + 1, p_2 + 1; p_3 + 1; z).$$

■ **Generalization.** ([38]) A broader generalization of the Gauss hypergeometric function is given by the generalized hypergeometric function ${}_3F_2$, which is introduced as follows:

$${}_3F_2(p_1, p_2, p_3; p_4, p_5; z) = \sum_{k=0}^{\infty} \frac{(p_1)_k (p_2)_k (p_3)_k}{(p_4)_k (p_5)_k} \frac{z^k}{k!}, \quad (1.9)$$

where the parameters satisfy

$$z \in \mathbb{C}, \quad |z| \leq 1, \quad \operatorname{Re}(p_4 + p_5 - p_1 - p_2 - p_3) > 0.$$

This class of functions extends many analytical properties of ${}_2F_1$ and frequently appears in advanced studies of special functions and fractional differential equations.

IV Fractional Calculus Operators

This section is devoted to the introduction of the main operators of fractional calculus, in particular the Riemann–Liouville fractional integrals and derivatives as well as the Caputo fractional derivatives. We also present their basic properties, which play a fundamental role in the study of fractional differential equations and boundary value problems ([12],[13],[23],[25],[34]).

1 Fractional Operators of Riemann–Liouville Type

Definition 1.13

Let $\Omega = [a, b]$ with $-\infty < a < b < +\infty$, and let $u \in L^1(a, b)$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$. The left-sided and right-sided Riemann–Liouville fractional integrals of order α are defined by

$$\mathcal{I}_{a+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{u(s)}{(t-s)^{1-\alpha}} ds, \quad \mathcal{I}_{b-}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{u(s)}{(s-t)^{1-\alpha}} ds. \quad (1.10)$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Proposition 3 *Let $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$. The Riemann–Liouville fractional integrals satisfy the following composition property for $t \in [a, b]$:*

$$\mathcal{I}_{a+}^{\alpha} (\mathcal{I}_{a+}^{\beta} u)(t) = \mathcal{I}_{a+}^{\alpha+\beta} u(t), \quad \mathcal{I}_{b-}^{\alpha} (\mathcal{I}_{b-}^{\beta} u)(t) = \mathcal{I}_{b-}^{\alpha+\beta} u(t).$$

Proof. We prove the left-sided case; the right-sided case follows a similar argument

$$\begin{aligned}
\mathcal{I}_{a+}^{\alpha}(\mathcal{I}_{a+}^{\beta}u)(t) &= \int_a^t \frac{1}{\Gamma(\alpha)(t-s)^{1-\alpha}} \left(\int_a^s \frac{u(\tau)}{\Gamma(\beta)(s-\tau)^{1-\beta}} d\tau \right) ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t u(\tau) \left(\int_{\tau}^t \frac{ds}{(t-s)^{1-\alpha}(s-\tau)^{1-\beta}} \right) d\tau \quad (\text{Fubini's theorem}) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t u(\tau) \left(\int_0^1 \frac{(t-\tau) dx}{((t-\tau)(1-x))^{1-\alpha}((t-\tau)x)^{1-\beta}} \right) d\tau \\
&\quad (s = \tau + (t-\tau)x, ds = (t-\tau) dx) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t u(\tau)(t-\tau)^{\alpha+\beta-1} \left(\int_0^1 (1-x)^{\alpha-1} x^{\beta-1} dx \right) d\tau \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t u(\tau)(t-\tau)^{\alpha+\beta-1} B(\alpha, \beta) d\tau \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-\tau)^{\alpha+\beta-1} u(\tau) d\tau \\
&= \mathcal{I}_{a+}^{\alpha+\beta} u(t).
\end{aligned}$$

■

Definition 1.14

Let $\alpha \in \mathbb{C}$ be such that $\operatorname{Re}(\alpha) > 0$, $u \in C^n[a, b]$ and set $n = \lfloor \operatorname{Re}(\alpha) \rfloor + 1$. The left- and right-sided Riemann–Liouville fractional derivatives of order α are defined by:

$$(\mathcal{D}_{a+}^{\alpha}u)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{u(s)}{(t-s)^{\alpha-n+1}} ds, \quad (1.11)$$

$$(\mathcal{D}_{b-}^{\alpha}u)(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt} \right)^n \int_t^b \frac{u(s)}{(s-t)^{\alpha-n+1}} ds. \quad (1.12)$$

When $\alpha = n \in \mathbb{N}$, these operators reduce to the classical integer-order derivatives:

$$\mathcal{D}_{a+}^n u(t) = u^{(n)}(t), \quad \mathcal{D}_{b-}^n u(t) = (-1)^n u^{(n)}(t). \quad (1.13)$$

Proposition 4 Let $\operatorname{Re}(\alpha) \geq 0$ and $\operatorname{Re}(\beta) > 0$. Then

$$\begin{aligned}
\mathcal{D}_{a+}^{\alpha}(t-a)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \\
\mathcal{D}_{b-}^{\alpha}(b-t)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{\beta-\alpha-1}.
\end{aligned} \quad (1.14)$$

Proof. We prove the left-sided case. Let $n-1 < \alpha < n$ and $u(t) = (t-a)^{\beta-1}$. Then

$$\mathcal{D}_{a+}^{\alpha}u(t) = \frac{d^n}{dt^n} \mathcal{I}_{a+}^{n-\alpha}u(t).$$

By definition, we have

$$\begin{aligned}
\mathcal{I}_{a+}^{n-\alpha}(t-a)^{\beta-1} &= \int_a^t \frac{(s-a)^{\beta-1}}{\Gamma(n-\alpha)(t-s)^{\alpha-n+1}} ds \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{(s-a)^{\beta-1}}{(t-s)^{\alpha-n+1}} ds \\
&\quad (u = \frac{s-a}{t-a}) \\
&= \frac{1}{\Gamma(n-\alpha)} (t-a)^{\beta+n-\alpha-1} \int_0^1 (1-u)^{n-\alpha-1} u^{\beta-1} du \\
&= \frac{1}{\Gamma(n-\alpha)} (t-a)^{\beta+n-\alpha-1} B(n-\alpha, \beta) \\
&= \frac{1}{\Gamma(n-\alpha)} (t-a)^{\beta+n-\alpha-1} \frac{\Gamma(n-\alpha)\Gamma(\beta)}{\Gamma(\beta+n-\alpha)} \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta+n-\alpha)} (t-a)^{\beta+n-\alpha-1}.
\end{aligned}$$

Applying the n -th derivative, we get

$$\begin{aligned}
\mathcal{D}_{a+}^\alpha(t-a)^{\beta-1} &= \frac{d^n}{dt^n} \left[\frac{\Gamma(\beta)}{\Gamma(\beta+n-\alpha)} (t-a)^{\beta+n-\alpha-1} \right] \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta+n-\alpha)} \frac{d^n}{dt^n} (t-a)^{\beta+n-\alpha-1} \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta+n-\alpha)} \frac{\Gamma(\beta+n-\alpha)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1} \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}.
\end{aligned}$$

The right-sided case follows by the change of variable $t \mapsto b-t$. ■

Lemma 1.1: [3]

Let $\alpha > 0$ and let $u \in C^n(a, b)$, where $n = [\alpha] + 1$. Then

$$\mathcal{I}_{a+}^\alpha \mathcal{D}_{a+}^\alpha u(t) = u(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n}, \quad c_i \in \mathbb{R}. \quad (1.15)$$

Proof. Let $n = [\alpha] + 1$, so that $n-1 < \alpha < n$. Set

$$v = \mathcal{I}_{a+}^{n-\alpha} u.$$

Then we have

$$\mathcal{D}_{a+}^\alpha u(t) = \frac{d^n}{dt^n} v(t).$$

Hence,

$$\begin{aligned}
 \mathcal{I}_{a+}^{\alpha} \mathcal{D}_{a+}^{\alpha} u(t) &= \mathcal{I}_{a+}^{\alpha} \frac{d^n}{dt^n} v(t) \\
 &= \frac{d^n}{dt^n} \mathcal{I}_{a+}^{\alpha} v(t) + \sum_{k=0}^{n-1} \frac{(t-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} v^{(n-k-1)}(a) \\
 &= v(t) + \sum_{k=0}^{n-1} \frac{(t-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} (\mathcal{I}_{a+}^{n-\alpha} u)^{(n-k-1)}(a).
 \end{aligned}$$

Since $v = \mathcal{I}_{a+}^{n-\alpha} u$, we conclude that

$$\mathcal{I}_{a+}^{\alpha} \mathcal{D}_{a+}^{\alpha} u(t) = u(t) + \sum_{i=1}^n c_i (t-a)^{\alpha-i}, \quad c_i \in \mathbb{R}.$$

2 Fractional Operators of Caputo Type

Definition 1.15

Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, $u \in AC^n([a, b])$ and let $n = \lfloor \operatorname{Re}(\alpha) \rfloor + 1$. The Caputo left-sided and right-sided fractional derivatives of order α are defined on $[a, b]$ by

$$({}^C \mathcal{D}_{a+}^{\alpha} u)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \quad t \in (a, b], \quad (1.16)$$

$$({}^C \mathcal{D}_{b-}^{\alpha} u)(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b \frac{(-1)^n u^{(n)}(s)}{(s-t)^{\alpha-n+1}} ds, \quad t \in [a, b). \quad (1.17)$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Proposition 5 *Let $\operatorname{Re}(\alpha) > 0$ and $\operatorname{Re}(\beta) > 0$. Then the Caputo fractional derivatives of power functions are given by*

$$\begin{aligned}
 {}^C \mathcal{D}_{a+}^{\alpha} (t-a)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}, \\
 {}^C \mathcal{D}_{b-}^{\alpha} (b-t)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (b-t)^{\beta-\alpha-1}.
 \end{aligned}$$

Moreover, for $k = 0, 1, \dots, n-1$, one has

$${}^C \mathcal{D}_{a+}^{\alpha} (t-a)^k = {}^C \mathcal{D}_{b-}^{\alpha} (b-t)^k = 0,$$

and in particular,

$${}^C \mathcal{D}_{a+}^{\alpha} 1 = {}^C \mathcal{D}_{b-}^{\alpha} 1 = 0.$$

Proof. Let $u(t) = (t - a)^{\beta-1}$ with $\operatorname{Re}(\beta) > 0$ and $n - 1 < \alpha < n$. Then

$$\begin{aligned}
{}^C\mathcal{D}_{a+}^{\alpha} u(t) &= \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{u^{(n)}(s)}{(t - s)^{\alpha-n+1}} ds \\
&= \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{\Gamma(\beta)}{\Gamma(\beta - n)} (s - a)^{\beta-n-1} (t - s)^{n-\alpha-1} ds \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta - n)\Gamma(n - \alpha)} \int_a^t (s - a)^{\beta-n-1} (t - s)^{n-\alpha-1} ds \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta - n)\Gamma(n - \alpha)} \int_0^1 (t - a)^{\beta-n-1} \tau^{\beta-n-1} (t - a)^{n-\alpha-1} (1 - \tau)^{n-\alpha-1} (t - a) d\tau \\
&\quad (s = a + (t - a)\tau, \quad ds = (t - a) d\tau) \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta - n)\Gamma(n - \alpha)} (t - a)^{\beta-\alpha-1} \int_0^1 \tau^{\beta-n-1} (1 - \tau)^{n-\alpha-1} d\tau \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta - n)\Gamma(n - \alpha)} (t - a)^{\beta-\alpha-1} B(\beta - n, n - \alpha) \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta - n)\Gamma(n - \alpha)} (t - a)^{\beta-\alpha-1} \frac{\Gamma(\beta - n)\Gamma(n - \alpha)}{\Gamma(\beta - \alpha)} \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} (t - a)^{\beta-\alpha-1}.
\end{aligned}$$

For $u(t) = (t - a)^k$, $k < n$, we have $u^{(n)}(t) = 0$, hence

$${}^C\mathcal{D}_{a+}^{\alpha} (t - a)^k = 0.$$

The right-sided case follows by the change $t \mapsto b - t$. ■

Lemma 1.2: [19]

Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$. If $u \in C^n([a, b])$, then

$$(\mathcal{I}_{a+}^{\alpha} {}^C\mathcal{D}_{a+}^{\alpha} u)(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!} (t - a)^k. \quad (1.18)$$

and

$$(\mathcal{I}_{b-}^{\alpha} {}^C\mathcal{D}_{b-}^{\alpha} u)(t) = u(t) - \sum_{k=0}^{n-1} \frac{(-1)^k u^{(k)}(b)}{k!} (b - t)^k. \quad (1.19)$$

In particular, if $0 < \operatorname{Re}(\alpha) \leq 1$ and $u \in C([a, b])$, then

$$(\mathcal{I}_{a+}^{\alpha} {}^C\mathcal{D}_{a+}^{\alpha} u)(t) = u(t) - u(a). \quad (\mathcal{I}_{b-}^{\alpha} {}^C\mathcal{D}_{b-}^{\alpha} u)(t) = u(t) - u(b). \quad (1.20)$$

Proof. We prove the left-sided case.

$$\begin{aligned}
{}^C\mathcal{D}_{a+}^\alpha u(t) &= \mathcal{I}_{a+}^{n-\alpha} \mathcal{D}^n u(t), \\
\mathcal{I}_{a+}^\alpha {}^C\mathcal{D}_{a+}^\alpha u(t) &= \mathcal{I}_{a+}^\alpha \mathcal{I}_{a+}^{n-\alpha} \mathcal{D}^n u(t), \\
&= \mathcal{I}_{a+}^n \mathcal{D}^n u(t), \\
&= u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!} (t-a)^k.
\end{aligned}$$

For the right-sided operator,

$$\begin{aligned}
{}^C\mathcal{D}_{b-}^\alpha u(t) &= (-1)^n \mathcal{I}_{b-}^{n-\alpha} \mathcal{D}^n u(t), \\
\mathcal{I}_{b-}^\alpha {}^C\mathcal{D}_{b-}^\alpha u(t) &= (-1)^n \mathcal{I}_{b-}^\alpha \mathcal{I}_{b-}^{n-\alpha} \mathcal{D}^n u(t), \\
&= (-1)^n \mathcal{I}_{b-}^n \mathcal{D}^n u(t), \\
&= u(t) - \sum_{k=0}^{n-1} \frac{(-1)^k u^{(k)}(b)}{k!} (b-t)^k.
\end{aligned}$$

If $0 < \operatorname{Re}(\alpha) \leq 1$, then $n = 1$, hence

$$\mathcal{I}_{a+}^\alpha {}^C\mathcal{D}_{a+}^\alpha u(t) = u(t) - u(a), \quad \mathcal{I}_{b-}^\alpha {}^C\mathcal{D}_{b-}^\alpha u(t) = u(t) - u(b).$$

■

Caputo Fractional Boundary Value Problems

This chapter studies Caputo fractional boundary value problems of order $1 < \alpha \leq 2$, including nonlinear nonlocal problems and two-point boundary value problems with convection terms. Existence results are established using Green's functions and fixed point techniques.

I Caputo Fractional Problem with Nonlocal Conditions

We study a fractional differential equation of order α with nonlocal boundary conditions given by

$$\begin{cases} {}^C\mathcal{D}_{0+}^\alpha u(t) = f(t, u(t)), & t \in [0, T], \quad 1 < \alpha \leq 2, \\ u(0) = k(u), \quad u(T) = u_T, \end{cases} \quad (2.1)$$

where ${}^C\mathcal{D}_{0+}^\alpha$ is the Caputo fractional derivative of order α . The function $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, the operator $k : C([0, T], \mathbb{R}) \rightarrow \mathbb{R}$ is a continuous functional, and $u_T \in \mathbb{R}$ [21].

Lemma 2.1: [21]

Let $1 < \alpha \leq 2$, and suppose that $g : [0, T] \rightarrow \mathbb{R}$ is a continuous function. Then a function u is a solution of the fractional boundary value problem

$${}^C\mathcal{D}_{0+}^\alpha u(t) = g(t), \quad t \in [0, T],$$

with the boundary conditions

$$u(0) = k(u), \quad u(T) = u_T,$$

if and only if u admits the integral representation

$$\begin{aligned} u(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s) ds \\ & + \left(1 - \frac{t}{T}\right) k(u) + \frac{t}{T} u_T. \end{aligned} \quad (2.2)$$

Proof. Since $1 < \alpha \leq 2$, the solution of the Caputo fractional differential equation

$${}^C\mathcal{D}_{0+}^\alpha u(t) = g(t)$$

can be written in the form

$$u(t) = \mathcal{I}_{0+}^\alpha g(t) + c_0 + c_1 t, \quad (2.3)$$

where c_0 and c_1 are real constants. Applying the condition $u(0) = k(u)$, we have

$$u(0) = k(u) \Rightarrow c_0 = k(u),$$

since $\mathcal{I}_{0+}^\alpha g(0) = 0$. Next, using the condition $u(T) = u_T$, we obtain

$$u(T) = u_T \Rightarrow \mathcal{I}_{0+}^\alpha g(T) + k(u) + c_1 T = u_T,$$

which leads to

$$c_1 = \frac{u_T - k(u) - \mathcal{I}_{0+}^\alpha g(T)}{T}.$$

Replacing c_0 and c_1 in (2.3), it follows that

$$\begin{aligned} u(t) &= \mathcal{I}_{0+}^\alpha g(t) + k(u) + \frac{t}{T} (u_T - k(u) - \mathcal{I}_{0+}^\alpha g(T)) \\ &= \mathcal{I}_{0+}^\alpha g(t) - \frac{t}{T} \mathcal{I}_{0+}^\alpha g(T) + \left(1 - \frac{t}{T}\right) k(u) + \frac{t}{T} u_T. \end{aligned}$$

Finally, by expressing the fractional integral explicitly, we get

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds \\ &\quad - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s) ds \\ &\quad + \left(1 - \frac{t}{T}\right) k(u) + \frac{t}{T} u_T. \end{aligned}$$

■

1

Existence and Uniqueness of Solutions

In this section, we derive sufficient conditions under which problem (2.1) admits a unique solution. The proof is based on the application of the Banach fixed point theorem.

Theorem 2.1: [21]

Let us assume the following conditions:

(H1) The function f satisfies a Lipschitz condition with respect to its second argument; that is, there exists a constant $L_f > 0$ such that

$$|f(t, v) - f(t, w)| \leq L_f |v - w|, \quad \forall t \in [0, T], \quad v, w \in \mathbb{R}.$$

(H2) The functional k is Lipschitz continuous on $C([0, T], \mathbb{R})$, i.e., there exists a constant $L_k > 0$ such that

$$|k(v) - k(w)| \leq L_k \|v - w\|, \quad \forall v, w \in C([0, T], \mathbb{R}).$$

If

$$\frac{2L_f T^\alpha}{\Gamma(\alpha + 1)} + L_k < 1,$$

then problem 2.1 admits a unique solution on $[0, T]$.

Proof. We reformulate the boundary value problem as an equivalent fixed point problem. We define the operator $A : C([0, T], \mathbb{R}) \rightarrow C([0, T], \mathbb{R})$ by

$$\begin{aligned} A(u)(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds \\ & - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, u(s)) ds \\ & - \left(\frac{t}{T} - 1 \right) k(u) + \frac{t}{T} u_T. \end{aligned}$$

It is known that fixed points of A correspond exactly to solutions of problem 2.1. We now prove that A is a contraction mapping.

Let $v, w \in C([0, T], \mathbb{R})$ and $t \in [0, T]$. Then

$$\begin{aligned}
 |A(v)(t) - A(w)(t)| &\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t-s)^{\alpha-1} (f(s, v(s)) - f(s, w(s))) ds \right| \\
 &\quad + \frac{t}{T\Gamma(\alpha)} \left| \int_0^T (T-s)^{\alpha-1} (f(s, v(s)) - f(s, w(s))) ds \right| \\
 &\quad + \left| \frac{t}{T} - 1 \right| |k(v) - k(w)| \\
 &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, v(s)) - f(s, w(s))| ds \\
 &\quad + \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s, v(s)) - f(s, w(s))| ds \\
 &\quad + \left| \frac{t}{T} - 1 \right| |k(v) - k(w)|.
 \end{aligned}$$

Using the Lipschitz condition of f , we get

$$|f(s, v(s)) - f(s, w(s))| \leq L_f |v(s) - w(s)| \leq L_f \|v - w\|.$$

Substituting into the previous inequality yields

$$\begin{aligned}
 |A(v)(t) - A(w)(t)| &\leq \frac{L_f}{\Gamma(\alpha)} \|v - w\| \int_0^t (t-s)^{\alpha-1} ds \\
 &\quad + \frac{tL_f}{T\Gamma(\alpha)} \|v - w\| \int_0^T (T-s)^{\alpha-1} ds + L_k \|v - w\| \\
 &= \frac{L_f}{\Gamma(\alpha)} \|v - w\| \frac{t^\alpha}{\alpha} + \frac{tL_f}{T\Gamma(\alpha)} \|v - w\| \frac{T^\alpha}{\alpha} + L_k \|v - w\| \\
 &= \frac{L_f t^\alpha}{\Gamma(\alpha+1)} \|v - w\| + \frac{L_f t T^{\alpha-1}}{\Gamma(\alpha+1)} \|v - w\| + L_k \|v - w\| \\
 &\leq \frac{L_f T^\alpha}{\Gamma(\alpha+1)} \|v - w\| + \frac{L_f T^\alpha}{\Gamma(\alpha+1)} \|v - w\| + L_k \|v - w\| \\
 &= \left(\frac{2L_f T^\alpha}{\Gamma(\alpha+1)} + L_k \right) \|v - w\|.
 \end{aligned}$$

Taking the supremum over $t \in [0, T]$, we obtain

$$\|A(v) - A(w)\| \leq \left(\frac{2L_f T^\alpha}{\Gamma(\alpha+1)} + L_k \right) \|v - w\|.$$

Therefore, A is a contraction mapping provided that

$$\frac{2L_f T^\alpha}{\Gamma(\alpha+1)} + L_k < 1.$$

Since the space $C([0, T], \mathbb{R})$, endowed with the supremum norm, is complete, the Banach contraction principle ensures the existence of a unique fixed point for the operator A . Consequently, problem (2.1) admits a unique solution. ■

2 Example

We study the following fractional boundary value problem.

$$\begin{cases} {}^c\mathcal{D}_{0+}^\alpha u(t) = \frac{1}{5}u(t) + e^{-t}, & t \in [0, 1], \\ u(0) = \frac{1}{4}u, \\ u(1) = 0, \end{cases} \quad (2.4)$$

where $1 < \alpha \leq 2$. We define

$$f(t, u) = \frac{1}{5}u + e^{-t}, \quad k(u) = \frac{1}{4}u.$$

We verify the hypotheses (H1) and (H2). Indeed, the function f is Lipschitz continuous with respect to the second variable since

$$|f(t, v) - f(t, w)| = \frac{1}{5}|v - w|,$$

which implies that $L_f = \frac{1}{5}$. Moreover, the functional k satisfies

$$|k(v) - k(w)| = \frac{1}{4}|v - w| \leq \frac{1}{4}\|v - w\|,$$

so that $L_k = \frac{1}{4}$. Using the uniqueness condition

$$\frac{2L_f T^\alpha}{\Gamma(\alpha + 1)} + L_k < 1,$$

and taking $T = 1$, we obtain

$$\frac{2 \cdot \frac{1}{5}}{\Gamma(\alpha + 1)} + \frac{1}{4} < 1.$$

Since $\Gamma(\alpha + 1) \geq \Gamma(2) = 1$ for $1 < \alpha \leq 2$, it follows that

$$\frac{2}{5} + \frac{1}{4} = 0.65 < 1.$$

Hence, all the conditions of the theorem are satisfied, and therefore problem (2.4) admits a unique solution on $[0, 1]$.

II Caputo Fractional Boundary Value Problem with Convection Term

We investigate the following two-point boundary value problem involving a Caputo fractional derivative and a constant convection term:

$$\begin{cases} -(^C\mathcal{D}_{0+}^\alpha u)(t) + \lambda u'(t) = f(t, u(t)), & t \in (a, b), & 1 < \alpha < 2, \\ u(a) - \beta_0 u'(a) = \gamma_0, \\ u(b) + \beta_1 u'(b) = \gamma_1. \end{cases} \quad (2.5)$$

Here, $^C\mathcal{D}_{0+}^\alpha$ is the Caputo fractional derivative of order α . The parameters λ , β_0 , β_1 , γ_0 , and γ_1 are real constants, and the function $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be continuous [36].

Definition 2.1: ([16],[17],[34])[Mittag-Leffler Function]

The Mittag-Leffler function is defined by

$$E_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0. \quad (2.6)$$

The two-parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0. \quad (2.7)$$

For convenience, we define

$$\mathcal{O}_\beta(x) = x^{\beta-1} E_{\alpha-1,\beta}(\lambda x^{\alpha-1}), \quad (2.8)$$

where $\lambda \in \mathbb{R}$ and $\alpha \in (1, 2)$. The following properties of $\mathcal{O}^\beta$ hold:

(P1) For every $\beta \geq 0$ and $x \geq 0$,

$$\frac{d}{dx} \mathcal{O}_{\beta+1}(x) = \mathcal{O}_\beta(x).$$

(P2) At the origin, the function satisfies

$$\mathcal{O}_1(0) = 1, \quad \mathcal{O}_\beta(0) = 0 \quad \text{for } \beta > 1.$$

(P3) The function $\mathcal{O}_1$ is positive on $(0, \infty)$, whereas $\mathcal{O}_2$ is monotone increasing on $[0, \infty)$.

(P4) For $x > 0$, one has

$$\mathcal{O}_{\alpha-1}(x) > 0,$$

and the function $\mathcal{O}_\alpha$ is increasing on $(0, \infty)$.

Lemma 2.2: [36]

Let $\lambda \in \mathbb{R}$ and let $\alpha > 0$. The following assertions are valid:

(1) For every function $r \in C([a, b], \mathbb{R})$, the series

$$\sum_{k=0}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} r(t)$$

is well defined and converges. Moreover, its limit can be expressed as

$$\sum_{k=0}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} r(t) = r(t) + \lambda \int_a^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) r(s) ds.$$

(2) The linear operator $\mathcal{I} - \lambda \mathcal{I}_{a+}^\alpha : C([a, b]) \longrightarrow C([a, b])$ admits an inverse. In addition, for every $r \in C([a, b])$, the inverse operator is represented by

$$(\mathcal{I} - \lambda \mathcal{I}_{a+}^\alpha)^{-1} r(t) = \sum_{k=0}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} r(t).$$

Proof. (1) By employing the fundamental properties of the Mittag-Leffler function and those of the two-parameter Mittag-Leffler function, we derive the following result.

$$\begin{aligned} \lambda \int_a^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) r(s) ds &= \lambda \int_a^t (t-s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\lambda^k (t-s)^{k\alpha}}{\Gamma(k\alpha + \alpha)} r(s) ds \\ &= \lambda \sum_{k=0}^{\infty} \int_a^t (t-s)^{\alpha-1} \frac{\lambda^k (t-s)^{k\alpha}}{\Gamma(k\alpha + \alpha)} r(s) ds \\ &= \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{\Gamma(k\alpha + \alpha)} \int_a^t (t-s)^{k\alpha + \alpha - 1} r(s) ds \\ &= \sum_{k=0}^{\infty} \lambda^{k+1} \mathcal{I}_{a+}^{k\alpha + \alpha} r(t) \\ &= \sum_{k=1}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} r(t). \end{aligned}$$

Hence, we obtain

$$\begin{aligned}
 r(t) + \sum_{k=1}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} r(t) &= \sum_{k=0}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} r(t). \\
 (\mathcal{I} - \lambda \mathcal{I}_{a+}^{\alpha}) \left(\sum_{k=0}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} r(t) \right) &= \mathcal{I} \left(\sum_{k=0}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} r(t) \right) - \lambda \mathcal{I}_{a+}^{\alpha} \left(\sum_{k=0}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} r(t) \right) \\
 &= \sum_{k=0}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} r(t) - \sum_{k=0}^{\infty} \lambda^{k+1} \mathcal{I}_{a+}^{\alpha} \mathcal{I}_{a+}^{k\alpha} r(t) \\
 &= \sum_{k=0}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} r(t) - \sum_{k=0}^{\infty} \lambda^{k+1} \mathcal{I}_{a+}^{(k+1)\alpha} r(t) \\
 &= \sum_{k=0}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} r(t) - \sum_{k=1}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} r(t) \\
 &= r(t).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \left(\sum_{k=0}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} \right) (\mathcal{I} - \lambda \mathcal{I}_{a+}^{\alpha}) r(t) &= \sum_{k=0}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} r(t) - \sum_{k=0}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} \lambda \mathcal{I}_{a+}^{\alpha} r(t) \\
 &= \sum_{k=0}^{\infty} \lambda^k \mathcal{I}_{a+}^{k\alpha} r(t) - \sum_{k=0}^{\infty} \lambda^{k+1} \mathcal{I}_{a+}^{(k+1)\alpha} r(t) \\
 &= r(t).
 \end{aligned}$$

Lemma 2.3: [36]

Suppose that $\beta_0, \beta_1 \geq 0$. Then the boundary value problem

$$-(^C\mathcal{D}_{0+}^{\alpha} u)(t) + \lambda u'(t) = h(t), \quad t \in (a, b), \quad 1 < \alpha < 2, \quad (2.9)$$

with the boundary conditions

$$u(a) - \beta_0 u'(a) = \gamma_0, \quad u(b) + \beta_1 u'(b) = \gamma_1,$$

is equivalent to the integral equation

$$u(t) = \int_a^b G(t, s) h(s) ds + \gamma_1 \sigma(t) + \gamma_0 (1 - \sigma(t)), \quad t \in [a, b],$$

where the auxiliary function σ is defined by

$$\sigma(t) = \frac{\beta_0 + \mathcal{O}_2(t-a)}{\beta_0 + \beta_1 \mathcal{O}_1(b-a) + \mathcal{O}_2(b-a)},$$

and the Green function $G(t, s)$ is given by

$$G(t, s) = \begin{cases} \sigma(t) \left[\mathcal{O}_\alpha(b-s) + \beta_1 \mathcal{O}_{\alpha-1}(b-s) \right] - \mathcal{O}_\alpha(t-s), & a \leq s \leq t \leq b, \\ \sigma(t) \left[\mathcal{O}_\alpha(b-s) + \beta_1 \mathcal{O}_{\alpha-1}(b-s) \right], & a \leq t \leq s \leq b. \end{cases}$$

With $\mathcal{O}_\alpha$ defined in (2.8).

Proof. Applying the fractional integral operator $\mathcal{I}_{a+}^\alpha$ to both sides of equation (2.9), we obtain

$$-\mathcal{I}_{a+}^\alpha ({}^C \mathcal{D}_{a+}^\alpha u)(t) + \lambda \mathcal{I}_{a+}^\alpha u'(t) = \mathcal{I}_{a+}^\alpha h(t).$$

Using the properties of fractional integrals and Lemma 1.1, we deduce

$$-u(t) + c_0 + c_1(t-a) - \frac{\lambda}{\Gamma(\alpha)} c_0(t-a)^{\alpha-1} + \lambda \mathcal{I}_{a+}^{\alpha-1} u(t) = \mathcal{I}_{a+}^\alpha h(t).$$

Setting $t = a$, we obtain $u(a) = c_0$. Hence,

$$(\mathcal{I} - \lambda \mathcal{I}_{a+}^{\alpha-1})u(t) = c_0 + c_1(t-a) - \frac{\lambda}{\Gamma(\alpha)} c_0(t-a)^{\alpha-1} - \mathcal{I}_{a+}^\alpha h(t).$$

By applying Lemma 2.2, we obtain

$$\begin{aligned} u(t) &= (\mathcal{I} - \lambda \mathcal{I}_{a+}^{\alpha-1})^{-1} \left[c_0 + c_1(t-a) - \frac{\lambda}{\Gamma(\alpha)} c_0(t-a)^{\alpha-1} - \mathcal{I}_{a+}^\alpha h(t) \right] \\ &= \sum_{k=0}^{\infty} \lambda^k \mathcal{I}_{a+}^{k(\alpha-1)} \left[c_0 + c_1(t-a) - \frac{\lambda}{\Gamma(\alpha)} c_0(t-a)^{\alpha-1} - \mathcal{I}_{a+}^\alpha h(t) \right]. \end{aligned}$$

Using the definition of the functions $\mathcal{O}_\beta$, we get

$$u(t) = c_0 \mathcal{O}_1(t-a) + c_1 \mathcal{O}_2(t-a) - \lambda c_0 \mathcal{O}_\alpha(t-a) - \int_a^t \mathcal{O}_\alpha(t-s) h(s) ds.$$

Differentiating, we obtain

$$u'(t) = c_1 \mathcal{O}_1(t-a) - \int_a^t \mathcal{O}_{\alpha-1}(t-s) h(s) ds.$$

Using the boundary conditions

$$u(a) - \beta_0 u'(a) = \gamma_0, \quad u(b) + \beta_1 u'(b) = \gamma_1,$$

we obtain the system

$$\begin{aligned} c_0 - \beta_0 c_1 &= \gamma_0, \\ c_0 + [\beta_1 \mathcal{O}_1(b-a) + \mathcal{O}_2(b-a)] c_1 &= \gamma_1 + \int_a^b [\mathcal{O}_\alpha(b-s) + \beta_1 \mathcal{O}_{\alpha-1}(b-s)] h(s) ds. \end{aligned}$$

Solving this system yields

$$\begin{aligned} c_0 &= \frac{\gamma_0 + \beta_0(\gamma_1 - \gamma_0) + \beta_0 \int_a^b (\mathcal{O}_\alpha(b-s) + \beta_1 \mathcal{O}_{\alpha-1}(b-s)) h(s) ds}{\beta_0 + \beta_1 \mathcal{O}_1(b-a) + \mathcal{O}_2(b-a)}, \\ c_1 &= \frac{\gamma_1 - \gamma_0 + \int_a^b (\mathcal{O}_\alpha(b-s) + \beta_1 \mathcal{O}_{\alpha-1}(b-s)) h(s) ds}{\beta_0 + \beta_1 \mathcal{O}_1(b-a) + \mathcal{O}_2(b-a)}. \end{aligned}$$

Finally, substituting c_0 and c_1 into $u(t)$, we obtain

$$u(t) = \int_a^b G(t, s) h(s) ds + \gamma_1 \sigma(t) + \gamma_0 (1 - \sigma(t)),$$

where $G(t, s)$ and $\sigma(t)$ are defined accordingly. ■

1 Nonnegativity of the Green Function

To analyze the sign behavior of the Green function, we first recall some fundamental notions of log-concavity and log-convexity. These properties play an important role in the study of special functions. A function $J(x)$ is called log-concave if $\ln J(x)$ is concave, which is equivalent to

$$(\ln J(x))'' \leq 0.$$

In contrast, $J(x)$ is said to be log-convex if $\ln J(x)$ is convex, that is,

$$(\ln J(x))'' \geq 0.$$

Lemma 2.4: [37]

Let $\tau \in (0, 1]$. Then, for all $x > 0$, each of the functions $x^\tau E_{\tau, \tau+1}(x^\tau)$ and $E_{\tau, 1}(x^\tau)$ is log-concave.

Lemma 2.5: [37]

Let $\tau \in (0, 1]$. For every $x > 0$, the function $x^\tau E_{\tau, \tau+1}(-x^\tau)$ is log-concave, whereas $E_{\tau, 1}(-x^\tau)$ and $x^{\tau-1} E_{\tau, \tau}(-x^\tau)$ are log-convex functions.

Theorem 2.2: [36]

Let $t \in [a, b]$ be fixed. For $a \leq s \leq t$, consider the function

$$f_t(s) := \frac{\mathcal{O}_\alpha(t-s)}{\mathcal{O}_\alpha(b-s) + \beta_1 \mathcal{O}_{\alpha-1}(b-s)}.$$

Then, $f_t(s)$ is monotone decreasing with respect to s on the interval $[a, t]$.

Proof. (I) If $\lambda = 0$, then

$$f_t(s) = \frac{\mathcal{O}_\alpha(t-s)}{\mathcal{O}_\alpha(b-s) + \beta_1 \mathcal{O}_{\alpha-1}(b-s)}.$$

It is easy to verify that

$$f'_t(s) = f_t(s) \left[-\frac{\alpha-1}{t-s} + \frac{\alpha-2}{b-s} + \frac{1}{b-s} + \frac{(\alpha-1)\beta_1}{\mathcal{O}_\alpha(b-s) + \beta_1 \mathcal{O}_{\alpha-1}(b-s)} \right].$$

Since all terms are well-defined on $a \leq s < t \leq b$, we deduce that

$$f'_t(s) < 0,$$

hence $f_t(s)$ is decreasing in s .

(II) If $\lambda \neq 0$, then by Lemmas 2.4 and (2.5), the function $|\lambda| \mathcal{O}_\alpha(x)$ is log-concave for $x > 0$. Thus,

$$[\ln(\mathcal{O}_\alpha(x))]'' \leq 0, \quad x > 0,$$

which implies that

$$\frac{\mathcal{O}_{\alpha-1}(x)}{\mathcal{O}_\alpha(x)} \text{ is decreasing for } x > 0.$$

Hence, for $a \leq s < t \leq b$, we obtain

$$\frac{\mathcal{O}_{\alpha-1}(t-s)}{\mathcal{O}_\alpha(t-s)} \geq \frac{\mathcal{O}_{\alpha-1}(b-s)}{\mathcal{O}_\alpha(b-s)},$$

which is equivalent to

$$\mathcal{O}_\alpha(t-s)\mathcal{O}_{\alpha-1}(b-s) - \mathcal{O}_{\alpha-1}(t-s)\mathcal{O}_\alpha(b-s) \leq 0.$$

Moreover, since the ratio

$$\frac{\mathcal{O}_\alpha(b-s)}{\mathcal{O}_\alpha(t-s)}$$

is increasing in s , and using similar arguments as in Lemma 2.4

$$\mathcal{O}_\alpha(t-s)\mathcal{O}_{\alpha-2}(b-s) - \mathcal{O}_{\alpha-1}(t-s)\mathcal{O}_{\alpha-1}(b-s) \leq 0.$$

Combining (4.1) and (4.2), the numerator of $f'_t(s)$ satisfies

$$\mathcal{O}_\alpha(t-s) [\mathcal{O}_{\alpha-1}(b-s) + \beta_1 \mathcal{O}_{\alpha-2}(b-s)] - \mathcal{O}_{\alpha-1}(t-s) [\mathcal{O}_\alpha(b-s) + \beta_1 \mathcal{O}_{\alpha-1}(b-s)] \leq 0.$$

Therefore,

$$f'_t(s) \leq 0,$$

which shows that $f_t(s)$ is decreasing on $[a, t]$. ■

Lemma 2.6: [36]

For $a \leq t \leq b$, the function

$$g(t) := \frac{\beta_1 \mathcal{O}_1(b-a) + \mathcal{O}_2(b-a) - \mathcal{O}_2(t-a)}{\beta_1 \mathcal{O}_{\alpha-1}(b-a) + \mathcal{O}_\alpha(b-a) - \mathcal{O}_\alpha(t-a)}$$

is an increasing function of t .

Proof. (I) In the case $\lambda = 0$, and for $\beta \geq 0$, we have

$$\mathcal{O}_\beta(x) = \frac{1}{\Gamma(\beta)} x^{\beta-1}.$$

Consequently,

$$g(t) = \frac{\beta_1 + (b-a) - (t-a)}{\beta_1(b-a)^{\alpha-2} + (b-a)^{\alpha-1} - (t-a)^{\alpha-1}} \cdot (\alpha-1).$$

A direct computation shows that

$$g'(t) > 0,$$

which implies that $g(t)$ is increasing with respect to t .

(II) When $\lambda > 0$, Lemma 2.4 implies that the function

$$\mathcal{O}_1(x) = \lambda \mathcal{O}_\alpha(x) + 1$$

is log-concave for $x > 0$, from which it follows that

$$\left(\frac{\mathcal{O}_1(x)}{\mathcal{O}_{\alpha-1}(x)} \right)' \geq 0.$$

and we obtain

$$[\mathcal{O}_2(b-a) - \mathcal{O}_2(t-a)]\mathcal{O}_{\alpha-1}(t-a) - \mathcal{O}_1(t-a)[\mathcal{O}_\alpha(b-a) - \mathcal{O}_\alpha(t-a)] \geq 0.$$

Hence the numerator of $g'(t)$ is non-negative, and therefore

$$g'(t) \geq 0,$$

which shows that $g(t)$ is increasing on $[a, b]$. ■

Theorem 2.3: [36]

Assume $\beta_1 \geq 0$. Then the Green's function $G(t, s)$ is nonnegative if and only if

$$\beta_0 \geq -\mathcal{O}_2(b-a) + \frac{\mathcal{O}_1(b-a)\mathcal{O}_{\alpha-1}(b-a)}{\mathcal{O}_\alpha(b-a)}.$$

Proof. The Green's function $G(t, s)$ is nonnegative on $[a, b] \times [a, b]$ if and only if

$$\sigma(t)[\mathcal{O}_\alpha(b-s) + \beta_1\mathcal{O}_{\alpha-1}(b-s)] - \mathcal{O}_\alpha(t-s) \geq 0, \quad a \leq s \leq t \leq b.$$

Equivalently,

$$\beta_0 \geq -\beta_1\mathcal{O}_1(b-a) - \mathcal{O}_2(b-a) + \frac{\beta_1\mathcal{O}_1(b-a) + \mathcal{O}_2(b-a) - \mathcal{O}_2(t-a)}{1 - \frac{\mathcal{O}_\alpha(t-s)}{\mathcal{O}_\alpha(b-s) + \beta_1\mathcal{O}_{\alpha-1}(b-s)}}. \quad (2.10)$$

By Theorem 2.2 and Lemma 2.6, inequality (2.10) is equivalent to

$$\beta_0 \geq \max_{a \leq s \leq t \leq b} \left[-\beta_1\mathcal{O}_1(b-a) - \mathcal{O}_2(b-a) + \frac{\beta_1\mathcal{O}_1(b-a) + \mathcal{O}_2(b-a) - \mathcal{O}_2(t-a)}{1 - f_t(s)} \right].$$

Since the maximum is attained at $s = a$, we obtain

$$\beta_0 \geq \max_{a \leq t \leq b} \left[-\beta_1\mathcal{O}_1(b-a) - \mathcal{O}_2(b-a) + (\mathcal{O}_\alpha(b-a) + \beta_1\mathcal{O}_{\alpha-1}(b-a))g(t) \right].$$

The maximum is attained at $t = b$, hence

$$\beta_0 \geq -\beta_1\mathcal{O}_1(b-a) - \mathcal{O}_2(b-a) + (\mathcal{O}_\alpha(b-a) + \beta_1\mathcal{O}_{\alpha-1}(b-a))\frac{\mathcal{O}_1(b-a)}{\mathcal{O}_{\alpha-1}(b-a)}.$$

Finally, we obtain

$$\beta_0 \geq -\mathcal{O}_2(b-a) + \frac{\mathcal{O}_1(b-a) \mathcal{O}_{\alpha-1}(b-a)}{\mathcal{O}_\alpha(b-a)}.$$

■

2 Existence Results

We establish the existence of a solution to problem (2.4) via the Schauder fixed point theorem. The Green function representation derived in the previous section allows us to reformulate the problem as a fixed point equation in a suitable Banach space, to which classical fixed point theory can be applied.

Let $E = C^1([a, b], \mathbb{R})$ be the Banach space of all continuously differentiable real-valued functions defined on $[a, b]$, endowed with the norm

$$\|u\|_{C^1} = \|u\|_\infty + \|u'\|_\infty.$$

Assumptions. We impose the following conditions on f :

(H1) The function $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

(H2) There exist constants $a_0, b_0 \geq 0$ such that

$$|f(t, x)| \leq a_0 + b_0|x|, \quad \forall (t, x) \in [a, b] \times \mathbb{R}.$$

(H3) The function $G(\cdot, s) \in C^1([a, b])$ for all $s \in [a, b]$, and $\sigma \in C^1([a, b])$. Moreover, there exists a constant $R > 0$ satisfying

$$R > \frac{|\gamma_1|(\|\sigma\|_\infty + \|\sigma'\|_\infty) + |\gamma_0|(\|1 - \sigma\|_\infty + \|\sigma'\|_\infty)}{1 - b_0(b-a)(\|G\|_\infty + \|G_t\|_\infty)},$$

provided that

$$b_0(b-a)(\|G\|_\infty + \|G_t\|_\infty) < 1,$$

and such that

$$(a_0 + b_0 R)(b-a)(\|G\|_\infty + \|G_t\|_\infty) + |\gamma_1|(\|\sigma\|_\infty + \|\sigma'\|_\infty) + |\gamma_0|(\|1 - \sigma\|_\infty + \|\sigma'\|_\infty) \leq R,$$

where

$$\|G\|_\infty = \sup_{(t,s) \in [a,b]^2} |G(t, s)|, \quad \|G_t\|_\infty = \sup_{(t,s) \in [a,b]^2} \left| \frac{\partial G}{\partial t}(t, s) \right|.$$

Remark 2.1 The condition $b_0(b-a)(\|G\|_\infty + \|G_t\|_\infty) < 1$ ensures that the denomi-

nator in (H3) is strictly positive, so that a valid $R > 0$ always exists. Indeed, setting

$$A = (b-a)(\|G\|_\infty + \|G_t\|_\infty), \quad C_0 = |\gamma_1|(\|\sigma\|_\infty + \|\sigma'\|_\infty) + |\gamma_0|(\|1-\sigma\|_\infty + \|\sigma'\|_\infty),$$

any

$$R \geq \frac{a_0 A + C_0}{1 - b_0 A}$$

satisfies the inequality in (H3), and is strictly positive since $a_0, C_0 \geq 0$ and $C_0 > 0$

Theorem 2.4

Assume that $\beta_0, \beta_1 \geq 0$ and that hypotheses (H_1) – (H_3) are satisfied. Then the boundary value problem (2.4) has at least one solution.

Proof. Define the operator

$$\mathcal{T} : C^1([a, b], \mathbb{R}) \rightarrow C^1([a, b], \mathbb{R})$$

by

$$(\mathcal{T}u)(t) = \int_a^b G(t, s) f(s, u(s)) ds + \gamma_1 \sigma(t) + \gamma_0 (1 - \sigma(t)).$$

Since $G(\cdot, s) \in C^1([a, b])$ and $\sigma \in C^1([a, b])$, we have $\mathcal{T}u \in C^1([a, b], \mathbb{R})$ for every $u \in C^1([a, b], \mathbb{R})$, and

$$(\mathcal{T}u)'(t) = \int_a^b \frac{\partial G}{\partial t}(t, s) f(s, u(s)) ds + (\gamma_1 - \gamma_0) \sigma'(t).$$

We verify the assumptions of the Schauder fixed point theorem.

Step 1: $\mathcal{T}$ maps B_R into itself.

Let

$$B_R = \{u \in C^1([a, b], \mathbb{R}) : \|u\|_{C^1} \leq R\}.$$

Let $u \in B_R$. For every $t \in [a, b]$, we have

$$|(\mathcal{T}u)(t)| \leq \int_a^b |G(t, s)| |f(s, u(s))| ds + |\gamma_1| |\sigma(t)| + |\gamma_0| |1 - \sigma(t)|.$$

Using hypothesis (H_2) and the fact that $\|u\|_{C^1} \leq R$, we obtain

$$|f(s, u(s))| \leq a_0 + b_0 |u(s)| \leq a_0 + b_0 R.$$

Since $|G(t, s)| \leq \|G\|_\infty$ for all $(t, s) \in [a, b]^2$, we get

$$\int_a^b |G(t, s)| ds \leq (b-a) \|G\|_\infty.$$

Hence,

$$|(\mathcal{T}u)(t)| \leq (a_0 + b_0 R) (b - a) \|G\|_\infty + |\gamma_1| \|\sigma\|_\infty + |\gamma_0| \|1 - \sigma\|_\infty.$$

Taking the supremum over $t \in [a, b]$:

$$\|\mathcal{T}u\|_\infty \leq (a_0 + b_0 R) (b - a) \|G\|_\infty + |\gamma_1| \|\sigma\|_\infty + |\gamma_0| \|1 - \sigma\|_\infty.$$

Moreover, since $\left| \frac{\partial G}{\partial t}(t, s) \right| \leq \|G_t\|_\infty$,

$$\begin{aligned} |(\mathcal{T}u)'(t)| &\leq \int_a^b \left| \frac{\partial G}{\partial t}(t, s) \right| |f(s, u(s))| ds + (|\gamma_1| + |\gamma_0|) |\sigma'(t)| \\ &\leq (a_0 + b_0 R) (b - a) \|G_t\|_\infty + (|\gamma_1| + |\gamma_0|) \|\sigma'\|_\infty. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\mathcal{T}u\|_{C^1} &= \|\mathcal{T}u\|_\infty + \|(\mathcal{T}u)'\|_\infty \\ &\leq (a_0 + b_0 R) (b - a) (\|G\|_\infty + \|G_t\|_\infty) \\ &\quad + |\gamma_1| (\|\sigma\|_\infty + \|\sigma'\|_\infty) \\ &\quad + |\gamma_0| (\|1 - \sigma\|_\infty + \|\sigma'\|_\infty) \\ &\leq R, \end{aligned}$$

by hypothesis (H_3) . Hence $\mathcal{T}(B_R) \subseteq B_R$.

Step 2: $\mathcal{T}$ is continuous.

Let $(u_n) \subset C^1([a, b], \mathbb{R})$ such that

$$u_n \rightarrow u \quad \text{in } C^1([a, b], \mathbb{R}).$$

Then, in particular,

$$u_n \rightarrow u \quad \text{uniformly on } [a, b].$$

By (H_1) ,

$$f(s, u_n(s)) \rightarrow f(s, u(s)), \quad \forall s \in [a, b].$$

Since (u_n) is bounded in C^1 , there exists $M > 0$ such that

$$\|u_n\|_{C^1} \leq M, \quad \forall n \in \mathbb{N}.$$

Using (H_2) ,

$$|f(s, u_n(s))| \leq a_0 + b_0 M.$$

By the Dominated Convergence Theorem,

$$\|\mathcal{T}u_n - \mathcal{T}u\|_\infty \rightarrow 0.$$

Similarly,

$$\begin{aligned} & |(\mathcal{T}u_n)'(t) - (\mathcal{T}u)'(t)| \\ & \leq \int_a^b \left| \frac{\partial G}{\partial t}(t, s) \right| |f(s, u_n(s)) - f(s, u(s))| ds. \end{aligned}$$

Hence,

$$\|(\mathcal{T}u_n)' - (\mathcal{T}u)'\|_\infty \rightarrow 0.$$

Therefore,

$$\|\mathcal{T}u_n - \mathcal{T}u\|_{C^1} \rightarrow 0.$$

Thus, $\mathcal{T}$ is continuous.

Step 3: $\mathcal{T}(B_R)$ is relatively compact.

From Step 1, the set $\mathcal{T}(B_R)$ is uniformly bounded in $C^1([a, b], \mathbb{R})$. It remains to prove that both $\mathcal{T}(B_R)$ and $\{(\mathcal{T}u)' : u \in B_R\}$ are equicontinuous.

Let $u \in B_R$ and let $t_1, t_2 \in [a, b]$. Then

$$\begin{aligned} |(\mathcal{T}u)(t_1) - (\mathcal{T}u)(t_2)| & \leq (a_0 + b_0 R) \int_a^b |G(t_1, s) - G(t_2, s)| ds \\ & \quad + (|\gamma_1| + |\gamma_0|) |\sigma(t_1) - \sigma(t_2)|. \end{aligned}$$

Similarly,

$$\begin{aligned} & |(\mathcal{T}u)'(t_1) - (\mathcal{T}u)'(t_2)| \\ & \leq (a_0 + b_0 R) \int_a^b \left| \frac{\partial G}{\partial t}(t_1, s) - \frac{\partial G}{\partial t}(t_2, s) \right| ds \\ & \quad + (|\gamma_1| + |\gamma_0|) |\sigma'(t_1) - \sigma'(t_2)|. \end{aligned}$$

Since $G(\cdot, s)$, $\partial_t G(\cdot, s)$, σ , $\sigma' \in C([a, b])$ are continuous on the compact set $[a, b]$, they are uniformly continuous.

Hence $\mathcal{T}(B_R)$ and $\{(\mathcal{T}u)' : u \in B_R\}$ are equicontinuous. Therefore, by the Arzelà–Ascoli theorem, $\mathcal{T}(B_R)$ is relatively compact in $C^1([a, b], \mathbb{R})$.

Finally, B_R is a nonempty, closed, bounded, and convex subset of the Banach space $C^1([a, b], \mathbb{R})$, and the operator $\mathcal{T} : B_R \rightarrow B_R$ is continuous with relatively compact range.

Hence, by the Schauder fixed point theorem, $\mathcal{T}$ admits at least one fixed point. ■

Corollary 2.1 Assume that $\beta_0, \beta_1 \geq 0$ and that

$$\beta_0 \geq -\mathcal{O}_2(b-a) + \frac{\mathcal{O}_1(b-a)\mathcal{O}_{\alpha-1}(b-a)}{\mathcal{O}_\alpha(b-a)}.$$

If $f(t, x) \geq 0$ for all $(t, x) \in [a, b] \times \mathbb{R}$ and $\gamma_0, \gamma_1 \geq 0$, then $h(t) = f(t, u(t)) \geq 0$, and consequently the solution of problem (2.5) is nonnegative on $[a, b]$.

Proof. Under the assumption on β_0 , Theorem 2.3 guarantees that

$$G(t, s) \geq 0, \quad \forall (t, s) \in [a, b] \times [a, b].$$

By Lemma 2.3, the unique solution is represented by

$$u(t) = \int_a^b G(t, s)h(s) ds + \gamma_1\sigma(t) + \gamma_0(1 - \sigma(t)).$$

We estimate each term separately.

First term. Since

$$G(t, s) \geq 0 \quad \text{and} \quad h(s) \geq 0, \quad \forall s \in [a, b],$$

we obtain

$$\int_a^b G(t, s)h(s) ds \geq 0.$$

Second term. By properties (P2) and (P3) of $\mathcal{O}_\beta$, we have

$$\mathcal{O}_2(t-a) \geq 0, \quad \forall t \geq a.$$

Since $\beta_0 \geq 0$, the numerator of

$$\sigma(t) = \frac{\beta_0 + \mathcal{O}_2(t-a)}{\beta_0 + \beta_1\mathcal{O}_1(b-a) + \mathcal{O}_2(b-a)}$$

is nonnegative. Moreover,

$$\beta_0 + \beta_1\mathcal{O}_1(b-a) + \mathcal{O}_2(b-a) > 0.$$

Therefore,

$$\sigma(t) \geq 0, \quad \forall t \in [a, b].$$

Since $\gamma_1 \geq 0$, it follows that

$$\gamma_1\sigma(t) \geq 0.$$

Third term. We claim that

$$\sigma(t) \leq 1, \quad \forall t \in [a, b].$$

Indeed, since $\mathcal{O}_2$ is increasing on $[0, \infty)$ by property (P3), we have

$$\mathcal{O}_2(t - a) \leq \mathcal{O}_2(b - a).$$

Hence,

$$\begin{aligned} \sigma(t) &= \frac{\beta_0 + \mathcal{O}_2(t - a)}{\beta_0 + \beta_1 \mathcal{O}_1(b - a) + \mathcal{O}_2(b - a)} \\ &\leq \frac{\beta_0 + \mathcal{O}_2(b - a)}{\beta_0 + \beta_1 \mathcal{O}_1(b - a) + \mathcal{O}_2(b - a)} \leq 1. \end{aligned}$$

Therefore,

$$1 - \sigma(t) \geq 0.$$

Since $\gamma_0 \geq 0$, we conclude that

$$\gamma_0(1 - \sigma(t)) \geq 0.$$

Conclusion. Combining the above estimates, we obtain

$$u(t) \geq 0, \quad \forall t \in [a, b].$$

■

Riemann–Liouville Fractional Boundary Value Problems

This chapter investigates a nonlinear Riemann–Liouville fractional boundary value problem of order $2 < \alpha \leq 3$ with mixed boundary conditions. Existence and uniqueness results are established by transforming the problem into an equivalent integral equation via the associated Green function and analyzing the resulting operator. Sufficient conditions are derived using a two-metric framework and Green function estimates, ensuring the existence of a unique fixed point corresponding to the solution.

I Problem Formulation

We study the following fractional boundary value problem (FBVP):

$$\begin{cases} \mathcal{D}_{c+}^{\alpha} u(t) = -\Psi(t, u(t)), & c < t < d, \quad 2 < \alpha \leq 3, \\ u(c) = p u'(c), \\ u'(d) = \gamma, \end{cases} \quad (3.1)$$

where $\mathcal{D}_{c+}^{\alpha}$ represents the Riemann–Liouville fractional derivative of order α . The function Ψ is assumed to be continuous on $[c, d] \times \mathbb{R}$, i.e. $\Psi \in C([c, d] \times \mathbb{R})$, while $p \in \mathbb{R} \setminus \{0\}$ and $\gamma \in \mathbb{R}$ are given constants [33].

Theorem 3.1: [33]

Let $c < t < d$ and $\alpha > 1$. Then, the following identities hold:

$$\begin{aligned} M_{\alpha}(t) &= \int_c^t (t-c)^{\alpha-1} (t-s)^{\alpha-1} (d-s)^{\alpha-2} ds \\ &= \frac{(d-c)^{\alpha-2} (t-c)^{2\alpha-1}}{\alpha} {}_2F_1(2-\alpha, 1; \alpha+1; \chi(t)), \end{aligned} \quad (3.2)$$

where

$$\chi(t) = \frac{t-c}{d-c}.$$

Moreover,

$$\int_c^d M_\alpha(t) dt = \frac{(d-c)^{3\alpha-2}}{(1-\alpha)(2\alpha-1)} + \frac{(1-3\alpha)(d-c)^{3\alpha-2}}{2\alpha^2(1-\alpha)} {}_3F_2(1-\alpha, 1, 2\alpha; \alpha+1, 2\alpha+1; 1). \quad (3.3)$$

Proof. [38] We compute $M_\alpha(t)$ by successive substitutions in order to reduce the integral into hypergeometric form.

$$\begin{aligned} M_\alpha(t) &= \int_c^t (t-c)^{\alpha-1} (t-s)^{\alpha-1} (d-s)^{\alpha-2} ds \\ &= (t-c)^{\alpha-1} \int_0^1 (u(t-c))^{\alpha-1} (d-(t-u(t-c)))^{\alpha-2} (t-c) du \\ &\quad (u = \frac{t-s}{t-c}, s = t-u(t-c), ds = -(t-c) du) \\ &= (t-c)^{2\alpha-1} \int_0^1 u^{\alpha-1} (d-t+u(t-c))^{\alpha-2} du \\ &= (d-c)^{\alpha-2} (t-c)^{2\alpha-1} \int_0^1 u^{\alpha-1} (1-\chi(t)(1-u))^{\alpha-2} du, \end{aligned}$$

where $\chi(t) = \frac{t-c}{d-c}$. Expanding using the binomial-type series,

$$(1-\chi(t)(1-u))^{\alpha-2} = \sum_{k=0}^{\infty} \frac{(2-\alpha)_k}{k!} \chi(t)^k (1-u)^k,$$

and interchanging summation and integration, we obtain

$$\begin{aligned} M_\alpha(t) &= (d-c)^{\alpha-2} (t-c)^{2\alpha-1} \sum_{k=0}^{\infty} \frac{(2-\alpha)_k}{k!} \chi(t)^k \int_0^1 u^{\alpha-1} (1-u)^k du \\ &= (d-c)^{\alpha-2} (t-c)^{2\alpha-1} \sum_{k=0}^{\infty} \frac{(2-\alpha)_k}{k!} \chi(t)^k B(\alpha, k+1). \end{aligned}$$

Using $B(\alpha, k+1) = \frac{\Gamma(\alpha)\Gamma(k+1)}{\Gamma(\alpha+k+1)}$ and $\Gamma(k+1) = k!$, we get

$$M_\alpha(t) = (d-c)^{\alpha-2} (t-c)^{2\alpha-1} \sum_{k=0}^{\infty} \frac{(2-\alpha)_k}{(\alpha+1)_k} \chi(t)^k.$$

Thus,

$$M_\alpha(t) = \frac{(d-c)^{\alpha-2} (t-c)^{2\alpha-1}}{\alpha} {}_2F_1(2-\alpha, 1; \alpha+1; \chi(t)).$$

Now integrate over $[c, d]$ and set $x = \chi(t)$:

$$\int_c^d M_\alpha(t) dt = \frac{(d-c)^{3\alpha-2}}{\alpha} \int_0^1 x^{2\alpha-1} {}_2F_1(2-\alpha, 1; \alpha+1; x) dx.$$

Using the power series expansion of the hypergeometric function, we obtain

$$\begin{aligned} \int_c^d M_\alpha(t) dt &= \frac{(d-c)^{3\alpha-2}}{\alpha} \int_0^1 x^{2\alpha-1} \sum_{k=0}^{\infty} \frac{(2-\alpha)_k}{(\alpha+1)_k} \frac{x^k}{k!} dx \\ &= \frac{(d-c)^{3\alpha-2}}{\alpha} \sum_{k=0}^{\infty} \frac{(2-\alpha)_k}{(\alpha+1)_k} \int_0^1 x^{2\alpha+k-1} dx \\ &= \frac{(d-c)^{3\alpha-2}}{\alpha} \sum_{k=0}^{\infty} \frac{(2-\alpha)_k}{(\alpha+1)_k} \frac{1}{2\alpha+k}. \end{aligned}$$

We now rewrite the factor

$$\frac{1}{2\alpha+k} = \frac{1}{2\alpha} \cdot \frac{(2\alpha)_k}{(2\alpha+1)_k},$$

which gives

$$\int_c^d M_\alpha(t) dt = \frac{(d-c)^{3\alpha-2}}{2\alpha^2} \sum_{k=0}^{\infty} \frac{(2-\alpha)_k (1)_k (2\alpha)_k}{(\alpha+1)_k (2\alpha+1)_k k!}.$$

Finally, using standard hypergeometric transformations, this expression can be rewritten as

$$\int_c^d M_\alpha(t) dt = \frac{(d-c)^{3\alpha-2}}{(1-\alpha)(2\alpha-1)} + \frac{(1-3\alpha)(d-c)^{3\alpha-2}}{2\alpha^2(1-\alpha)} {}_3F_2(1-\alpha, 1, 2\alpha; \alpha+1, 2\alpha+1; 1).$$

■

Lemma 3.1: [1]

Let $2 < \alpha \leq 3$ and $z \in (0, 1)$. Define

$$H_\alpha(z) = {}_2F_1(2-\alpha, 1; \alpha+1; z). \quad (3.4)$$

Then, the function $H_\alpha(z)$ is strictly decreasing on $(0, 1)$ and satisfies

$$\frac{\alpha}{2(\alpha-1)} \leq H_\alpha(z) \leq 1. \quad (3.5)$$

Proof. The proof is divided into monotonicity and boundedness.

Monotonicity. Starting from the differentiation formula of the Gauss hypergeometric function, we have

$$\begin{aligned} H'_\alpha(z) &= \frac{d}{dz} {}_2F_1(2 - \alpha, 1; \alpha + 1; z) \\ &= \frac{2 - \alpha}{\alpha + 1} {}_2F_1(3 - \alpha, 2; \alpha + 2; z) \\ &= \frac{2 - \alpha}{\alpha + 1} \sum_{k=0}^{\infty} \frac{(3 - \alpha)_k (2)_k}{(\alpha + 2)_k} \frac{z^k}{k!}. \end{aligned}$$

Since for $2 < \alpha \leq 3$ and $z \in (0, 1)$ all terms in the series satisfy

$$\frac{(3 - \alpha)_k (2)_k}{(\alpha + 2)_k} \frac{z^k}{k!} > 0,$$

it follows that

$${}_2F_1(3 - \alpha, 2; \alpha + 2; z) > 0.$$

Therefore,

$$H'_\alpha(z) = \frac{2 - \alpha}{\alpha + 1} {}_2F_1(3 - \alpha, 2; \alpha + 2; z) < 0, \quad \forall z \in (0, 1),$$

which implies that $H_\alpha(z)$ is strictly decreasing on $(0, 1)$.

Bounds. Since H_α is continuous and strictly decreasing on $(0, 1)$, we have

$$H_\alpha(0) = {}_2F_1(2 - \alpha, 1; \alpha + 1; 0) = 1,$$

and

$$\begin{aligned} \lim_{z \rightarrow 1^-} H_\alpha(z) &= {}_2F_1(2 - \alpha, 1; \alpha + 1; 1) \\ &= \frac{\Gamma(\alpha + 1)\Gamma(2\alpha - 2)}{\Gamma(2\alpha - 1)\Gamma(\alpha)} \\ &= \frac{\alpha}{2(\alpha - 1)}. \end{aligned}$$

Hence,

$$\frac{\alpha}{2(\alpha - 1)} \leq H_\alpha(z) \leq 1, \quad \forall z \in (0, 1).$$

■

Remark 3.1 Let $z = \chi(t)$ as defined in Theorem 3.1. By the previous result, we have

$$H_\alpha(z) \geq \frac{\alpha}{2(\alpha-1)} > 0, \quad \text{for } 2 < \alpha \leq 3.$$

In particular,

$$H_\alpha(z) = {}_2F_1(2-\alpha, 1; \alpha+1; z) > 0, \quad \forall z \in (0, 1).$$

Moreover, since

$$\frac{(d-c)^{\alpha-2}(t-c)^{2\alpha-1}}{\alpha} \geq 0, \quad \forall t \in [c, d],$$

it follows that the integral $M_\alpha(t)$ defined in Theorem 3.1 satisfies

$$M_\alpha(t) \geq 0, \quad \forall t \in [c, d].$$

II

Integral Representation of the Solution

In this section, we establish the equivalent integral formulation of the fractional boundary value problem defined in (3.1). This representation will play a crucial role in the subsequent analysis.

Lemma 3.2

The solution of the fractional boundary value problem (3.1) is given by the following integral equation:

$$u(t) = \frac{\gamma(t-c)^{\alpha-1}}{(\alpha-1)(d-c)^{\alpha-2}} + \int_c^d \Omega(s, t) \Psi(s, u(s)) ds, \quad (3.6)$$

where $\Omega(s, t)$ is the associated Green function given by

$$\Omega(s, t) = \begin{cases} \frac{(t-c)^{\alpha-1}(d-s)^{\alpha-2}}{\Gamma(\alpha)(d-c)^{\alpha-2}} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{for } c \leq s \leq t \leq d, \\ \frac{(t-c)^{\alpha-1}(d-s)^{\alpha-2}}{\Gamma(\alpha)(d-c)^{\alpha-2}}, & \text{for } c \leq t \leq s \leq d. \end{cases} \quad (3.7)$$

Proof. Applying the fractional integral operator $\mathcal{I}_{c+}^\alpha$ to both sides of

$$\mathcal{D}_{c+}^\alpha u(t) = -\Psi(t, u(t)),$$

we obtain the general solution

$$u(t) = C_1(t-c)^{\alpha-1} + C_2(t-c)^{\alpha-2} + C_3(t-c)^{\alpha-3} - \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} \Psi(s, u(s)) ds.$$

Differentiating, we get

$$u'(t) = C_1(\alpha - 1)(t - c)^{\alpha-2} + C_2(\alpha - 2)(t - c)^{\alpha-3} + C_3(\alpha - 3)(t - c)^{\alpha-4} \\ - \frac{1}{\Gamma(\alpha - 1)} \int_c^t (t - s)^{\alpha-2} \Psi(s, u(s)) ds.$$

Using the boundary condition $u(c) = p u'(c)$, we note that the terms $(t - c)^{\alpha-3}$ and $(t - c)^{\alpha-4}$ become singular at $t = c$ unless their coefficients vanish. Hence,

$$C_2 = 0, \quad C_3 = 0.$$

Thus,

$$u(t) = C_1(t - c)^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_c^t (t - s)^{\alpha-1} \Psi(s, u(s)) ds,$$

and

$$u'(t) = C_1(\alpha - 1)(t - c)^{\alpha-2} - \frac{1}{\Gamma(\alpha - 1)} \int_c^t (t - s)^{\alpha-2} \Psi(s, u(s)) ds.$$

Using the condition $u'(d) = \gamma$, we obtain

$$\gamma = C_1(\alpha - 1)(d - c)^{\alpha-2} - \frac{1}{\Gamma(\alpha - 1)} \int_c^d (d - s)^{\alpha-2} \Psi(s, u(s)) ds.$$

Solving for C_1 , we get

$$C_1 = \frac{\gamma}{(\alpha - 1)(d - c)^{\alpha-2}} + \frac{1}{(\alpha - 1)(d - c)^{\alpha-2} \Gamma(\alpha - 1)} \int_c^d (d - s)^{\alpha-2} \Psi(s, u(s)) ds.$$

Substituting C_1 into $u(t)$ yields

$$u(t) = \frac{\gamma(t - c)^{\alpha-1}}{(\alpha - 1)(d - c)^{\alpha-2}} \\ + \frac{(t - c)^{\alpha-1}}{(\alpha - 1)(d - c)^{\alpha-2} \Gamma(\alpha - 1)} \int_c^d (d - s)^{\alpha-2} \Psi(s, u(s)) ds \\ - \frac{1}{\Gamma(\alpha)} \int_c^t (t - s)^{\alpha-1} \Psi(s, u(s)) ds \\ = \frac{\gamma(t - c)^{\alpha-1}}{(\alpha - 1)(d - c)^{\alpha-2}} \\ + \int_c^d \frac{(t - c)^{\alpha-1} (d - s)^{\alpha-2}}{(\alpha - 1)(d - c)^{\alpha-2} \Gamma(\alpha - 1)} \Psi(s, u(s)) ds \\ - \int_c^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \Psi(s, u(s)) ds.$$

$$\int_c^t (t - s)^{\alpha-1} \varphi(s, u(s)) ds = \int_c^d (t - s)^{\alpha-1} \mathbf{1}_{\{s \leq t\}} \Psi(s, u(s)) ds.$$

Finally, rewriting the integrals over the common interval $[c, d]$, we obtain

$$u(t) = \frac{\gamma(t-c)^{\alpha-1}}{(\alpha-1)(d-c)^{\alpha-2}} + \int_c^d \Omega(s, t) \Psi(s, u(s)) ds,$$

where $\Omega(s, t)$ is The Green function. ■

Lemma 3.3: [33]

For all $2 < \alpha \leq 3$, the Green function $\Omega(s, t)$ satisfies the following relations:

1.

$$\max_{t \in [c, d]} \left(\int_c^d |\Omega(s, t)|^2 ds \right) \leq \frac{4(\alpha-1)(d-c)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-3)(2\alpha-1)}, \quad (3.8)$$

2.

$$\int_c^d \left(\int_c^d |\Omega(s, t)|^2 ds \right) dt = \sqrt{\Upsilon(\alpha, c, d)}, \quad (3.9)$$

where

$$\begin{aligned} \sqrt{\Upsilon(\alpha, c, d)} &= \frac{(d-c)^{2\alpha}}{\Gamma^2(\alpha)(2\alpha-3)(2\alpha-1)} + \frac{(d-c)^{2\alpha}}{\Gamma^2(\alpha)(4\alpha^2-2\alpha)} \\ &\quad - \frac{2(d-c)^{2\alpha}}{\Gamma^2(\alpha)(1-\alpha)(2\alpha-1)} \\ &\quad - \frac{(1-3\alpha)(d-c)^{2\alpha}}{\Gamma^2(\alpha)\alpha^2(1-\alpha)} {}_3F_2(1-\alpha, 1, 2\alpha; \alpha+1, 2\alpha+1; 1). \end{aligned}$$

Proof. Let $\Pi(t)$ and C be defined by

$$\Pi(t) = \frac{(t-c)^{2\alpha-2}}{\Gamma^2(\alpha)(d-c)^{2\alpha-4}}, \quad C = \frac{2}{\Gamma^2(\alpha)(d-c)^{\alpha-2}}.$$

From the expression of the Green function $\Omega(s, t)$, we obtain for $c \leq s \leq t \leq d$

$$\begin{aligned} |\Omega(s, t)|^2 &= \frac{(t-c)^{2\alpha-2}(d-s)^{2\alpha-4}}{\Gamma^2(\alpha)(d-c)^{2\alpha-4}} - \frac{2(t-s)^{\alpha-1}(t-c)^{\alpha-1}(d-s)^{\alpha-2}}{\Gamma^2(\alpha)(d-c)^{\alpha-2}} \\ &\quad + \frac{(t-s)^{2\alpha-2}}{\Gamma^2(\alpha)} \\ &= \Pi(t)(d-s)^{2\alpha-4} - C(t-c)^{\alpha-1}(d-s)^{\alpha-2}(t-s)^{\alpha-1} + \frac{(t-s)^{2\alpha-2}}{\Gamma^2(\alpha)}. \end{aligned}$$

and for $c \leq t \leq s \leq d$ we have

$$\begin{aligned} |\Omega(s, t)|^2 &= \frac{(t-c)^{2\alpha-2}(d-s)^{2\alpha-4}}{\Gamma^2(\alpha)(d-c)^{2\alpha-4}} \\ &= \Pi(t)(d-s)^{2\alpha-4}. \end{aligned}$$

For all $t \in [c, d]$, we have

$$\begin{aligned}
\int_c^d |\Omega(s, t)|^2 ds &= \int_c^t |\Omega(s, t)|^2 ds + \int_t^d |\Omega(s, t)|^2 ds \\
&= \Pi(t) \int_c^t (d-s)^{2\alpha-4} ds - C \int_c^t (t-c)^{\alpha-1} (d-s)^{\alpha-2} (t-s)^{\alpha-1} ds \\
&\quad + \int_c^t \frac{(t-s)^{2\alpha-2}}{\Gamma^2(\alpha)} ds + \Pi(t) \int_t^d (d-s)^{2\alpha-4} ds \\
&= \Pi(t) \left[\int_c^t (d-s)^{2\alpha-4} ds + \int_t^d (d-s)^{2\alpha-4} ds \right] \\
&\quad + \int_c^t \frac{(t-s)^{2\alpha-2}}{\Gamma^2(\alpha)} ds - C \int_c^t (t-c)^{\alpha-1} (d-s)^{\alpha-2} (t-s)^{\alpha-1} ds \\
&= \Pi(t) \int_c^d (d-s)^{2\alpha-4} ds + \int_c^t \frac{(t-s)^{2\alpha-2}}{\Gamma^2(\alpha)} ds - CM_\alpha(t).
\end{aligned}$$

where $M_\alpha(t)$ denotes the integral given previously.

Since $CM_\alpha(t) \geq 0$, it follows that

$$\begin{aligned}
\int_c^d |\Omega(s, t)|^2 ds &\leq \Pi(t) \int_c^d (d-s)^{2\alpha-4} ds + \int_c^t \frac{(t-s)^{2\alpha-2}}{\Gamma^2(\alpha)} ds \\
&= \Pi(t) \left[\frac{(d-c)^{2\alpha-3}}{2\alpha-3} \right] + \frac{(t-c)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} \\
&\leq \Pi(d) \left[\frac{(d-c)^{2\alpha-3}}{2\alpha-3} \right] + \frac{(d-c)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} \\
&= \frac{(d-c)^{2\alpha-2}}{\Gamma^2(\alpha)(d-c)^{2\alpha-4}} \cdot \frac{(d-c)^{2\alpha-3}}{2\alpha-3} + \frac{(d-c)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} \\
&= \frac{(d-c)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-3)} + \frac{(d-c)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} \\
&= \frac{4(\alpha-1)(d-c)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-3)(2\alpha-1)}.
\end{aligned}$$

Hence,

$$\max_{t \in [c, d]} \left(\int_c^d |\Omega(s, t)|^2 ds \right) \leq \frac{4(\alpha-1)(d-c)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-3)(2\alpha-1)}.$$

Finally, integrating over $t \in [c, d]$, we obtain

$$\begin{aligned}
\int_c^d \left(\int_c^d |\Omega(s, t)|^2 ds \right) dt &= \int_c^d \left[\frac{(d-c)^{2\alpha-3}}{2\alpha-3} \Pi(t) + \frac{(t-c)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} \right] dt - C \int_c^d M_\alpha(t) dt \\
&= \sqrt{\Upsilon(\alpha, c, d)},
\end{aligned}$$

where $\sqrt{\Upsilon(\alpha, c, d)}$ is a positive constant depending on α , c , and d . ■

III A sufficient condition for existence and uniqueness of the solution

This section is devoted to establishing new sufficient conditions that guarantee the existence and uniqueness of solutions for the fractional boundary value problem considered in this study.

Theorem 3.2: [33]

Let $K > 0$ be a given constant, and assume that the function $\Psi(t, u)$ is continuous and satisfies a Lipschitz condition with respect to the second variable. More precisely, for all $(t, u), (t, v) \in [c, d] \times \mathbb{R}$, we have

$$|\Psi(t, u) - \Psi(t, v)| \leq K|u - v|. \quad (3.10)$$

Moreover, if the condition

$$K \sqrt{\Upsilon(\alpha, c, d)} < 1 \quad (3.11)$$

holds, where $\Upsilon(\alpha, c, d)$ is defined in Lemma 3.3, then the considered boundary value problem possesses a unique solution in the space $C([c, d])$.

Proof. In the context of this work, we take $E = C([c, d])$ be the space of continuous real-valued functions on $[c, d]$ endowed with the metrics

$$\rho(u, v) = \max_{t \in [c, d]} |u(t) - v(t)|,$$

and

$$\sigma(u, v) = \left(\int_c^d |u(t) - v(t)|^2 dt \right)^{1/2}.$$

It is well known that (E, ρ) is a complete metric space, while (E, σ) is not complete. Moreover, for all $u, v \in E$, we have

$$\sigma(u, v) \leq \sqrt{d - c} \rho(u, v).$$

Assume that the operator $T : E \rightarrow E$ satisfies:

$$\rho(Tu, Tv) \leq \delta \sigma(u, v), \quad \sigma(Tu, Tv) \leq \theta \sigma(u, v), \quad \theta \in (0, 1).$$

We study the fractional boundary value problem with its equivalent integral representation.

Using this formulation, we introduce the operator $T : C([c, d]) \rightarrow C([c, d])$ defined by

$$T[u](t) = \frac{\gamma(t-c)^{\alpha-1}}{(\alpha-1)(d-c)^{\alpha-2}} + \int_c^d \Omega(s, t) \Psi(s, u(s)) ds, \quad \forall t \in [c, d].$$

We aim to prove that T admits a unique fixed point in $C([c, d])$ by verifying the assumptions of Rus's fixed point theorem.

Verification of the Lipschitz-type estimate (Condition 1). Let ρ denote the L^∞ norm and σ the L^2 norm on $C([c, d])$, i.e.,

$$\rho(u, v) = \sup_{t \in [c, d]} |u(t) - v(t)|, \quad \sigma(u, v) = \left(\int_c^d |u(t) - v(t)|^2 dt \right)^{1/2}.$$

Let $u, v \in C([c, d])$. For any $t \in [c, d]$, we compute:

$$\begin{aligned} |T[u](t) - T[v](t)| &= \left| \int_c^d \Omega(s, t) (\Psi(s, u(s)) - \Psi(s, v(s))) ds \right| \\ &\leq \int_c^d |\Omega(s, t)| |\Psi(s, u(s)) - \Psi(s, v(s))| ds \\ &\leq K \int_c^d |\Omega(s, t)| |u(s) - v(s)| ds \\ &= K \int_c^d |\Omega(s, t)| |u(s) - v(s)| \cdot 1 ds \\ &\leq K \left(\int_c^d |\Omega(s, t)|^2 ds \right)^{\frac{1}{2}} \left(\int_c^d |u(s) - v(s)|^2 ds \right)^{\frac{1}{2}}. \end{aligned} \quad (3.12)$$

Taking the supremum over $t \in [c, d]$, ([11],[27]) we obtain

$$\rho(T[u], T[v]) \leq K \sup_{t \in [c, d]} \left(\int_c^d |\Omega(s, t)|^2 ds \right)^{1/2} \sigma(u, v).$$

Using Lemma 3.3, we estimate the kernel by

$$\sup_{t \in [c, d]} \int_c^d |\Omega(s, t)|^2 ds = \frac{4(\alpha-1)(d-c)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-3)(2\alpha-1)}.$$

Therefore,

$$\rho(T[u], T[v]) \leq \delta \sigma(u, v),$$

where

$$\delta = K \left(\frac{4(\alpha-1)(d-c)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-3)(2\alpha-1)} \right)^{1/2}.$$

Using the equivalence between ρ and σ norms on $C([c, d])$, we deduce

$$\rho(T[u], T[v]) \leq \delta \sqrt{d - c} \rho(u, v).$$

This shows that T is Lipschitz continuous.

Verification of the second condition. From the previous estimate, we obtain a constant δ such that

$$\rho(T[u], T[v]) \leq \delta \sigma(u, v), \quad \forall u, v \in C([c, d]).$$

Contractive condition in σ -metric. Starting from

$$|T[u](t) - T[v](t)| \leq K \left(\int_c^d |\Omega(s, t)|^2 ds \right)^{1/2} \sigma(u, v),$$

we obtain

$$|T[u](t) - T[v](t)|^2 \leq K^2 \left(\int_c^d |\Omega(s, t)|^2 ds \right) \sigma^2(u, v).$$

Integrating over $t \in [c, d]$,

$$\int_c^d |T[u](t) - T[v](t)|^2 dt \leq K^2 \int_c^d \int_c^d |\Omega(s, t)|^2 ds dt \sigma^2(u, v).$$

Hence,

$$\sigma(T[u], T[v]) \leq K \left(\int_c^d \int_c^d |\Omega(s, t)|^2 ds dt \right)^{1/2} \sigma(u, v).$$

Using Lemma 3.3

$$\Upsilon(\alpha, c, d) = \int_c^d \int_c^d |\Omega(s, t)|^2 ds dt,$$

we get

$$\sigma(T[u], T[v]) \leq K \sqrt{\Upsilon(\alpha, c, d)} \sigma(u, v) = \theta \sigma(u, v), \quad \theta = K \sqrt{\Upsilon(\alpha, c, d)}.$$

Since

$$K \sqrt{\Upsilon(\alpha, c, d)} < 1,$$

we conclude that $\theta < 1$, hence T is a contraction in σ . All the hypotheses of Rus's fixed point theorem are fulfilled. Hence, the operator T has a unique fixed point in

$C([c, d])$, which implies that the fractional boundary value problem under consideration possesses a unique solution. ■

IV The Special Case $\alpha = 3$

When $\alpha = 3$, the fractional boundary value problem (3.1) reduces to the classical third-order differential problem

$$\begin{cases} u'''(t) = -\Psi(t, u(t)), & c < t < d, \\ u(c) = u'(c) = 0, \\ u'(d) = \gamma, \end{cases} \quad (3.13)$$

where $\gamma \in \mathbb{R}$. This problem was previously studied by Laadjal [20], who established a uniqueness result under the condition

$$d - c < \sqrt[3]{\frac{12}{K}}. \quad (3.14)$$

In what follows, we derive explicit integral estimates for the Green function at $\alpha = 3$ and obtain an alternative sufficient condition for uniqueness that improves upon (3.14).

1 Integral Estimates for $\alpha = 3$

The following corollary is an immediate consequence of Lemma 3.3 applied to the case $\alpha = 3$.

Corollary 3.1 *For $\alpha = 3$, the Green function $\Omega(s, t)$ satisfies*

$$\max_{t \in [c, d]} \int_c^d |\Omega(s, t)|^2 ds \leq \frac{8(d - c)^5}{15}, \quad (3.15)$$

$$\Upsilon(3, c, d) = \int_c^d \int_c^d |\Omega(s, t)|^2 ds dt = \frac{(d - c)^6}{315}. \quad (3.16)$$

As a direct application of Theorem 3.2 and Corollary 2, we obtain the following uniqueness criterion.

Corollary 3.2 *Let $\Psi \in C([c, d] \times \mathbb{R})$ satisfy the Lipschitz condition*

$$\|\Psi(t, u) - \Psi(t, v)\| \leq K \|u - v\|, \quad \forall t \in [c, d], \quad u, v \in \mathbb{R}.$$

If

$$K \frac{(d-c)^3}{\sqrt{315}} < 1, \quad (3.17)$$

then problem (3.13) admits a unique solution in $C([c, d])$.

Proof. Setting $\alpha = 3$ in the contraction parameter defined in Theorem 3.2 and using (3.16), we obtain

$$\theta = K \sqrt{\Upsilon(3, c, d)} = K \frac{(d-c)^3}{\sqrt{315}}.$$

The condition (3.17) is therefore equivalent to $\theta < 1$, and the conclusion follows by Rus's fixed point theorem. ■

Remark 3.2 Since $\sqrt{315} \approx 17.748$, condition (3.17) can be equivalently expressed as

$$d - c < \sqrt[3]{\frac{\sqrt{315}}{K}}.$$



Application of the Results

To illustrate the applicability of the main theorem, we consider the fractional boundary value problem

$$\mathcal{D}_{c+}^{\alpha} u(t) = -5 \cos(u(t)) + t^3 - 1, \quad c < t < d, \quad 2 < \alpha \leq 3, \quad (3.18)$$

with the same boundary conditions as in (3.1). Setting $\Psi(t, u) = 5 \cos(u) - t^3 + 1$, we verify the assumptions of Theorem 3.2.

Verification of the Lipschitz condition. Since the terms $-t^3 + 1$ are independent of u , they do not affect the Lipschitz property. For any $u, v \in \mathbb{R}$ and $t \in [c, d]$, we have

$$|\Psi(t, u) - \Psi(t, v)| = 5 |\cos(u) - \cos(v)| \leq 5 |u - v|,$$

where we used the standard inequality $|\cos(u) - \cos(v)| \leq |u - v|$. Hence, Ψ satisfies a global Lipschitz condition with constant $K = 5$.

Existence and uniqueness. By Theorem 3.2, the integral operator associated with (3.18) is a contraction on $C([c, d])$ provided that

$$K \sqrt{\Upsilon(\alpha, c, d)} < 1 \quad \Longleftrightarrow \quad \sqrt{\Upsilon(\alpha, c, d)} < \frac{1}{5}.$$

Under this condition, all hypotheses of Rus's fixed point theorem are satisfied, and the problem (3.18) therefore admits a unique solution in $C([c, d])$.

1

Comparison with Laadjal's Condition

For the special case $\alpha = 3$ with $K = 5$, Laadjal's condition (3.14) gives

$$d - c < \sqrt[3]{\frac{12}{5}} \approx 1.34,$$

whereas Corollary 3 yields the less restrictive bound

$$d - c < \sqrt[3]{\frac{\sqrt{315}}{5}} \approx 1.42.$$

More generally, denoting by $D_L(\alpha)$ and $D(\alpha)$ the threshold constants appearing in Laadjal's condition and in the present condition, respectively, one has

$$D(\alpha) > D_L(\alpha), \quad \forall 2 < \alpha \leq 3,$$

which confirms that the criterion established in this work is applicable to a strictly wider class of problems. This improvement is a consequence of the L^2 -norm framework combined with the Green function estimates, in contrast to the L^∞ -based approach of Laadjal.

Conclusion

This thesis has investigated two main classes of nonlinear fractional boundary value problems involving Caputo and Riemann–Liouville fractional derivatives. Using Green’s function representations, each problem was reformulated as an equivalent integral equation, allowing the application of fixed point techniques. For the Caputo problems, existence and uniqueness results were established via the Banach contraction principle and the Schauder fixed point theorem. For the Riemann–Liouville problem, a new sufficient condition for the existence of a unique solution was derived by applying Rus’s fixed point theorem within a two-metric framework, and was shown to improve upon existing results in the literature.

Appendix: Main Results in Fixed Point Theory

This appendix collects some basic results in fixed point theory. We begin with the Banach fixed point theorem for contraction mappings, a fundamental tool in nonlinear analysis. This is followed by the Rus fixed point theorem, and we conclude with the Schauder fixed point theorem.

A.1 Banach Contraction Principle

This section focuses on the Banach contraction principle, which is a central result in fixed point theory. It is widely used to establish both existence and uniqueness of fixed points for contraction-type operators in complete metric spaces [6].

Definition 1.1

Let $T : X \rightarrow X$ be a mapping. A point $x \in X$ is called a **fixed point** of T if it satisfies

$$T(x) = x.$$

Definition 1.2

Let (X, d) be a metric space. The mapping $T : X \rightarrow X$ is said to be a **contraction** if there exists a constant $k \in (0, 1)$ such that

$$d(T(x), T(y)) \leq k d(x, y), \quad \forall x, y \in X.$$

Theorem 1.3: [Banach Fixed Point Theorem]

Let $(E, \|\cdot\|)$ be a Banach space and let $T : E \rightarrow E$ be a contraction mapping with constant $k \in (0, 1)$. Then T has a unique fixed point in E .

Proof. Let $T : E \rightarrow E$ be a contraction, so there exists a constant $k \in (0, 1)$ such that

$$\|T(x) - T(y)\| \leq k\|x - y\| \quad \forall x, y \in E.$$

Fix an arbitrary point $x_0 \in E$ and define the sequence (x_n) by

$$x_{n+1} = T(x_n), \quad n \geq 0.$$

We first show that (x_n) is a Cauchy sequence. Indeed, for $n \geq 1$,

$$\|x_{n+1} - x_n\| = \|T(x_n) - T(x_{n-1})\| \leq k\|x_n - x_{n-1}\|.$$

By induction, this implies

$$\|x_{n+1} - x_n\| \leq k^n \|x_1 - x_0\|.$$

Now, for $m > n$,

$$\begin{aligned} \|x_m - x_n\| &\leq \sum_{j=n}^{m-1} \|x_{j+1} - x_j\| \\ &\leq \sum_{j=n}^{m-1} k^j \|x_1 - x_0\| \\ &\leq \|x_1 - x_0\| \sum_{j=n}^{\infty} k^j. \end{aligned}$$

Since $k \in (0, 1)$, the geometric series converges, hence (x_n) is a Cauchy sequence. As E is complete, there exists $x \in E$ such that $x_n \rightarrow x$.

We now show that x is a fixed point of T . By continuity of T ,

$$T(x) = T\left(\lim_{n \rightarrow \infty} x_n\right) = \lim_{n \rightarrow \infty} T(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x.$$

Finally, for uniqueness, suppose $x, y \in E$ are fixed points. Then

$$\|x - y\| = \|T(x) - T(y)\| \leq k\|x - y\|.$$

Since $k < 1$, it follows that $\|x - y\| = 0$, hence $x = y$.

Therefore, T admits a unique fixed point in E . ■

A.2 Rus Fixed Point Theorem

The following theorem, establishes existence and uniqueness of fixed points via a two-metric framework. It is particularly well-suited to integral operators arising in boundary value problems.

Theorem 1.4: [Rus Fixed Point Theorem]

Let E be a nonempty set endowed with two metrics ρ and σ such that (E, ρ) is a complete metric space. Assume that:

1. The operator $T : E \rightarrow E$ is continuous with respect to the metric ρ ;
2. There exists a constant $\delta > 0$ such that

$$\rho(Tu, Tv) \leq \delta \sigma(u, v), \quad \forall u, v \in E;$$

3. There exists a constant $\theta \in (0, 1)$ such that

$$\sigma(Tu, Tv) \leq \theta \sigma(u, v), \quad \forall u, v \in E.$$

Then, the operator T admits a unique fixed point $w \in E$, that is,

$$T(w) = w.$$

Proof. Let $u_0 \in E$ be arbitrary and define the iterative sequence

$$u_{n+1} = Tu_n, \quad n \geq 0.$$

Using condition (3), we obtain

$$\sigma(u_{n+1}, u_n) = \sigma(Tu_n, Tu_{n-1}) \leq \theta \sigma(u_n, u_{n-1}),$$

and by induction,

$$\sigma(u_{n+1}, u_n) \leq \theta^n \sigma(u_1, u_0).$$

Let $m > n$. Using the triangle inequality and the above estimate, we get

$$\sigma(u_m, u_n) \leq \sum_{k=n}^{m-1} \sigma(u_{k+1}, u_k) \leq \sigma(u_1, u_0) \sum_{k=n}^{m-1} \theta^k.$$

Since $\theta \in (0, 1)$, the geometric series implies that (u_n) is a Cauchy sequence in (E, σ) . Moreover, using inequality (2),

$$\rho(u_{n+1}, u_n) = \rho(Tu_n, Tu_{n-1}) \leq \delta \sigma(u_n, u_{n-1}) \leq \delta \theta^{n-1} \sigma(u_1, u_0).$$

Hence (u_n) is also Cauchy in (E, ρ) . Since (E, ρ) is complete, there exists $w \in E$ such

that

$$u_n \rightarrow w \quad \text{in } \rho.$$

Now, using the continuity of T with respect to ρ , we obtain

$$Tu_n \rightarrow Tw \quad \text{in } \rho.$$

But $Tu_n = u_{n+1}$ and $u_{n+1} \rightarrow w$, hence

$$Tw = w.$$

Thus, w is a fixed point of T .

Uniqueness: Let $w_1, w_2 \in E$ be two fixed points. Then

$$\sigma(w_1, w_2) = \sigma(Tw_1, Tw_2) \leq \theta \sigma(w_1, w_2).$$

Since $\theta \in (0, 1)$, it follows that $\sigma(w_1, w_2) = 0$, hence $w_1 = w_2$. Therefore, the fixed point is unique. ■

A.3

Compactness and Fixed Point Results

In this subsection, we recall key results linking compactness and fixed point theory. We state the Schauder fixed point theorem and the Arzelà–Ascoli theorem, which will be essential in the analysis of existence results.

Theorem 1.5: [Schauder Fixed Point Theorem [9]]

Let E be a Banach space and let $K \subset E$ be a nonempty, closed, and convex subset. If $T : K \rightarrow K$ is a continuous mapping such that $T(K)$ is relatively compact in E , then T has at least one fixed point in K .

Proof. Since $T(K)$ is relatively compact in E , its closure $\overline{T(K)}$ is compact. Let $C = \overline{\text{conv}} T(K)$ denote the closed convex hull of $T(K)$. By Mazur's theorem, C is compact. Moreover, since K is closed and convex and $T(K) \subset K$, we have $C \subset K$. Thus T maps C into $T(K) \subset C$, and $T : C \rightarrow C$ is a continuous mapping on a nonempty compact convex subset of the Banach space E .

For each $n \geq 1$, let $\{y_1, \dots, y_{k_n}\} \subset C$ be a finite $\frac{1}{n}$ -net of C , and set

$$E_n = \text{span}\{y_1, \dots, y_{k_n}\}, \quad C_n = \overline{\text{conv}}\{y_1, \dots, y_{k_n}\} = C \cap E_n.$$

Each C_n is a nonempty compact convex subset of the finite-dimensional space E_n .

Define the Schauder projection $P_n : C \rightarrow C_n$ by

$$P_n(x) = \frac{\sum_{i=1}^{k_n} \mu_i(x) y_i}{\sum_{i=1}^{k_n} \mu_i(x)}, \quad \mu_i(x) = \max\left(0, \frac{1}{n} - \|x - y_i\|\right).$$

Then P_n is continuous, $P_n(C) \subset C_n$, and

$$\|P_n(x) - x\| < \frac{1}{n}, \quad \forall x \in C.$$

The composition $T_n = P_n \circ T : C_n \rightarrow C_n$ is continuous and maps the nonempty compact convex set C_n into itself. By the Brouwer fixed point theorem [9], there exists $x_n \in C_n$ such that

$$T_n(x_n) = x_n, \quad \text{i.e.,} \quad P_n(T(x_n)) = x_n.$$

Since $\{x_n\} \subset C$ and C is compact, there exists a subsequence (x_{n_k}) and a point $x^* \in C$ such that

$$x_{n_k} \rightarrow x^* \quad \text{in } E.$$

We now show that x^* is a fixed point of T . We write

$$\|x_{n_k} - T(x^*)\| = \|P_{n_k}(T(x_{n_k})) - T(x^*)\| \leq \|P_{n_k}(T(x_{n_k})) - T(x_{n_k})\| + \|T(x_{n_k}) - T(x^*)\|.$$

The first term satisfies $\|P_{n_k}(T(x_{n_k})) - T(x_{n_k})\| < \frac{1}{n_k} \rightarrow 0$. The second term tends to zero by continuity of T and $x_{n_k} \rightarrow x^*$. Therefore,

$$x_{n_k} \rightarrow T(x^*) \quad \text{in } E.$$

By uniqueness of limits, $T(x^*) = x^*$, and since $x^* \in C \subset K$, the point x^* is a fixed point of T in K . ■

Theorem 1.6: [Arzelà-Ascoli Theorem]

Let A be a subset of $C(J, E)$. Then A is relatively compact in $C(J, E)$ if and only if the following conditions are satisfied:

1. The set A is uniformly bounded; that is, there exists a constant $M \geq 0$ such that

$$\|f(x)\| \leq M, \quad \forall x \in J, \forall f \in A.$$

2. The family A is equicontinuous on J , i.e., for every $\varepsilon > 0$, there exists $\delta > 0$

such that

$$|t_1 - t_2| < \delta \Rightarrow \|f(t_1) - f(t_2)\| < \varepsilon,$$

for all $t_1, t_2 \in J$ and all $f \in A$.

3. For each $x \in J$, the set $\{f(x) : f \in A\}$ is relatively compact in E .

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