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Legendre-Chebyshev pseudo-spectral method for the diffusion equation with non-classical boundary conditions

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ABSTRACT. The present paper is devoted to the numerical approximation for the diffusion equation subject to non-local boundary conditions. For the space discretization, we apply the Legendre-Chebyshev pseudo-spectral method, so that, the problem under consideration is reduced to a system of ODEs which can be solved by the second order Crank-Nicolson schema. Optimal error estimates for the semi-discrete scheme are derived in L^2-norm. Numerical tests are included to demonstrate the effectiveness of the proposed method.

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1. Introduction

Partial differential equations with non-local boundary conditions (NLBCs) have attracted considerable interest due to their appearance in the mathematical modelling of numerous phenomena, in particular, models require that the value of the unknown data at the boundary is connected to its value inside of the domain.

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The primary aim of this paper is to investigate the use of Legendre-Chebyshev pseudo-spectral method (LC-PSM) for the numerical resolution of the following non-local boundary value problem,

$$\begin{aligned}
&\partial_t u(x,t) - \partial_{xx} u(x,t) = f(x,t), \quad x \in \Lambda,\ t \in (0,T] \\
&\partial_x u(-1,t) = \int_{-1}^{1} K_1(x) u(x,t) \mathrm{d}x \\
&\partial_x u(1,t) = \int_{-1}^{1} K_2(x) u(x,t) \mathrm{d}x
\end{aligned} \tag{1.1}$$

with the initial condition:

$$u(x,0) = u_0(x), \quad x \in [-1,1] \tag{1.2}$$

Where Λ represents the open interval $]-1,1[$, and $f(x,t)$ and $u_0(x)$ are two prescribed functions.

As has been mentioned, parabolic equations with non-local boundary conditions have a wide range of applications in various fields of sciences and engineering. Consider, heat conduction [3], thermodynamics [9], thermoelasticity [8], medical science [5], and biotechnology [7]. Various approaches have been put forward to study the theoretical aspects of these models and to seek for analytical and numerical solutions. There have been numerous studies in the literature devoted to investigate the existence, uniqueness and also continuous dependence on data for the solutions of above non-classical problem and some other similar problems [19, 20, 21].

As for the numerical resolution for this kind of problems, the classical algorithms have been applied with some extensions and modifications, we mention but not limiting to, finite difference schemes [6, 10, 17, 22], finite element procedures[12], Galerkin approaches[2], operational matrices method [1, 23]

In the same context, researchers have shown an increased interest in spectral/pseudo-spectral methods [4, 14, 18], as an accurate and efficient method for the numerical approximation of various types of evolution problems. However, a search of the literature revealed few papers which are devoted to develop and implement those methods for PDEs with non-local boundary conditions.

The work of Hu, Ma and Shen [15] can be considered as the most significant contribution in this direction, in which the authors analysed the use of Legendre spectral method and its error analysis for a one-dimensional elliptic equation with Neumann-type non-local boundary conditions. In [13] a Chebyshev pseudo-spectral method is used to handle a one-dimensional parabolic equation with Robin-type integral boundary conditions. Dehghan and Shamsi in [11] proposed a technique based on the Legendre pseudo-spectral method for the two-dimensional linear parabolic equation. It should be noted that the papers [11, 13] does not contain any theoretical analysis for the stability and convergence.

The present work follows the methodology as in [15] in attempt to implement the Legendre-Chebyshev pseudo-spectral method for the one-dimensional linear parabolic equations subject to (NLBCs). The main objectives of this paper are, firstly, to construct (LC-PSM) for the diffusion equation subject to Neumann type non-local boundary conditions, and secondly, to carry out optimal error estimates of the proposed schema.

Let $L^2(\Lambda)$ be the square integrable functions space equipped with L^2-scalar product and its associated norm, defined as

$$(v,w)_{L^2} = \int_\Lambda v(x)w(x)\mathrm{d}x\ , \quad \|v\| = \sqrt{(v,v)} = \left(\int_\Lambda |v(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}$$

We denote by $H^r(\Lambda), r \in \mathbb{N}$ the standard Sobolev space:

$$H^r(\Lambda) = \{u \in L^2(\Lambda) : \frac{\partial^k u}{\partial x^k} \in L^2(\Lambda), 0 \leq k \leq r\}$$

The norm and semi-norm in $H^r(\Lambda)$ are denoted by $\|.\|_r$ and $|.|_r$ respectively. In purpose to construct the proposed Legendre-Chebyshev pseudo-spectral method, we first reformulate the problem (1.1)-(1.2) into its weak form,
Find $u(t) \in H^1(\Lambda)$, such that for all $\varphi \in H^1(\Lambda)$,

$$(\partial_t u(t), \varphi) + (\partial_x u(t), \partial_x \varphi) - \mathcal{K}(u(t), \varphi) = (f(t), \varphi), \quad 0 < t \leq T \tag{1.3}$$

where,

$$\mathcal{K}(u(t), \varphi) = \varphi(1)\left(\int_\Lambda K_2(x)u(x,t)\mathrm{d}x\right) - \varphi(-1)\left(\int_\Lambda K_1(x)u(x,t)\mathrm{d}x\right) \tag{1.4}$$

Solvability of the variational problem (1.3) has been investigate in [21]

The rest of this paper is organized as follows. First, in Section (2), we quote the essential notions and formulas for the use of Legendre-Chebyshev pseudo-spectral method, followed by, the implementation of the proposed method for the resolution of the problem under consideration. The error analysis of the semi-discrete scheme is given in Section (3). Extension to nonlinear problems is discussed in Section (4). Finally, series of numerical tests are used to ensure the efficacy of the proposed method in Section (5). We conclude the paper with a short conclusion which includes some remarks.

2. Legendre-Chebyshev pseudo-spectral method

In this section, we present Legendre-Chebyshev pseudo-spectral method to solve the nonlocal boundary value problem (1.1). Denote by $\mathbb{P}_N(\Lambda)$ the set of all algebraic polynomials of degree at most N. We consider the interpolation operator at the Chebyshev-Gauss-Lobatto (CGL) nodes $\xi_j = \cos\frac{j\pi}{N}, 0 \leq j \leq N$, denoted by Π_N^C, which satisfy $\Pi_N^C v \in \mathbb{P}_N(\Lambda)$ for all $v \in L^2(\Lambda)$, and

$$\Pi_N^C v(\xi_j) = v(\xi_j)\ , 0 \leq j \leq N$$

2.1. Implement of the ChebyshevLegendre pseudo-spectral method.

The weak form (1.3) leads to the standard Galerkin approximation, which can be read as: Find $u_N(t) \in \mathbb{P}_N(\Lambda)$ such that, for all $\varphi \in \mathbb{P}_N(\Lambda)$,

$$(\partial_t u_N(t), \varphi) + (\partial_x u_N(t), \partial_x \varphi) - \mathcal{K}(u_N(t), \varphi) = (f, \varphi), \quad 0 < t \leq T \tag{2.1}$$

For practical and theoretical reasons, we approximate the source term f and the initial data u_0 by $\Pi_N^C f$ and $\Pi_N^C u_0$, respectively. Therefore, the Legendre-Chebyshev pseudo-spectral approximation for (1.1)-(1.2) reads as, find $u_N(t) \in \mathbb{P}_N(\Lambda)$ such that for all $\varphi \in \mathbb{P}_N(\Lambda)$,

$$\begin{cases} (\partial_t u_N(t), \varphi) + (\partial_x u_N(t), \partial_x \varphi) - \mathcal{K}(u_N(t), \varphi) = (\Pi_N^C f, \varphi) \\ u_N(0) = \Pi_N^C u_0 \end{cases} \tag{2.2}$$

Let us denote $L_n(x)$ the nth degree Legendre polynomial, which is defined by the following three-term recurrence formula:

$$\begin{aligned} &L_0(x) = 1 \quad \text{and} \quad L_1(x) = x \\ &L_{n+1}(x) = \frac{(2n+1)x}{n+1} L_n(x) \frac{n}{n+1} L_{n-1}(x) \quad , n \geq 1 \end{aligned} \tag{2.3}$$

We recall that the Legendre polynomials satisfy the orthogonality relation,

$$\int_{-1}^{1} L_k(x) L_j(x) \mathrm{d}x = \frac{1}{k+1} \delta_{k,j} \tag{2.4}$$

Let N be positive integer, we define

$$\begin{aligned} \phi_k(x) &= L_k(x) - L_{k+2}(x), \quad 0 \leq k \leq N-2 \\ \phi_{N-1}(x) &= \frac{1}{2}\left(L_0(x) + L_1(x)\right) \\ \phi_N(x) &= \frac{1}{2}\left(L_0(x) - L_1(x)\right) \end{aligned} \tag{2.5}$$

It is easy to verify using linear algebra arguments, that the set $(\phi_k)_{k=0}^N$ forms a basis for the space $\mathbb{P}_N(\Lambda)$. Now, we shall state the following lemma which is main ingredient of our algorithm.

Lemma 2.1 ([24]). *For two integer $j, k \in \mathbb{N}$, let us denote,*

$$\begin{aligned} m_{j,k} = m_{k,j} &= (\phi_j, \phi_k) = \int_{-1}^{1} \phi_j(x) \phi_k(x) \mathrm{d}x \\ p_{j,k} = p_{k,j} &= (\phi_j', \phi_k') = \int_{-1}^{1} \phi_j'(x) \phi_k'(x) \mathrm{d}x \end{aligned}$$

Then, for $0 \leq j, k \leq N-2$:

$$m_{j,k} = m_{k,j} = \begin{cases} \dfrac{1}{4k+6}\left(\dfrac{2}{2k+1} + \dfrac{2}{2k+5}\right) & , j = k \\ -\dfrac{1}{\sqrt{4k+6}} \cdot \dfrac{1}{\sqrt{4(k+2)+6}} \cdot \dfrac{2}{2k+5} & , j = k \pm 2 \\ 0 & , \text{otherwise} \end{cases}$$

and

$$p_{j,k} = p_{k,j} = \delta_{jk} = \begin{cases} 1 & , j = k \\ 0 & , \text{otherwise} \end{cases}$$

Now, for all $t \in (0, T]$ we seek an approximate solution $u_N(t)$ for variational problem (2.2) in $\mathbb{P}_N(\Lambda)$, and since $(\phi_k)_{k=0}^N$ forms a basis of the admissible space, thus, $u_N(t)$ can be written as a linear combination of ϕ_k with time-dependent coefficients, namely,

$$u_N(x,t) = \sum_{k=0}^{N} \alpha_k(t)\phi_k(x) \tag{2.6}$$

Inserting (2.6) into (2.2) and taking $\varphi = \phi_j$ $(j = 0, \cdots, N)$ yields,

$$\sum_{k=0}^{N} \frac{\mathrm{d}\alpha_k(t)}{\mathrm{d}t}(\phi_k, \phi_j) + \sum_{k=0}^{N} \alpha_k(t)(\phi_k', \phi_j') - \sum_{k=0}^{N} \alpha_k(t)\mathcal{K}(\phi_k, \phi_j) = (\Pi_N^C f(t), \phi_j) \tag{2.7}$$

starting with,

$$\sum_{k=0}^{N} \alpha_k(0)(\phi_k, \phi_j) = (\Pi_N^C u_0, \phi_j) \tag{2.8}$$

Denote,

$$\begin{aligned}
&\mathbf{U}(t) = (\alpha_0, \alpha_1, \ldots, \alpha_N)^t \\
&\mathbf{U}_0 = (u_0^0, u_1^0, \ldots, u_N^0)^t, \quad u_j^0 = \int_\Lambda \Pi_C^N u_0(x)\phi_j(x)\mathrm{d}x \\
&\mathbf{F}(t) = (f_0(t), f_1(t), \ldots, f_N(t))^t, \quad f_j(t) = \int_\Lambda \Pi_C^N f(x,t)\phi_j(x)\mathrm{d}x \\
&\mathbf{M} = [m_{i,k}]_{0\le j,k\le N}, \quad m_{j,k} = \int_\Lambda \phi_j(x)\phi_k(x)\mathrm{d}x \\
&\mathbf{P} = [p_{i,k}]_{0\le j,k\le N}, \quad p_{j,k} = \int_\Lambda \phi_j'(x)\phi_k'(x)\mathrm{d}x \\
&\mathbf{B} = [b_{i,k}]_{0\le j,k\le N}, \quad b_{j,k} = \mathcal{K}(\phi_j, \phi_k)
\end{aligned}$$

Then, the initial value problem (2.7)-(2.8) can be written in matrix formulation as follow,

$$\begin{aligned}
&\mathbf{M}\mathbf{U}'(t) + (\mathbf{P} - \mathbf{B})\mathbf{U}(t) = \mathbf{F}(t) \\
&\mathbf{U}(0) = \mathbf{U}_0
\end{aligned} \tag{2.9}$$

The value entries of the matrices **M** and **P** can be easily determined by using Lemma (2.1). In the other hand, since $L_k(1) = 1$ and $L_k(-1) = (-1)^k$, then

$$\phi_j(-1) = \phi_j(1) = 0, \quad 0 \le j \le N-2$$

So the matrix **B** is almost-null matrix except the two last rows, with entries:

$$\begin{aligned}
b_{N-1,j} &= \int_{-1}^{1} K_2(x)\phi_j(x)\mathrm{d}x \\
b_{N,j} &= \int_{-1}^{1} K_1(x)\phi_j(x)\mathrm{d}x
\end{aligned}$$

which can be calculated directly or by the aid of suitable quadratic formula.

2.2. Time integration. As a final step, using the second-order Crank-Nicolson schema we discretize the obtained system (2.9), that means, to approximate the values of $\alpha_k(t)$ at selected mesh points t_i. To this aim, for a positive integer M we define the step time $\Delta t = \frac{T}{M}$, and let $t_i = i\Delta t$, denote α_k^i the approximation of $\alpha_k(t_i)$. Then, we arrive at the following algebraic system:

$$\begin{aligned}&(\mathbf{M} + \Delta t(\mathbf{P} - \mathbf{B}))\,\mathbf{U}^{i+1} = (\mathbf{M} - \Delta t(\mathbf{P} + \mathbf{B}))\,\mathbf{U}^{i} + \Delta t(\mathbf{F}^{i+1} + \mathbf{F}^{i}) \quad ,i \geq 1\\ &\mathbf{M}\mathbf{U}^0 = \mathbf{U}(0)\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}\mathbf{U}^i &= (\alpha_0^i, \alpha_1^i, \cdots, \alpha_N^i)^t\\ \mathbf{F}^i &= (f_0(t_i), f_2(t_i), \cdots, f_N(t_i))^t\end{aligned}$$

System (2.10) can be solved easily using either direct or iterative methods. We preferred to apply QR factorization method, given its ease of implementation.

3. Error Estimates

In this section, we derive some error estimates for the semi-discrete scheme of the problem (1.1)-(1.2). First, in the next subsection we recall some essential lemmas related to spectral approximation method, which are needed in error analysis later.

3.1. Preliminaries. This subsection is devoted to introduce some basic notions and results that will be used in the sequel. Let $P_N : L^2(\Lambda) \to \mathbb{P}_N(\Lambda)$ be the L^2-orthogonal projection, namely,

$$(P_N v, \varphi) = (v, \varphi), \quad \forall \varphi \in \mathbb{P}_N(\Lambda)$$

Define $P_N^1 : H^1(\Lambda) \to \mathbb{P}_N(\Lambda)$, such that

$$P_N^1 v(x) = v(-1) + \int_{-1}^{x} P_{N-1}\partial_y v(y)\mathrm{d}y$$

According to the definition of P_N^1, it is readily to check that,

$$(\partial_x P_N^1 - \partial_x v, \partial_x \varphi) = 0 \quad ,\forall \varphi \in \mathbb{P}_N(\Lambda)$$

Next, we give the approximation property for the projection operator P_N^1.

Lemma 3.1 ([16]). *If $v \in H^r(\Lambda)$ with $r \geq 1$, then the following estimate holds*

$$\|v - P_N^1 v\|_l \leq CN^{l-r}\|v\|_r \quad ,0 \leq l \leq 1 \tag{3.1}$$

We also need the following lemma to control the error between v and $\Pi_N^C v$.

Lemma 3.2 ([24]). *Let $v \in H^1(\Lambda)$, there exists a positive constant C independent on N such that*

$$N\|\Pi_N^C v - v\| + |\Pi_N^C v|_1 \leq C\|v\|_1 \tag{3.2}$$

Moreover, if $v \in H^r(\Lambda)$ for $r \leq 1$, then the following estimate holds

$$\|v - \Pi_N^C v\|_r \leq CN^{r-s}\|v\|_s \quad ,0 \leq r \leq 1 \tag{3.3}$$

Remark 3.3. *Under the same assumptions of Lemma* (3.2)*, we can obtain from* (3.2) *the inequality,*

$$\|\Pi_N^C v\| \leq C\|v\|_1 r \tag{3.4}$$

Next, we derive a basic estimate that will be used later in our proofs.

Lemma 3.4. *With $\mathcal{K}(.,.)$ defined by* (1.4)*. Assume that $K_1, K_2 \in L^2(\Lambda)$. Then, for any $w, v \in H^1(\Lambda)$, the following estimate holds*

$$|\mathcal{K}(w,v)| \leq C_\varepsilon\big(\|w\|^2 + \|v\|^2\big) + \varepsilon|v|_1^2 \tag{3.5}$$

Proof. We start from the inequality

$$|\mathcal{K}(w,v)| \leq \left|v(1)\left(\int_\Lambda K_2(x)w(x)\mathrm{d}x\right)\right| + \left|v(-1)\left(\int_\Lambda K_1(x)w(x)\mathrm{d}x\right)\right| \tag{3.6}$$

It is readily checked that,

$$|\mathcal{K}(u,v)| \leq |v(1)|^2 + |v(-1)|^2 + C\|w\|^2 \tag{3.7}$$

Applying the the trace inequality,

$$\|z\|_\Gamma^2 \leq \varepsilon|z|_1^2 + C_\varepsilon\|z\|^2, \forall z \in H^1(\Lambda),\ 0 < \varepsilon < \varepsilon_0 \tag{3.8}$$

we obtain

$$|\mathcal{K}(w,v)| \leq C_\varepsilon\big(\|w\|^2 + \|v\|^2\big) + \varepsilon|v|_1^2 \tag{3.9}$$

which achieves the proof.

3.2. Error estimates. In this subsection, we consider the semi-discrete approximation (2.2). We give first a stability result for the solution $u_N(t)$ (2.2).

Theorem 3.5. *Let $u_0 \in H^1(\Lambda)$ and $f \in C^1\left(0, T; H^1(\Lambda)\right)$, then the solution $u_N(t)$ of* (2.2) *satisfies:*

$$\|u_N(t)\|^2 + \int_0^t |u_N(s)|_1^2 \mathrm{d}s \leq C\left(\int_0^t \|f(s)\|_1^2 \mathrm{d}s + \|u_0\|_1^2\right) \quad, 0 \leq t \leq T \tag{3.10}$$

Proof. The proof is straightforward, we only illustrate the mains steps. By virtue of (2.2) in which we set $\varphi = u_N(t)$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N(t)\|^2 + |u_N(t)|_1^2 = (\Pi_N^C f(t), u_N(t)) + \mathcal{K}(u_N(t), u_N(t)) \tag{3.11}$$

The first term on the right hand-side of (3.11) can be estimated by the aid of Cauchy inequality and inequality (3.4),

$$|(\Pi_N^C f(t), u_N(t))| \leq \|u_N(t)\|^2 + C\|f(t)\|_1^2 \tag{3.12}$$

We now estimate the second term on the right hand-side of (3.11). Using (3.5), we can obtain,

$$|\mathcal{K}(u_N(t), u_N(t))| \leq C_\varepsilon\|u(t)_N\|^2 + \varepsilon|u_N(t)|_1 \tag{3.13}$$

Putting things together and choosing $0 < \varepsilon < 1$ yields,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N(t)\|^2 + |u_N(t)|_1^2 \leq C_1\|u_N(t)\|^2 + C_2\|f(t)\|_1^2 \tag{3.14}$$

Integrating both sides of (3.14) over $(0,t)$ and applying Gronwall inequality, gives

$$\|u_N(t)\|^2 + \int_0^t |u_N(s)|_1^2 \mathrm{d}s \le C_4 \left(\int_0^t \|f(s)\|_1^2 \mathrm{d}s + \|u_0\|_1^2 \right) \quad ,0 \le t \le T \tag{3.15}$$

which is the desired result.

Now, we turn to the error estimates for the semi-discretization (2.2). Let $u(t)$ and $u_N(t)$ be the solutions of (1.3) and (2.2), respectively. We write the error $e_N(t) = u(t) - u_N(t)$ as a sum of two term, namely,

$$u(t) - u_N(t) = \left(u(t) - P_N^1 u(t) \right) + \left(P_N^1 u(t) - u_N(t) \right) = \rho_N(t) + \theta_N(t) \tag{3.16}$$

According to the approximation properties of the operator P_N^1, we only need to estimate the term $\theta_N(t)$. That is the objective of the following lemma.

Lemma 3.6. *If $u \in C^1\left(0,T;H^r(\Lambda)\right), r \ge 2$, then the following error estimate holds,*

$$\|\theta_N(t)\| \le CN^{-r},\ 0 \le t \le T \tag{3.17}$$

Proof. Subtract (2.2) from (1.3), we obtain the following error equation,

$$(\partial_t \theta_N(t), \varphi) + (\partial_x \theta_N(t), \partial_x \varphi) = \left(\Pi_N^C f - f, \varphi \right) - (\partial_t \rho_N(t), \varphi) + \mathcal{K}(\theta_N(t) + \rho_N(t), \varphi) \tag{3.18}$$

Taking $\varphi = \theta_N(t) \in \mathbb{P}_N$ in (3.18), we may have,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_N(t)\|^2 + |\theta_N(t)|_1^2 = I_1 + I_2 + I_3 \tag{3.19}$$

where

$$\begin{aligned} I_1 &= \left(\Pi_N^C f - f, \theta_N(t) \right) \\ I_2 &= -\left(\partial_t \rho_N(t), \theta_N(t) \right) \\ I_3 &= \mathcal{K}(\theta_N(t) + \rho_N(t), \theta_N(t)) \end{aligned} \tag{3.20}$$

Now, w shall estimate the terms I_1, I_2, and I_3 separately. Due to Cauchy-Schwarz inequality,

$$|I_1| = \left| \left(\Pi_N^C f - f, \theta_N(t) \right) \right| \le \|\Pi_N^C f - f\| \|\theta_N(t)\| \tag{3.21}$$

According to Lemma (3.2),

$$|I_1| \le C_1 N^{-r} \|f\|_r \|\theta_N(t)\| \tag{3.22}$$

In a similar way, using Lemma (3.1) and Cauchy-Schwarz inequality we can get,

$$|I_2| \le C_2 N^{-r} \|\partial_t u\|_r \|\theta_N(t)\| \tag{3.23}$$

The estimate of the term I_3 is a direct consequence of Lemma (3.4). Taking $w = \theta_N(t) + \rho_N(t)$ and $v = \theta_N(t)$ in (3.5) yields,

$$\begin{aligned} |I_3| &= |\mathcal{K}(\theta_N(t) + \rho_N(t), \theta_N(t))| \\ &\le C_\varepsilon \left(\|\theta_N(t) + \rho_N(t)\|^2 + \|\theta_N(t)\|^2 \right) + \varepsilon |\theta_N(t)|_1^2 \end{aligned} \tag{3.24}$$

using the triangular inequality,

$$|I_3| \le C_\varepsilon\big(\|\rho_N(t)\|^2 + \|\theta_N(t)\|^2\big) + \varepsilon|\theta_N(t)|_1^2 \tag{3.25}$$

hence, due to Lemma (3.1), on can obtain,

$$|I_3| \le C_\varepsilon\|\theta_N(t)\|^2 + \varepsilon|\theta_N(t)|^2 + C_3 N^{-2r}\|u\|_r^2 \tag{3.26}$$

Combining the above estimates, then equality (3.19) becomes,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_N(t)\|^2 + |\theta_N(t)|_1^2 \le C_4 N^{-2r}\left(\|f\|_r^2 + \|\partial_t u\|_r^2 + \|u\|_r^2\right) + C_\varepsilon\|\theta_N(t)\|_r^2 + \varepsilon|\theta_N(t)|_1^2 \tag{3.27}$$

By taking ε sufficiently small and integrating (3.27) over $(0,t)$, we obtain,

$$E(t) \le p(t) + C\int_0^t E(s)\mathrm{d}s \tag{3.28}$$

where

$$\begin{aligned} E(t) &= \|\theta_N(t)\|^2 + \int_0^t |\theta_N(s)|_1^2 \\ p(t) &= CN^{-2r}\int_0^t \left(\|f(s)\|_r^2 + \|\partial_t u(s)\|_r^2 + \|u(s)\|_r^2\right)\mathrm{d}s + \|\theta_N(0)\|^2 \end{aligned} \tag{3.29}$$

Gronwalls inequality implies,

$$E(t) \le p(t)e^{Ct},\ 0 < t \le T \tag{3.30}$$

Take into account,

$$\theta_N(0) = P_N^1 u_0 - \Pi_N^C u_0 = \left(P_N^1 u_0 - u_0\right) + \left(u_0 - \Pi_N^C u_0\right)$$

and approximation results (3.1) and (3.3), we get,

$$\|\theta_N(0)\|^2 \le CN^{-2r}\|u_0\|_r^2 \tag{3.31}$$

Inserting (3.31) into (3.28) yields,

$$\|\theta_N(t)\|^2 + \int_0^t |\theta_N(s)|^2\mathrm{d}x \le C\left(\int_0^t \left(\|f(s)\|_r^2 + \|\partial_t u(s)\|_r^2 + \|u(s)\|_r^2\right)\mathrm{d}s + \|u_0\|_r^2\right)$$

for all $0 \le t \le T$, which is the desired result.

Theorem 3.7. *Let $u(t)$ and $u_N(t)$ be the solution of* (1.3) *and* (2.1) *respectively. If $u \in C^1\left(0,T;H^r(\Lambda)\right)$ with $r \ge 1$, then the following error estimate holds,*

$$\|u(t) - u_N(t)\| \le CN^{-r},\ 0 \le t \le T \tag{3.32}$$

Proof. Applying triangular inequality to equality (3.16) yields,

$$\|u(t) - u_N(t)\| \le \|\theta_N(t)\| + \|\rho_N(t)\| \tag{3.33}$$

Using Lemma (3.1) and Lemma (3.6),

$$\|u(t) - u_N(t)\| \le CN^{-r} + C'N^{-r} \tag{3.34}$$

This completes the proof.

Remark 3.8. *It should be mentioned that the proposed method can be applied to the problem* (1.1) *with more general conditions, namely,*

$$\partial_x u(-1,t) + \alpha_1 u(-1,t) = \int_{-1}^{1} K_1(x,t)u(x,t)\mathrm{d}x + p_1(t)$$
$$\partial_x u(1,t) + \alpha_2 u(1,t) = \int_{-1}^{1} K_2(x,t)u(x,t)\mathrm{d}x + p_2(t)$$

4. Extension to nonlinear parabolic problems

The method we presented in previous sections for linear parabolic problems is well-suited for nonlinear parabolic problems. As an example, let us consider the following parabolic problem:

$$\partial_t u(x,t) - \partial_{xx} u(x,t) + G(u)(x,t) = F(x,t)$$
$$\partial_x u(-1,t) = \int_{-1}^{1} K_1(x)u(x,t)\mathrm{d}x$$
$$\partial_x u(1,t) = \int_{-1}^{1} K_2(x)u(x,t)\mathrm{d}x$$

where $G(\cdot)$ and $F(\cdot)$ are two given functions with sufficient smoothness. Using the same notations as in Section (2), then the LegendreChebyshev pseudo-spectral method for (3) leads to the following nonlinear differential system:

$$\begin{aligned}&\mathbf{M}\mathbf{U}'(t) + (\mathbf{P}-\mathbf{B})\mathbf{U}(t) + \mathbf{G}(\mathbf{U}(\mathbf{t})) = \mathbf{F}(t)\\ &\mathbf{U}(0) = \mathbf{U}_0\end{aligned} \tag{4.1}$$

where

$$\mathbf{G}(\mathbf{U}(t)) = (g_0(t), g_1(t), \cdots, g_N(t))^t, \quad g_j(t) = \int_\Lambda \Pi_N^C G(u(x,t))\phi_j(x)\mathrm{d}x$$

Applying the Crank-Nicolson/leapfrog schema leads to,

$$\left(\mathbf{M} + \Delta t(\mathbf{P}-\mathbf{B})\right)\mathbf{U}^{i+1} = \left(\mathbf{M} - \Delta t(\mathbf{P}-\mathbf{B})\right)\mathbf{U}^{i-1} + \Delta t(\mathbf{F}^{i+1} + \mathbf{F}^{i-1}) + 2\Delta t(\mathbf{G}(\mathbf{U}^{\mathbf{i}})) \quad , i \geq 1 \tag{4.2}$$

starting with

$$\begin{cases}\left(\mathbf{M} + \dfrac{\Delta t}{2}(\mathbf{P}-\mathbf{B})\right)\mathbf{U}^1 = \left(\mathbf{M} - \dfrac{\Delta t}{2}(\mathbf{P}-\mathbf{B})\right)\mathbf{U}^0 + \dfrac{\Delta t}{2}(\mathbf{F}^1 + \mathbf{F}^0) + \Delta t(\mathbf{G}(\mathbf{U}^{\mathbf{0}}))\\ \mathbf{M}\mathbf{U}^0 = \mathbf{U}(0)\end{cases} \tag{4.3}$$

where

$$\mathbf{G}(\mathbf{U}^i) = (g_0^i, g_1^i, \cdots, g_N^i)^t, \quad g_j^i = \int_\Lambda \Pi_N^C G(u_i(x))\phi_j(x)\mathrm{d}x$$

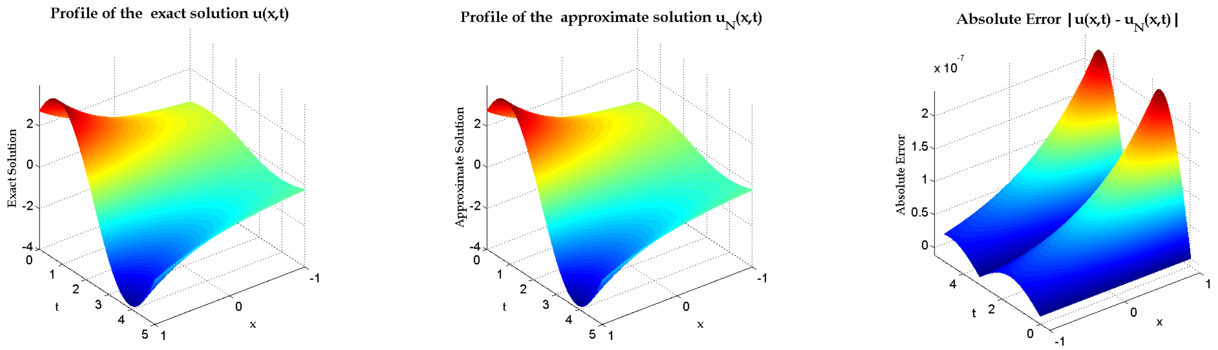

FIGURE 1. Numerical solution to (5.1) for $N = 16$, $\Delta t = 10^{-3}$ and $T = 5$

	t = 1		t = T	
N	L^2-error	L^∞-error	L^2-error	oL^∞-error
4	1.0180e-003	1.2236e-003	4.2676e-003	5.9152e-003
8	2.0043e-008	2.3128e-008	9.7785e-009	1.1314e-008
12	1.3449e-009	1.9224e-009	9.4592e-010	1.3513e-009
16	1.3448e-009	1.9221e-009	9.4207e-010	1.3458e-009
20	1.3440e-009	1.9219e-009	9.2043e-010	1.3149e-009

TABLE 1. L^2 and L^∞-errors of the (LG-SM) for solving Example (1) with $\Delta t = 10^{-4}$.

5. Numerical Validations

In this part, three numerical examples are provided to present the computational results obtained by the proposed (LC-PSM) to support the theoretical results and to illustrate the robustness and effectiveness of the method presented in the previous sections.

Example 1. Consider problem:

$$\begin{aligned}
\partial_t u(x,t) - \partial_{xx} u(x,t) &= -2\sin(t)e^x \\
\partial_x u(-1,t) &= \int_{-1}^{1} \frac{1}{2} e^{-(x+1)} u(x,t)\mathrm{d}x \\
\partial_x u(-1,t) &= \int_{-1}^{1} \frac{1}{2} e^{-x+1} u(x,t)\mathrm{d}x
\end{aligned} \tag{5.1}$$

with the exact solution

$$u^*(x,t) = (\sin(t) + \cos(t))e^x$$

In order to obtain approximate solution for (5.1), we take $\Delta t = 10^{-3}$ and $N = 16$ polynomial degree besides $T = 5$. Profiles of exact and approximate solutions, and the absolute errors function are plotted in Figure (1). The results confirm the high efficacy of the proposed method.

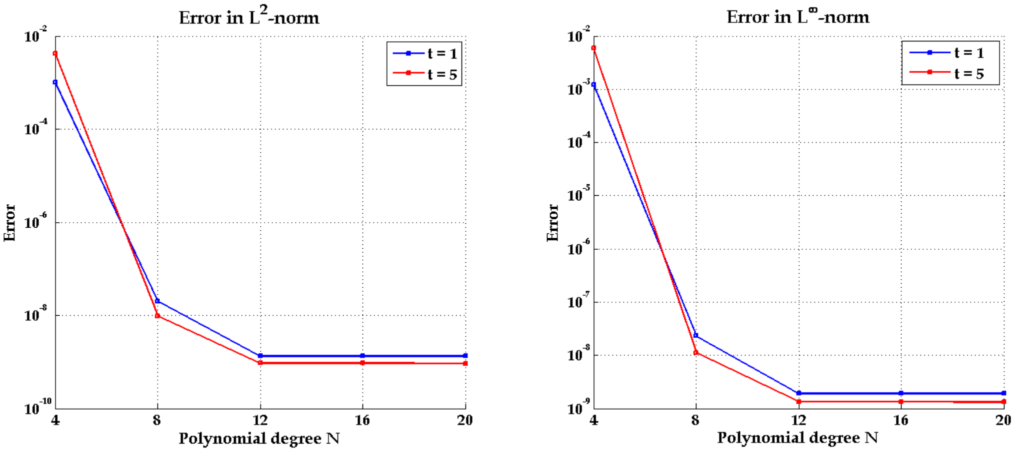

FIGURE 2. L_2 and L_∞-errors versus N at: $t = 1$ and $t = 5$

To investigate the convergence behaviour of numerical solutions we first examine the spatial discretization errors. For this aim, we fix step time $\Delta t = 10^{-4}$ so that the temporal discretization error is negligible as compared with the spatial discretization error, computational results are reported in Table (1). In Figure (2) , we plot the errors $E = uu_N$ in L_2 and L^∞ norms as a function of N at $t = 1$ and $t = T$. As usual in spectral methods, the errors in logarithm scale dropped rapidly, until the time discretization errors dominate.

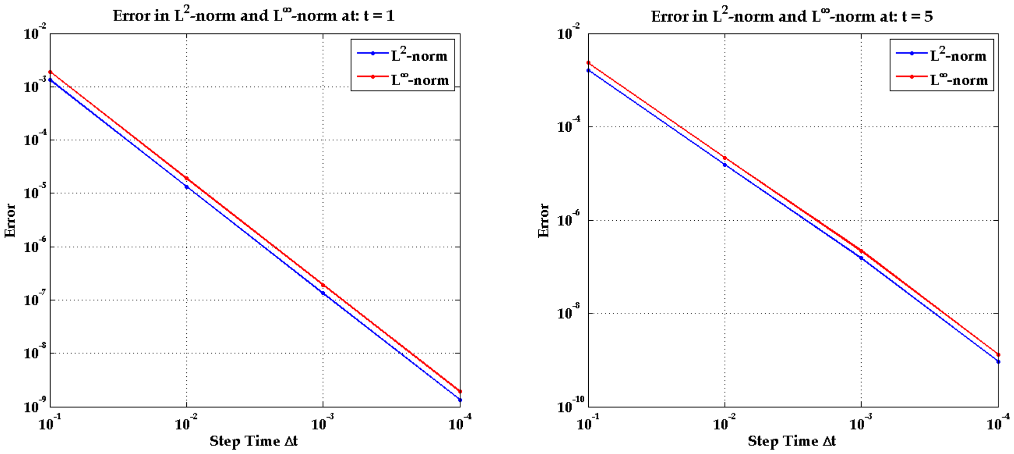

FIGURE 3. L_2 and L_∞-errors versus Δt at: $t = 1$ and $t = 5$

Now we investigate the time discretization errors, we fix the polynomial degree $N = 20$ large enough such that the space discretization errors are negligible, and make vary the step time. In Figure (3) , we plot the errors in L_2 and L^∞ norms as functions of Δt at t=1 and t=T. As expected, one can observe the second-order accuracy, which supports the theoretical results

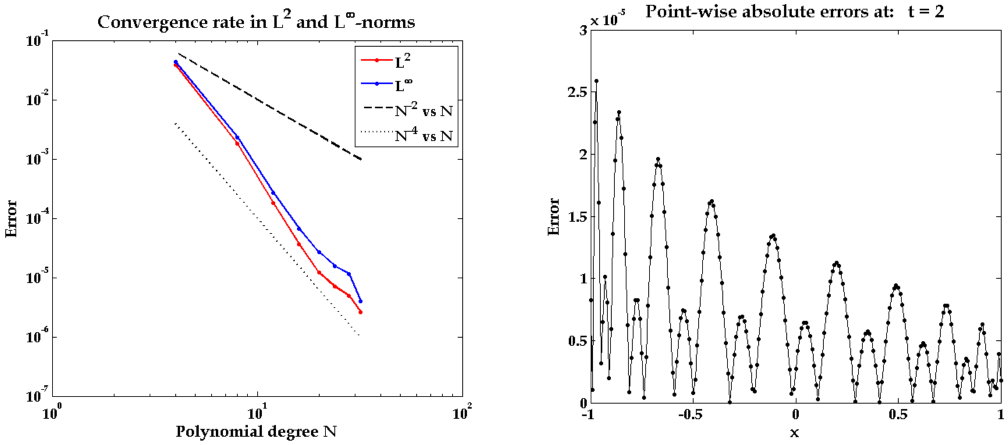

Figure 4. Left: $\log_{10}$(error) versus N at: $t = 2$ for Example (5.2). Right: Point-wise absolute errors for Example (5.2) with $N = 20$ and $\Delta t = 10^{-3}$

Example 2. Consider the initial boundary problem

$$\begin{aligned}
\partial_t u(x,t) - \partial_{xx} u(x,t) &= \sqrt{\frac{x+1}{2}}\left(4x^4 - 71x^2 + 14x + 21\right)e^t \\
\partial_x u(-1,t) &= \int_{-1}^{1} x(x+1)^{1/2}(1-x)u(x,t)\mathrm{d}x \\
\partial_x u(-1,t) &= \int_{-1}^{1} \sqrt{1-x}(x^3 - x)u(x,t)\mathrm{d}x
\end{aligned} \tag{5.2}$$

with the exact solution

$$u^*(x,t) = e^t(x+1)^{\frac{5}{2}}(x-1)^2$$

In this example, we consider a problem with an exact solution which has a low regularity in order to highlight the relation between the convergence rate and the regularity of the solution. For this purpose, we fix a step time sufficiently small, say $\Delta t = 10^{-4}$ so that the temporal discretization error is negligible, and make vary the polynomial degree N. In Figure, we show the decay rates of L^2 and L^∞-errors versus N in a log-scale. For a close comparison, we also plot the lines of decay rates N^{-3} and N^{-4}.

Overall, the computational results show that the L^2-error of (LG-SM) for the test problem (5.2) decay with a rate between N^{-3} and N^{-4}, which is exactly match with Theorem (3.7) since $u \in H^3(\Lambda)$ and $u \notin H^4(\Lambda)$

Example 3. Now, let us consider the following nonlinear parabolic problem:

$$\begin{aligned}
\partial_t u(x,t) - \partial_{xx} u(x,t) + G(u)(x,t) &= F(x,t) \\
\partial_x u(-1,t) &= \frac{15}{8}\int_{-1}^{1} (x^2+1)u(x,t)\mathrm{d}x \\
\partial_x u(-1,t) &= \int_{-1}^{1} \frac{8e^{2x}}{e^2 - 5e^{-2}} u(x,t)\mathrm{d}x
\end{aligned}$$

where, $G(u) = u^2 - u$ and $F(x,t)\ \frac{x^4-(t+1)^2(x^2+2)-2(t+1)x^2}{(t+1)^4}$. The exact solution to above problem is given as

$$u^*(x,t) = \frac{x^2}{t^2+2t+1}$$

Proceeding in a similar manner as of previous numerical experiments, the effectiveness and accuracy are tested by performing a comparison between numerical and exact solutions.

t variable	L^2-error	L^∞-error
$t = 0.4$	6.0211e-009	6.8643e-009
$t = 0.8$	5.6313e-009	5.9256e-009
$t = 1.2$	5.0165e-009	5.2393e-009
$t = 1.6$	4.5893e-009	4.8128e-009
$t = 2.0$	4.0734e-009	3.9158e-009

TABLE 2. Computational results for Example (3) using (LC-PSM) with $N = 8$ and $\Delta t = 10^{-3}$

Table (2) shows errors in L^2 and L^∞-norms between the numerical solutions obtained for $N = 8$ and $\Delta t = 10^{-3}$ and the exact solutions. We clearly observe that numerical solutions obtained by the proposed method are nearly identical with the exact solutions even for moderate N, that demonstrates the validity of the proposed (LC-PSM) for nonlinear parabolic equations with non-local boundary conditions.

Conclusion

Legendre-Chebyshev pseudo-spectral method has been developed to solve a linear one-dimensional parabolic equation subject to Neumann-type non-local boundary conditions. The proposed method reduces the problem under consideration to linear differential system by means of (LC-PSM) for the space discretization, and then the second-order Crank-Nicolson scheme is used to approximate the resulting system. The advantage of the present method is that it gives excellent accurate results even for a small polynomial degree N. At the same time, this approach is simple and can be easy applied to nonlinear problems.

In a future work, we plan to look for ways to reduce the storage requirement due to the time memory effect. Other further works include extending the proposed approach to nonlinear partial differential equations with integral boundary conditions.

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