Solution of Final Exam (Numerical method)

Exercise 01: 6 points

Let $f(x) = x^3 + 4x^2 - 10$ on $[1,2]$, then

1. Existance and uniqueness: (The IVT conditions) **(2 pts)**
   a. $\begin{cases} f(1) = -5 \\ f(2) = 14 \end{cases} \Rightarrow f(1).f(2) < 0.$ **(0.5)**
   b. Since the function f is polynomial so it is continuous on $\mathbb{R}$, hence it is continuous on $[1,2]$. **(0.5)**
   c. The function f is monotone since
   $$f'(x) = 3x^2 + 8x > 0, \text{ for all } x \in [1,2].$$
   Then f' increasing function (↗) hence monotone. **(0.5)**

By the IVT there exists a unique root $\alpha \in [1,2,]$ such that $f(\alpha) = 0$. **(0.5)**

2. The convergence conditions of Fixed point method **(2 pts)**
   - For $x = g_1(x) = x - x^3 - 4x^2 + 10$
     a. $g_1([1,2]) \subset [1,2]$? We observe that g_1 is well define and continuous on interval $[1,2]$, and
     $\begin{cases} g_1(1) = 6 \notin [1,2] \\ g_1(2) = -12 \notin [1,2] \end{cases} \Rightarrow g_1([1,2]) \not\subset [1,2]$ **(01)**
     So $g_1(x)$ does not satisfy the first condition
     b. $|g'_1(x)| \leq k < 1$ for all $x \in [1,2]$ and $k = \max_{x\in[1,2]} |g'_1(x)|$:
     We have $g'_1(x) = 1 - 3x^2 - 8x < 0, \forall x \in [1,2]$ then g'_1 ↘ (deceasing) and we obtain
     $$|g'_1(2)| \geq |g'_1(x)| \geq |g'_1(1)|$$
     Implies
     $$27 \geq 10 > 1. \quad \textbf{(01)}$$
     Thus, g'_1 does not satisfy the second condition for the fixed point method.
   - For $x = g_2(x) = \left(\frac{10}{4+x}\right)^{1/2}$ **(2 pts)**
     a. $g_2([1,2]) \subset [1,2]$? We observe that g_1 is well define and continuous on interval $[1,2]$, and
     $\begin{cases} g_2(1) = \sqrt{2} = 1.41 \in [1,2] \\ g_2(2) = \sqrt{\frac{5}{3}} = 1.29 \in [1,2] \end{cases} \Rightarrow g_2([1,2]) \subset [1,2].$ **(01)**
     So $g_2(x)$ is stable on $[1,2]$.
     b. $|g'_2(x)| \leq k < 1$ for all $x \in [1,2]$ and $k = \max_{x\in[1,2]} |g'_2(x)|$:

We have $g'_2(x) = \frac{-5}{\sqrt{10}(x+4)^{3/2}} < 0, \forall x \in [1,2]$ then $g'_2 \searrow$ (deceasing) and we obtain

$$|g'_2(2)| \leq |g'_2(x)| \leq |g'_2(1)|$$

Implies

$$0.1 \leq |g'_2(x)| \leq 0.14. \qquad \textbf{(01)}$$

Thus, $k = |g'_2(1)| = 0.14 < 1$ then $g'_2(x)$ satisfy the fixed point conditions.

Exercise 02: 7 points

Consider the differential equation with initial condition:

$$\begin{cases} y' = -y + t + 1 \\ y(0) = 1. \end{cases}$$

1. Existance and uniqueness of solution: **(2 pts)**
   - The function $f(t, y) = -y + t + 1$ is continuous on rectangle $R = \{[0,1] \times \mathbb{R}\}$.) **(0.5)**
   - Also the function $f(t, y)$ satisfy the Lipchitz condition on the second variable: Let y_1, y_2 then we have

     $$|f(t, y_1) - f(t, y_2)| = |-y_1 + t + 1 + y_2 - t - 1| = |-y_1 + y_2| = |-1||y_1 - y_2| \leq 1.|y_1 - y_2|$$

   So $\exists L = 1 > 0$.

   or

   $$\max_{t \in [0,1]} \left|\frac{\partial f(t,y)}{\partial y}\right| = 1 \leq 1 = L. \qquad \textbf{(01)}$$

   Thus the Cauchy problem has a unique solution. **(0.5)**
2. Let $h = 0.1$ so the Runge-Kutta of order 4 is given as **(4 pts)**

   $$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2(k_2 + k_3) + k_4), \qquad i = 0,1,2, \ldots \textbf{(0.5)}$$

   Such that

   $$\begin{cases} k_1 = f(t_i, y_i) \\ k_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right) \\ k_3 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right) \\ k_4 = f(t_i + h, y_i + h\, k_3) \end{cases} \qquad i = 0,1,2, \ldots \textbf{(0.5)}$$

   - For $i = 0$, we have

     $$\begin{cases} k_1 = f(t_0, y_0) = -1 + 0 + 1 = 0 \\ k_2 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1\right) = -y_0 - \frac{h}{2}k_1 + t_0 + \frac{h}{2} + 1 = 0.05 \\ k_3 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_3\right) = -y_0 - \frac{h}{2}k_2 + t_0 + \frac{h}{2} + 1 = 0.04750 \\ k_2 = f(t_0 + h, y_0 + h\, k_3) = -y_0 - hk_3 + t_0 + h + 1 = 0.095250 \end{cases} \qquad \textbf{(01)}$$

     Hence

$$y_1 = y(0.1) = y_0 + \frac{h}{6}(k_1 + 2(k_2 + k_3) + k_4) = 1 + \frac{0.1}{6}(0 + 2(0.05 + 0.04750) + 0.095250) = 1.004837.$$ **(0.5)**

- For $i = 1$, we have

$$\begin{cases} k_1 = f(t_1, y_1) = -y_1 + t_1 + 1 = 0.09516 \\ k_2 = f\left(t_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_1\right) = -y_1 - \frac{h}{2}k_1 + t_1 + \frac{h}{2} + 1 = 0.140404 \\ k_3 = f\left(t_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_3\right) = -y_1 - \frac{h}{2}k_2 + t_1 + \frac{h}{2} + 1 = 0.13814 \\ k_2 = f(t_1 + h, y_1 + h\,k_3) = -y_1 - hk_3 + t_1 + h + 1 = 0.18730 \end{cases}$$ **(01)**

  Hence

  $y_2 = y(0.2) = y_1 + \frac{h}{6}(k_1 + 2(k_2 + k_3) + k_4) = 1.004837 + \frac{0.1}{6}(0.09516 + 2(0.140404 + 0.13814) + 0.18730) = 1.01882946$ **(0.5)**

3. We have $y_{Exact} = e^{-t} + t$ so the absolute error is given as **(1 pts)**
   $|y_2(0.2) - y_{exact}(0.2)| = |1.01882946 - 1.01873075| = 0.0000987069.$ **(01)**

Exercise 03: 7 points

Let the following integral

$$I = \int_0^1 e^x\, dx$$

1. Calculate the integral I using the generalized trapezoid method with 8 intervals. **(03 pts)**

   $h = \frac{b-a}{n} = \frac{1-0}{8} = 0.125$ **(0.5)**, then

x_i	$x_0 = 0$	$x_1 = 0.125$	$x_2 = 0.250$	$x_3 = 0.375$	$x_4 = 0.5$	$x_5 = 0.625$	$x_6 = 0.750$	$x_7 = 0.875$	$x_8 = 1.000$
$f(x_i)$	1.0000	0.1331	1.2840	1.4550	1.6487	1.8682	2.1170	2.3989	2.7183

(Table with dividing the interval) **(1.5)**,

So $I_{GT8} = \frac{h}{2}\Big(f(x_0) + 2\big(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7)\big) + f(x_8)\Big) = \frac{0.125}{2}(1 + 2(0.1331 + 1.2840 + 1.4550 + 1.6487 + 1.8682 + 2.1170 + 2.3989) + 2.7183) = 1.7205$ **(1)**,

2. Compare the obtained result with the exact value: **(1.5 pts)**,

$$I_{Exact} = \int_0^1 e^x dx = e^1 - e^0 = 1.7183$$ **(0.5)**

   Hence, the absolute error is

$$|I_{Exact} - I_{GT8}| = |1.7183 - 1.7205| = 0.0022$$ **(1)**

3. The maximum error committed in this case is given by **(2.5 pts)**,

$$|E_{GT8}| \leq E_{max} = \frac{n\,h^3}{12} M \textbf{ (0.5)}$$

Where

$$M = \max_{x\in[0.1]} |f''(x)| \textbf{(0.5)}$$

With $f'(x) = e^x$, and $= e^x > 0, \forall x \in [0,1]$, then f'' is increasing **(0.5)** and we obtain

$$M = \max_{x\in[0.1]} |e^x| = f''(1) = e \textbf{(0.5)},$$

Thus,

$$E_{max} = \frac{(1-0)^3}{12.8^2}\ e = 0.0035. \textbf{ (0.5)}$$