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The exponential stability of nonlinear differential systems with time varying delay

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Introduction

In mathematics, delay differential equations (DDEs) are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. DDEs are also called time-delay systems, systems with aftereffect or dead-time, hereditary systems, equations with deviating argument, or differential-difference equations. They belong to the class of systems with the functional state, i.e. partial differential equations (PDEs) which are infinite dimensional, as opposed to ordinary differential equations (ODEs) having a finite dimensional state vector. Four points may give a possible explanation of the popularity of DDEs:

- Aftereffect is an applied problem: it is well known that, together with the increasing expectations of dynamic performances, engineers need their models to behave more like the real process. Many processes include aftereffect phenomena in their inner dynamics. In addition, actuators, sensors, and communication networks that are now involved in feedback control loops introduce such delays. Finally, besides actual delays, time lags are frequently used to simplify very high order models. Then, the interest for DDEs keeps on growing in all scientific areas and, especially, in control engineering.

- Delay systems are still resistant to many classical controllers: one could think that the simplest approach would consist in replacing them by some finite-dimensional approximations. Unfortunately, ignoring effects which are adequately represented by DDEs is not a general alternative: in the best situation (constant and known delays), it leads to the same degree of complexity in the control design. In worst cases (time-varying delays, for instance), it is potentially disastrous in terms of stability and oscillations.

- Voluntary introduction of delays can benefit the control system.
- In spite of their complexity, DDEs often appear as simple infinite-dimensional models in the very complex area of partial differential equations (PDEs).

A general form of the time-delay differential equation for $x(t) \in \mathbb{R}^n$ is

$$\frac{d}{dt}x(t) = f(t, x(t), x_t)$$

where $x_t = \{x(s) : s \leq t\}$represents the trajectory of the solution in the past. In this equation, f is a functional operator from $\mathbb{R} \times \mathbb{R}^n \times C^1(\mathbb{R} \times \mathbb{R}^n)$to $\mathbb{R}^n$.

Notation

Notation

Set notation

1. $\mathbb{N}$: the set of natural numbers, $\{1, 2, 3, \ldots\}$

2. $\mathbb{Z}$: the set of integers, $\{0, \pm1, \pm2, \pm3, \ldots\}$

3. $\mathbb{Q}$: the set of rational numbers, $\left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$

4. $\mathbb{R}$: the set of real numbers

5.$\mathbb{R}^+$: the set of positive real numbers

6. $\mathbb{C}$: the set of complex numbers

7. $[a, b]$: the closed interval $\{x \in \mathbb{R} : a \leq x \leq b\}$

8. $[a, b)$: the interval $\{x \in \mathbb{R} : a \leq x < b\}$

9. $(a, b]$: the interval $\{x \in \mathbb{R} : a < x \leq b\}$

10. (a, b) : the open interval $\{x \in \mathbb{R} : a < x < b\}$

11.H_0, H_1 : null and ilternative hypotheses for a hypothesis test

Miscellaneous symbols

1. $=$: is equal to

2. $\neq$: is not equal to

3. $\equiv$: is identical to or is congruent to

4. $\approx$: is approximately equal to

5. ˜ : is distributed as

6. $\cong$: is isomorphic to

7. $<$: is less than

8. $\leq$: is less than or equal to

9. $>$: is greater than

10. $\geq$: is greater than or equal to

11. ∞ : infinity

12. $\Rightarrow$: implies

13. $\Leftrightarrow$: implies and is implied by (is equivalent to)

14. $\sum_{i=1}^{n} a_i$: $a_1 + a_2 + ... + a_n$

15. $|a|$: the modulus of a

16. $f(x)$: the value of the function f at x

17. $f : x \mapsto y$: the function f maps the element x to the element y

18. f^{-1} : the inverse function of the one-one function f

19. $\lim_{x\to a} f(x)$: the limit of $f(x)$ as x tends to a

20. $\frac{dy}{dx}$: the derivative of y with respect to x

21. $\frac{d^n y}{dx^n}$: the nth derivative of y with respect to x

22. $f^{'}(x), f^{''}(x), \ldots, f^{(n)}(x)$: the first,second,…,nth derivatives of $f(x)$with respect to x

23. $\int_{b}^{b} ydx$: the definite integral of y with respect to x between the limits $x = a$ and $x = b$

24. $\dot{x}, \ddot{x}, ...$: the first,second,… derivatives of x with respect to t

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Abstract

This thesis is devoted to the study of the new criteria of the exponential stability of two forms of nonlinear deferential systems with time varying delay.

First, we presented some definitions and basic results on matrices. Then, we give theorems of existence of solution of these systems and different notions of stability. Next, we presented the explicit criteria for exponential stability and for two forms of nonlinear deferential systems with time varying delay. Finally, we present some illustrative examples.

Chapter 1

Definitions and preliminary results

Let $\mathbb{N}$ be the set of all natural numbers. For given $m \in \mathbb{N}$, let us denote $\underline{m} := \{1, 2, ..., m\}$. For integers $l, q \geq 1$, $\mathbb{R}^l$ denotes the l-dimensional vector space over $\mathbb{R}$ and $\mathbb{R}^{l \times q}$ stands for the set of all $l \times q-$matrices with entries in $\mathbb{R}$.

1.1 Matrix operations

1.1.1 Addition of Matrices

If $A = (a_{ij})$ and $B = (b_{ij})$ are two matrices of the same order then their sum $A + B$ is a matrix, and each element of that matrix is the sum of the corresponding elements.i.e $A + B = (a_{ij} + b_{ij})$.

Consider the two matrices A and B of order 2×2. Then the sum is given by:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

1.1.2 Properties of Matrix Addition

Proposition 1 *If A, B and C are matrices of same order, then*

1. Commutative Law: $A + B = B + A$

2. Associative Law: $(A+B)+C = A+(B+C)$

3. Identity of the Matrix: $A+O = O+A = A$, where O is zero matrix which is additive identity of the matrix,

4. Additive Inverse: $A+(-A) = 0 = (-A)+A$, where $(-A)$is obtained by changing the sign of every element of A which is additive inverse of the matrix.

5. $\begin{cases} A+B = A+C \\ B+A = C+A \end{cases} \Longrightarrow B = C$

6. $tr\,(A \pm B) = tr\,(A) \pm tr\,(B)$

7. If $A+B = 0 = B+A$, then B is called additive inverse of A and also A is called the additive inverse of A.

1.1.3 Subtraction of Matrices

If A and B are two matrices of the same order, then we define $A - B = A + (-B)$.

Consider the two matrices A and B of order 2×2. Then the difference is given by:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} - \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 & b_1 - b_2 \\ c_1 - c_2 & d_1 - d_2 \end{pmatrix}$$

We can subtract the matrices by subtracting each element of one matrix from the corresponding element of the second matrix. i.e. $A - B = (a_{ij} - b_{ij})$

1.1.4 Multiplication of Matrices

If A and B be any two matrices, then their product AB will be defined only when the number of columns in A is equal to the number of rows in B.

If $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $B = (b_{ij}) \in \mathbb{R}^{n\times p}$ then their product $AB = C = (c_{ij}) \in \mathbb{R}^{m\times p}$will be a matrix of order $m \times p$ where $c_{ij} = \sum\limits_{r=1}^{n} a_{ir} b_{rj}$.

1.1.5 Properties of matrix multiplication

1. Matrix multiplication is not commutative in general, i.e. in general $AB \neq BA$.
2. Matrix multiplication is associative, i.e $(AB)\,C = A\,(BC)$.
3. Matrix multiplication is distributive over matrix addition, i.e

$$A.\,(B + C) = A.B + A.C \quad \text{and} \quad (A + B)\,.C = AC + BC.$$

4. If A is an $m \times n$ matrix, then $I_m A = A = AI_n$.
5. The product of two matrices can be a null matrix while neither of them is null, i.e. if $AB = 0$, it is not necessary that either $A = 0$ or $B = 0$
6. If A is an $m \times n$ matrix and O is a null matrix then $A_{m\times n}.O_{n\times p} = O_{m\times p}$. i.e. the product of the matrix with a null matrix is always a null matrix.
7. If $AB = 0$ (It does not mean that $A = 0$ or $B = 0$, again the product of two non-zero matrices may be a zero matrix).
8. If $AB = AC$, $B \neq C$ (Cancellation Law is not applicable).
9. $tr\,(AB) = tr\,(BA)$.
10. There exist a multiplicative identity for every square matrix such $AI = IA = A$.

1.2 Inequalities between real matrices

- Inequalities between real matrices or vectors will be understood componentwise, i.e. for two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{l\times q}$, we write $A \geq B$ if $a_{ij} \geq b_{ij}$ for $i = 1, ..., l$, $j = 1, ..., q$.
- In particular, if $a_{ij} b_{ij}$ for $i = 1, ..., l$, $j = 1, ..., q$, then we write $A \gg B$ instead of $A \geq B$.

- A matrix $A = (a_{ij})$in $\mathbb{R}^{l\times q}$ is called positive if all elements of A are non-negative, i.e.$a_{ij} \geq 0$ for $i = 1, ..., l, j = 1, ..., q$ and denoted by$A \geq 0$. We denoted by $\mathbb{R}_+^{l\times q}$ the set of all positive matrices.
- A positive matrix is called non-negative matrix.
- A matrix is called strictly positive and noted $A > 0$ if all its elements are strictly positive.
- Similar notation are adopted for vectors.

Example 1 *Consider $A \in \mathbb{R}^{2\times 3}; B \in \mathbb{R}^{3\times 4}$ such that*

$$A = \begin{pmatrix} 2 & 7 & 3 \\ 0 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & \frac{1}{2} & 3 & 4 \\ \frac{1}{5} & 6 & 7 & 3 \\ 1 & \frac{3}{4} & 1 & \frac{1}{2} \end{pmatrix}$$

- A is a positive matrix
- B is a strictly positive matrix

1.3 Metzler matrix

Definition 1

- A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a Metzler matrix if all off-diagonal entries of M are non negative, $a_{ij} \geq 0$ for $i \neq j, \;\; i, j = 1, ..., n$
- A matrix $A \in \mathbb{R}^{n\times n}$ is called a strict Metzler matrix if:

$$a_{ij} \geq 0, \forall i, j = 1, ..., n \text{ and } a_{ij} < 0 \text{ if } i = j$$

Example 2 *Consider two matrices $A \in \mathbb{R}^{3\times 3}, A \in \mathbb{R}^{2\times 2}$ such that*

$$A = \begin{pmatrix} -1 & 0 & 3 \\ 0 & -2 & 1 \\ 5 & 2 & 3 \end{pmatrix}, B = \begin{pmatrix} -5 & 4 \\ 1 & -3 \end{pmatrix}$$

- A is a Metzler matrix.
- B is a strict Metzler matrix.

1.4 Matrix norms

For every $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, we have

- The 1-norm $\|x\|_1$, defined such that $\|x\|_1 = |x_1| + |x_2| + ... + |x_n|$.
- The euclidean norm $\|x\|_2$, defined such that $\|x\|_2 = (|x_1|^2 + |x_2|^2 + ... + |x_n|^2)^{1/2}$.
- The sup-norm $\|x\|_1$, defined such that $\|x\|_\infty = \{|x_1| + |x_2| + ... + |x_n|\}$.

more generally, we defined the p-norm for $p \geq 0$ by $\|x\|_p = (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{1/p}$.

A matrix norm $\|.\|$ on $M_n(k)$ with $K = \mathbb{R}$ or $\mathbb{C}$ is a norm on the vector space $M_n(k)$ with the additional property that for all $A, B \in M_n(k)$

$$\|A.B\| \leq \|A\| \, \|B\|$$

Proposition 2 *For every square matrix* $A = (a_{ij}) \in M_n(k)$, *we have*

1. $\|A\|_1 = \sup_{\substack{x \in \mathbb{C}^n \\ \|x\|_1 = 1}} \|Ax\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$
2. $\|A\|_\infty = \sup_{\substack{x \in \mathbb{C}^n \\ \|x\|_\infty = 1}} \|Ax\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$
3. $\|A\|_2 = \sup_{\substack{x \in \mathbb{C}^n \\ \|x\|_2 = 1}} \|Ax\|_2 = \sqrt{\rho(A^*A)} = \sqrt{\rho(AA^*)}$

The norm $\|A\|_2$ is often called the spectral norm.

Example 3 *Consider* $A \in \mathbb{R}^{3\times 3}$ *such that*

$$A = \begin{pmatrix} 3 & -1 & 4 \\ 1 & 5 & -9 \\ 2 & 6 & 5i \end{pmatrix}$$

The row sums are 8,15,13. The column sums are 6,12,18.

$$\|A\|_1 = 18 \qquad , \|A\|_\infty = 15 \qquad , \|A\|_2 \approx 13.5824$$

- For $x \in \mathbb{R}^n$ we define $|x| = (|x_i|)$
- For $P \in \mathbb{R}^{l\times q}$ we define $|P| = (|P_{ij}|)$
- A norm in $\mathbb{R}^n$ is said to be monotonic if $\|x\| \leq \|y\|$ whenever $x, y \in \mathbb{R}^n; |x| \leq |y|$.

Lemma 3 *Every p-norm on* $\mathbb{R}^n, 1 \leq p < \infty$, *and the sup-norm is monotonic.*

For any matrix $M \in \mathbb{R}^{n\times n}$ the spectral abscissa of M is denoted by

$$\mu(M) = \max\{\Re\lambda : \lambda \in \sigma(M)\},$$

where $\sigma(M) := \{z \in \mathbb{C} : \det(zI_n - M) = 0\}$ is the spectrum of M.

1.5 Properties of Metzler matrices

We present in this paragraph some properties of metzler matrices

Theorem 4 *Suppose* $M \in \mathbb{R}^{n\times n}$ *is a Metzler matrix. Then*

1. (Perron-Frobenius) $\mu(M)$ is an eigenvalue of M and there exists a nonnegative eigenvector $x \neq 0$ such that $Mx = \mu(M)x$
2. Given $\alpha \in \mathbb{R}$, there exists a nonzero vector $x \geq 0$ such that $Mx \geq \alpha x$ if and only if $\mu(M) \geq \alpha$.

3. $(tI_n - M)^{-1}$ exists and is nonnegative if and only if $t > \mu(M)$.

4. Given $B \in \mathbb{R}_+^{n\times n}, C \in \mathbb{C}^{n\times n}$.Then

$$|C| \leq B \Rightarrow \mu(M+C) \leq \mu(M+B).$$

Theorem 5 *Let $M \in \mathbb{R}^{n\times n}$ be a Metzler matrix. Then the following statements are equivalent*

1. $\mu(M) < 0$;

2. $Mp \ll 0$ for some $p \in \mathbb{R}_+^n$;

3. M is invertible and $M^{-1} \leq 0$;

4. For given $b \in \mathbb{R}^n, b \gg 0$ there exists $x \in \mathbb{R}_+^n$, such that $Mx + b = 0$;

5. For any $x \in \mathbb{R}_+^n \setminus \{0\}$, the row vector $x^T M$ has at least one negative entry.

Definition 2 *The upper dini derivative, which is also called on upper right-hand derivative, of a countinuous function* $f : \mathbb{R} \rightarrow \mathbb{R}$,*is denoted by* f'_+ *and defined by*

$f'_+(t) = \limsup_{h\to 0^+} \frac{f(t+h)-f(t)}{h}$

where lim sup is the supremun limit and the limit is a one-sided limit. The lower dini dirivative, $f\,'$, is defined by

$f\,'_-(t) = \liminf_{h\to 0^+} \frac{f(t)-f(t-h)}{h}$,where lim inf is the infimum limit.

Definition 3 *A function is said to be homogeneous of degree r,if multiplication of each of its independent variables by a constant j will alter the value of the function by the proportion j^r,that is $f(jx_1, ..., jx_n) = j^r f(x_1, ..., x_n)$In general. j can take any value. However, in order for the preceding equation to make sense, $(jx_1, ..., jx_n)$ must not lie outside the domain of the function f. For this reasan. in econmic applications the constant j is usually taken to be positive, as most economic variables do not admit negative values.*

Chapter 2

Stability of non-linear systems with time varying delay

Let $\mathbb{k}^{m\times n}$ be endowed with the norm $\|.\|$and let J be an interval of $\mathbb{R}$. Denote by $C\left(J,\mathbb{k}^{m\times n}\right)$, the vector space of all continuous functions on J with values in $\mathbb{k}^{m\times n}$.

In particular,$\mathbb{C}\left(\left[\alpha,\beta\right],\mathbb{C}^{m\times n}\right)$ is a Banach space endowed with the norm

$$\|\varphi\| := max_{\theta\in[\alpha,\beta]}\left\|\varphi\left(\theta\right)\right\|.$$

In what follows, we write C instead of $C\left(\left[-h,0\right],\mathbb{R}^{n}\right)$ and denote $C_r := \{\varphi\in C : \|\varphi\|\leq r\}$, for given $r>0$.

Definition 4 *For a matrix function $\varphi\left(.\right) : J \longrightarrow \mathbb{R}^{m\times n}$, we say that $\varphi\left(.\right)$ nonnegative and denote it by $\varphi\geq 0$ if $\varphi\left(\theta\right)\geq 0$ for all $\theta\in J$.*

2.1 Systems of the first forme

Consider a nonlinear differential system with time-varying delay of the form

$$\dot{x}(t) = F(t; x\left(t\right), x\left(t-\tau(t)\right), \int_{-h(t)}^{0} B(t,s)x(t+s)ds), \tag{2.1}$$

where

(H_1) $\tau(.), h(.) : \mathbb{R} \to \mathbb{R}$ are given continuous functions such that $0 \le h(t), \tau(t) \le h, \forall t \in \mathbb{R}$, for some $h > 0$ and $B(.;.) : \mathbb{R} \times [-h, 0] \to \mathbb{R}^{n\times n}$ is a given matrix-valued continuous function.

(H_2) $F(.;.,.,.) : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \times\mathbb{R}^n \to \mathbb{R}^n$, is a given continuous function such that $F(t; 0, 0, 0) = 0; \forall t \in \mathbb{R}$ and $F(t; x, y, z)$ is (locally) Lipschitz continuous with respect to x, y, z on each compact subset of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times\mathbb{R}^n$ .Then (2.1) can be represented in the form

$$\dot{x}(t) = W(t; x_t),$$

where for each $t \in \mathbb{R}$, $x_t(\theta) := x(t + \theta)$,$\theta \in [-h, 0]$and $W(.;.) : \mathbb{R} \times \mathbb{C} \to \mathbb{R}^n$ is defined by

$$W(t; \varphi) := F(t; \varphi(0), \varphi(-\tau(t)), \int_{-h(t)}^{0} B(t, s)\varphi(s)ds).$$

Let system (2.1) satisfying (H_1) and(H_2), it follows that $W(., .)$is continuous on $\mathbb{R} \times \mathbb{C}$ and is Lipschitzian in φ in each compact subset of $\mathbb{R} \times \mathbb{C}$. Thus for a fixed $\sigma \in \mathbb{R}$ and a given $\varphi \in \mathbb{C}$, there exists a unique local solution of (2.1) satisfying the initial condition

$$x(\sigma + s) := \varphi(s), s \in [-h, 0]. \qquad (2.2)$$

This solution is denoted by $x(.; \sigma, \varphi)$ and $x(.; \sigma, \varphi)$ is continuous on $[\sigma - h, \gamma)$for some $\gamma > \sigma$ and satisfies(2.1) for every $t \in [\sigma, \gamma)$. Furthermore,if the interval $[\sigma - h, \gamma)$ is the maximum interval of existence of $x(.; \sigma, \varphi)$ then $x(.; \sigma, \varphi)$ is said to be noncontinuable. The existence of a noncontinuable solution follows from Zorn's lemma and the maximum interval of existence must be open.

2.1.1 Notations of stability

Definition 5 *The zero solution of*(2.1) *is said to be locally exponentially stable if there exist positive numbers* r, K, β *such that for each* $\sigma \in \mathbb{R}^+$ *and each* $\varphi \in C_r$ *, the solution* $x(.; \sigma, \varphi)$ *of* $(2.1) - (2.2)$ *exists on*$[\sigma - h, \infty)$ *and furthermore satisfies*

$$\|x(t; \sigma, \varphi)\| \le Ke^{-\beta(t-\sigma)}, \forall t \ge \sigma.$$

Definition 6 *The zero solution of (2.1) is said to be globally exponentially stable if there exist positive numbers K, β such that for each $\sigma \in \mathbb{R}$ and each $\varphi \in \mathbb{C}$, the solution $x(.;\sigma,\varphi)$ of (2.1)–(2.2) exists on $[\sigma - h, \infty)$ and furthermore satisfies*

$$\|x(t;\sigma,\varphi)\| \leq Ke^{-\beta(t-\sigma)} \|\varphi\|, \forall t \geq \sigma.$$

Definition 7 *When the zero solution of (2.1) is locally exponentially stable, globally exponentially stable then we also say that (2.1) is locally exponentially stable, globally exponentially stable, respectively.*

2.1.2 Explicit criteria for exponential stability

Theorem 6 *Let $F(t;.,y,z)$ be continuously differentiable on $\mathbb{R}^n$ for any $t \in \mathbb{R}$ and any $y, z \in \mathbb{R}^n$. Assume that*

$$\sup \int_{-h}^{0} \|B(t,s)\| \, ds < \infty \qquad (2.3)$$

and there exist $\delta > 0$ and matrix-valued continuous functions $A_0(.) := \left(a_{ij}^{(0)}(.)\right) : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $A_1(.), A_2(.) : \mathbb{R} \longrightarrow \mathbb{R}_+^{n\times n}$ such that

$$\frac{\partial F_i}{\partial x_i}(t;x,y,z) \leq a_{ii}^{(0)}(t); \left|\frac{\partial F_i}{\partial x_j}(t;x,y,z)\right| \leq a_{ij}^{(0)}(t), i \neq j, i,j \in \underline{n} \qquad (2.4)$$

for any $t \in \mathbb{R}$ and any $x, y, z \in B_\delta$ and

$$|F(t;0,y,z)| \leq A_1(t)|y| + A_2(t)|z|, \forall t \in \mathbb{R}, \forall y, z \in B_\delta. \qquad (2.5)$$

Furthermore, suppose there exist $\beta > 0$ and $p \in \mathbb{R}^n, p \gg 0$ so that

$$A_0(t) + A_1(t)e^{\beta\tau(t)} + A_2(t)\int_{-h(t)}^{0} |B(t,s))| \, e^{-\beta s}ds)p \ll -\beta p, \forall t \in \mathbb{R} \qquad (2.6)$$

Then

(i) The zero solution of (2.1) is exponentially stable.

(ii) In addition, if (2.4) holds for any $t \in \mathbb{R}$ and any $x, y, z \in \mathbb{R}^n$ and (2.5) holds for any $t \in \mathbb{R}$ and any $y, z \in \mathbb{R}^n$, Then the zero solution of(2.1) is globally exponentially stable, regardless of (2.3).

Let $\varphi \in \mathbb{C}$ be given and let $x(t; \sigma, \varphi), t \in [\sigma - h, \gamma)$ be a noncontinuable solution of (2.1) − (2.2). We divide the proof into three steps.

Step I. There exists $r > 0$ such that for any $\varphi \in C_r$ and any $\sigma \in \mathbb{R}$, $x(.) := x(t; \sigma, \varphi)$ satisfies

$$\|x(t)\| \leq \delta, \forall t \in [\sigma, \gamma) ; \int_{-h}^{0} \|B(t, s)\| \|x(t + s)\| ds \leq \delta, \forall t \in [\sigma, \gamma) . \tag{2.7}$$

Without loss of generality, let $\mathbb{R}^n$ be endowed with the maximum norm $\|.\|_\infty$. Since $p \gg 0$, (2.6) holds for any vector $kp \in \mathbb{R}^n$,$k > 0$. Therefore, we can assume further that

$$\|p\| \leq \min \left\{ \frac{\delta}{2}, \frac{\delta}{2 \sup_{t \in R} \int \|B(t, s)\| ds} \right\} . \tag{2.8}$$

Let $p := (p_1, p_2, ..., p_n)^T$, $p_i > 0$, $\forall i \in \underline{n}$ and choose $r \in \{0, \min_{1 \leq i \leq n} p_i\}$.

Then $|\varphi(t)| \ll p$, for any $t \in [-h, 0]$ and for any $\varphi \in C_r$.

Note that $x(\sigma + t) = \varphi(t), t \in [-h, 0]$ and $|x(\sigma)| = |\varphi(0)| \ll p$:

We claim that $|x(t)| \leq p, \forall t \in [\sigma, \gamma)$.

Assume on the contrary that there exists $t_0 > \sigma$ such that $|x(t_0)| p$.

Set $t_1 := \inf \{t \in (\sigma, \gamma) : |x(t)| p\}$. By continuity, $t_1 > \sigma$ and there is $i_0 \in n$ such that

$$|x(t)| \leq p \forall t \in [\sigma, t_1) ; | x_{i0}(t_1) |= p_{i0}; | x_{i0}(t_1) |> p_{i0}, \forall t \in (t_1, t_1 + \varepsilon), \tag{2.9}$$

for some $\varepsilon > 0$.

Let $A_0(t) := \left(a_{ij}^{(0)}(t) \right)$; $A_1(t) := \left(a_{ij}^{(1)}(t) \right), t \in \mathbb{R}$ and let $A_2(t) |B(t, s)| := (b_{ij}(t, s)), s \in [-h, 0], t \in \mathbb{R}$.. By the mean value Theorem , we have for each $t \in \mathbb{R}$ and for each $i \in \underline{n}$

$$\begin{aligned}\dot{x_i}(t) &= (F_i(t; x\left(t\right), x\left(t-\tau(t)\right)), \int_{-h(t)}^{0} B(t,s)x(t+s)ds) - F_i(t; 0, , x\left(t-\tau(t)\right)), \\ &\quad \int_{-h(t)}^{0} B(t,s)x(t+s)ds)) + \ F_i(t; 0, , x\left(t-\tau(t)\right)), \int_{-h(t)}^{0} B(t,s)x(t+s)ds)\end{aligned}$$

$$\begin{aligned}&= \sum_{j=1}^{n}\left(\int_0^1 \frac{\partial F_i}{\partial x_j}\left(t, \xi x\left(t\right), x\left(t-\tau(t)\right)\right), \int_{-h(t)}^{0} B(t,s)x(t+s)ds)d\xi\right) x_j\left(t\right) \\ &\quad + F_i(t; 0, , x\left(t-\tau(t)\right)), \int_{-h(t)}^{0} B(t,s)x(t+s)ds).\end{aligned}$$

On the other hand, it follows from (2.8)-(2.9) that

$$\left\|x\left(t\right)\right\| \leq \frac{\delta}{2}, \quad t \in \left[-h+\sigma, t_1\right]; \qquad \int_{-h}^{0} \left\|B(t,s)\right\| \left\|x(t+s)\right\| ds \leq \frac{\delta}{2}, \qquad t \in \left[\sigma, t_1\right]. \tag{2.10}$$

By continuity,

$$\left\|x\left(t\right)\right\| \leq \delta, \quad t \in \left[-h+\sigma, t_1+\varepsilon_0\right); \qquad \int_{-h}^{0} \left\|B(t,s)\right\| \left\|x(t+s)\right\| ds \leq \delta, t \in \left[\sigma, t_1+\varepsilon_0\right]. \tag{2.11}$$

for some sufficiently small $\varepsilon_0 > 0$. Thus,

$$\frac{\partial}{\partial t}\left|x_i\left(t\right)\right| = sgn\left(x_i\left(t\right)\right)\dot{x}_i\left(t\right) \leq (\int_0^1 \frac{\partial F_i}{\partial x_j}\left(t, \xi x\left(t\right), x\left(t-\tau(t)\right)\right), \int_{-h(t)}^{0} B(t,s)x(t+s)ds)d\xi)\left|x_i\left(t\right)\right|$$

$$\begin{aligned}&+ \sum_{j=1, j\neq i}^{n}\int_0^1 \left|\frac{\partial F_i}{\partial x_j}\left(t, \xi x\left(t\right), x\left(t-\tau(t)\right)\right), \int_{-h(t)}^{0} B(t,s)x(t+s)ds)\right| d\xi\left|x_j\left(t\right)\right| + \\ &\left|F_i(t; 0, , x\left(t-\tau(t)\right)), \int_{-h(t)}^{0} B(t,s)x(t+s)ds\right|\end{aligned}$$

$$\leq a_{ii}^{0}\left(t\right)\left|x_{i}\left(t\right)\right|+\sum_{j=1,j\neq i}^{n}a_{ij}^{(0)}\left(t\right)\left|x_{j}\left(t\right)\right|+\sum_{j=1}^{n}a_{ij}^{(1)}\left(t\right)\left|x_{j}\left(t-\tau\left(t\right)\right)\right|+\sum_{j=1}^{n}\int_{-h}^{0}b_{ij}\left(t,s\right)\left|x_{j}\left(t+s\right)\right|ds,$$

for almost any $t\in\left[\sigma,t_{1}+\varepsilon_{0}\right]$. It follows that

$$\begin{aligned}
D^{+}\left|x_{i}\left(t\right)\right| \quad &: \quad =\limsup\frac{\left|x_{i}\left(t+h\right)\right|-\left|x_{i}\left(t\right)\right|}{h}\\
&= \quad \limsup\frac{1}{h}\int_{t}^{t+h}\frac{d}{ds}\left|x_{i}\left(s\right)\right|ds\leq a_{ii}^{0}\left(t\right)\left|x_{i}\left(t\right)\right|+\sum_{j=1,j\neq i}^{n}a_{ij}^{(0)}\left(t\right)\left|x_{j}\left(t\right)\right|
\end{aligned}$$

$$+\sum_{j=1}^{n}a_{ij}^{(1)}\left(t\right)\left|x_{j}\left(t-\tau\left(t\right)\right)\right|+\sum_{j=1}^{n}\int_{-h}^{0}b_{ij}\left(t,s\right)\left|x_{j}\left(t+s\right)\right|ds$$

for any $t\in\left[\sigma,t_{1}+\varepsilon_{0}\right]$, where D^{+} denotes the Dini upper-right derivative.In particular, we have

$$D^{+}\left|x_{i_{0}}\left(t_{1}\right)\right|\leq a_{i_{0}i_{0}}^{(0)}\left(t_{1}\right)\left|x_{i_{0}}\left(t_{1}\right)\right|+\sum_{j=1,j\neq i}^{n}a_{i_{0}j}^{(0)}\left(t_{1}\right)\left|x_{j}\left(t_{1}\right)\right|+\sum_{j=1}^{n}a_{i_{0}j}^{(1)}\left(t_{1}\right)\left|x_{j}\left(t_{1}-\tau\left(t_{1}\right)\right)\right|$$

$$+\sum_{j=1}^{n}\int_{-h}^{0}b_{i_{0}j}\left(t_{1},s\right)\left|x_{j}\left(t_{1}+s\right)\right|ds\leq\sum_{j=1}^{n}a_{i_{0}j}^{(0)}\left(t_{1}\right)p_{j}+\sum_{j=1}^{n}a_{i_{0}j}^{(1)}\left(t_{1}\right)p_{j}+\sum_{j=1}^{n}\int_{-h}^{0}b_{i_{0}j}\left(t_{1},s\right)p_{j}ds$$

$$\leq\sum_{j=1}^{n}a_{i_{0}j}^{(0)}\left(t_{1}\right)p_{j}+\sum_{j=1}^{n}a_{i_{0}j}^{(1)}\left(t_{1}\right)e^{Bh}p_{j}+\sum_{j=1}^{n}\int_{-h}^{0}b_{i_{0}j}\left(t_{1},s\right)e^{-Bs}p_{j}ds<-\beta p_{i_{0}}<0.$$

However, this conflicts with (2.9). Therefore

$$|x(t;\sigma,\varphi)|=\left|x\left(t\right)\right|\leq p,\forall t\in\left[\sigma,\gamma\right);\forall\varphi\in C_{r}.$$

By the monotonicity of vector norms, this yields

$$\|x(t;\sigma,\varphi)\| \leq \|p\| \leq \delta, \forall t \in [\sigma,\gamma)\,; \forall \varphi \in C_r. \tag{2.12}$$

It follows from (2.8) and (2.12) that

$$\int_{-h}^{0} \|B(t,s)\| \, \|x(t+s)\| \, ds \leq \delta, \forall t \in [\sigma,\gamma)\,.$$

Step II. We show that

$$\|x(t;\sigma,\varphi)\| \leq Ke^{-\beta(t-\sigma)} \forall t \in [\sigma,\gamma)\,; \forall \varphi \in C_r. \tag{2.13}$$

where $\beta > 0$ satisfies(2.6) and r is determined in Step I and K depends on β, r. Choose $K > 0$ such that $|\varphi(t)| \ll Ke^{-\beta t}p$, for any $t \in [-h, 0]$ and for any $\varphi \in C_r$.

Define $v(t) := Ke^{-\beta(t-\sigma)}p, t \in [\sigma - h, \infty)$. We have, $|x(t)| \ll v(t)$ $\forall t \in [\sigma - h, \sigma)$.

We claim that $|x(t)| \leq v(t)$ $\forall t \in [\sigma, \gamma)$.

Assume on the contrary that there exists $t_* > \sigma$ such that $|x(t_*)| v(t_*)$.

Set $t_c := \inf\{t \in (\sigma,\gamma) : |x(t)| v(t)\}$. By continuity, we have $t_c > \sigma$. Then there is $i_0 \in \underline{n}$ such that $|x(t)| \leq v(t)$

$$\forall t \in [\sigma, t_c)\,; |x_{i_0}(t_c)| = v_{i_0}(t_c)\,,\ |x_{i_0}(t)|\ > v_{i_0}(t)\,, \forall t \in [t_c, t_c + \varepsilon)\,, \tag{2.14}$$

for some $\varepsilon > 0$. Taking $(2.4) - (2.5), (2.7)$ into account, by a similar argument as in Step I, we have

$$\begin{aligned} D^+ |x_i(t)| \ &\leq\ a_{ii}^{(0)}(t)\,|x_i(t)| + \sum_{j=1, j\neq i}^{n} a_{ij}^{(0)}(t)\,|x_j(t)| + \sum_{j=1}^{n} a_{ij}^{(1)}(t)\,|x_j(t-\tau(t))| \\ &\quad + \sum_{j=1}^{n} \int_{-h}^{0} b_{ij}(t,s)\,|x_j(t+s)|\,ds \end{aligned}$$

for any $t \in [\sigma, \gamma)$. In particular, it follows from (2.14) that

$$D^{+}\left|x_{i_0}\left(t_c\right)\right| \leq Ke^{-\beta(t_c-\sigma)}\left(\sum_{j=1}^{n} a_{i_0 j}^{(0)}\left(t_c\right) p_j + \sum_{j=1}^{n} a_{i_0 j}^{(1)}\left(t_c\right) e^{\beta h} p_j + \sum_{j=1}^{n}\int_{-h}^{0} b_{i_0 j}\left(t_c, s\right) e^{-Bs} p_j ds\right)$$

$$< -\beta K e^{-\beta(t_c-\sigma)} p_{i_0} = D^{+} v_{i_0}\left(t_c\right).$$

However, this conflicts with (2.14). Therefore

$$|x(t;\sigma,\varphi)| \leq v\left(t\right) = Ke^{-\beta(t-\sigma)}p, \forall t \in [\sigma,\gamma)\,; \forall \varphi \in C_{r.}$$

By the monotonicity of vector norms, this yields

$$\|x(t;\sigma,\varphi)\| \leq K_1 e^{-\beta(t-\sigma)}, \forall t \in [\sigma,\gamma)\,; \forall \varphi \in C_{r.}$$

where $K_1 := K\left\|p\right\| > 0$.

Step III. We claim that $\gamma = \infty$.

Seeking a contradiction, we assume that $\gamma < \infty$. Then it follows from (2.13) that $x(.;\sigma,\varphi)$ is bounded on$[\sigma,\gamma)$.

By continuity of F and B, this together with (2.1) imply that $\dot{x}$ (.)is bounded on$[\sigma,\gamma)$ .Thus$\dot{x}$ (.) is uniformly continuous on$[\sigma,\gamma)$ .Therefore, $\lim_{t\to\gamma} - x\left(t\right)$ exists and x (.) can be extended to a continuous function on$[\sigma,\gamma)$. Moreover the closureof $\{x_t : t \in [\sigma,\gamma)\}$ is a compact set in $C_{r.}$, by Arzela-Ascoli theorem .

Note that$\{(t,x_t) : t \in [\sigma,\gamma)\} \subset [\sigma,\gamma] \times$ the closure of $\{x_t : t \in [\sigma,\gamma)\}$ .Thus, the closure of $\{(t,x_t) : t \in [\sigma,\gamma)\}$ is compact set in $\mathbb{R}_+ \times \mathbb{C}_r$ .Since (γ, x_γ) belongs to this compact set, one can find a solution of (2.1) through this point to the right of γ. This contradicts the noncontinuability of $x\,(.)$. Thus γ must be equal to ∞.

Finally, if (2.4) holds for any $t \in \mathbb{R}$,any $x, y, z \in \mathbb{R}^n$ and (2.5) holds for any $t \in \mathbb{R}$, any $y, z \in \mathbb{R}^n$, Then Step I can be omitted.Thus, (2.3) can be removed.

By applying the Theorem to the invariant time delay system, we have for any $\varphi \in$

C_1,

$$\|y(t;\sigma,\varphi)\| \leq K_1 e^{-\beta(t-\sigma)}, \forall t \geq \sigma, \tag{2.15}$$

for some $K_1 > 0$,$\beta > 0$. From linearity of the invariant time elay system , it follows that

$$\frac{1}{\|\varphi\|}\|y(t;\sigma,\varphi)\| = \left\| y(t;\sigma,\frac{\varphi}{\|\varphi\|})\right\| \leq K_1 e^{-\beta(t-\sigma)}, \forall t \geq \sigma, \forall \varphi \in C, \varphi \neq 0.$$

Thus,

$$\|y(t;\sigma,\varphi)\| \leq K_1 e^{-\beta(t-\sigma)}\|\varphi\|, \forall t \geq \sigma, \forall \varphi \in C. \tag{2.16}$$

For given $\varphi \in C$, let $x(.) := x(.;\sigma,\varphi)$ be the solution of (2.1)−(2.2) and let $y(.) := y(.;\sigma,|\varphi|)$, where $|\varphi|(t) := |\varphi(t)|, \forall t \in [\sigma-h,\sigma)$. Let $\zeta > 0$ be arbitrary but fixed.

Note that $|x(t)| \ll y(t) + \zeta p$,$t \in [\sigma-h,\sigma)$. Then the same argument as in step II of the proof of (i) yields

$$|x(t)| \leq y(t) + \zeta p, \forall t \geq \sigma.$$

By the monotonicity of vector norms,

$$\|\ x(t)\| = \|\ |x(t)|\| \leq \|y(t) + \zeta p\| \leq \|y(t)\| + \zeta\|p\|, \forall t \geq \sigma. \tag{2.17}$$

Letting ζ tend to zero in (2.17), we obtain

$$\|\ x(t)\| = \|\ |x(t)|\| \leq \|y(t)\|, \forall t \geq \sigma. \tag{2.18}$$

Then (2.16) and (2.18) imply that the zero solution of (2.1) is globally exponentially stable. Furthermore, the arguments presented above show that the zero solution of (2.1) is globally exponentially stable regardless of (2.3). This completes the proof.

Corollary 7 *Let $F(t;.,y,z)$ be continuously differentiable on $\mathbb{R}^n$ for any $t \in \mathbb{R}$ and any $y,z \in \mathbb{R}^n$. Assume that (3) holds and there exist $\delta > 0$ and $A_0 = \left(a_{ij}^{(0)}\right) \in \mathbb{R}^{n\times n}$ and matrix-valued* *continuous functions* $A_1(.), A_2(.) : \mathbb{R} \to \mathbb{R}_+^{n\times n}$ *such that (5) holds and*

$$\frac{\partial F_i}{\partial x_i}(t; x, y, z) \leq a_{ii}^{(0)}; \left|\frac{\partial F_i}{\partial x_j}(t; x, y, z)\right| \leq a_{ij}^{(0)}, \quad i \neq j, \qquad i, j = 1, ..., n \qquad (2.19)$$

for any $t \in \mathbb{R}$ *and any* $x, y, z \in B_\delta$ *.Suppose there exists* $B_0 \in \mathbb{R}_+^{n\times n}$ *such that*

$$A_1(t) + A_2(t)\int_{-h}^{0} |B(t,s)|\, ds \leq B_0, \forall t \in \mathbb{R}, \qquad (2.20)$$

and $A_0 + B_0$ is Hurwitz stable. Then

(i) The zero solution of (2.1) is exponentially stable.

(ii) In addition, if (2.5) holds for any $t \in \mathbb{R}$ and any $y, z \in \mathbb{R}^n$ and (2.19) holds for any $t \in \mathbb{R}$ and any $x, y, z \in \mathbb{R}^n$ and for any $t \in \mathbb{R}$, then the zero solution of (2.1) is globally exponentially stable, regardless of (2.3).

Proof. Let $A_0(t) = A_0, t \in \mathbb{R}$. It remains to show that there exist $\beta > 0$and $p \in \mathbb{R}^n, p \gg 0$ so that (2.6) holds.

Since $A_0 + B_0$ is a Hurwitz stable Metzler matrix, there exists $p \in \mathbb{R}^n$, $p \gg 0$ so that $(A_0 + B_0)\, p \ll 0$, by the precedent Theorem . By continuity, this gives

$$\left(A_0 + e^{h\beta} B_0\right) p \ll -\beta p, \qquad (2.21)$$

■

for some sufficiently small $\beta > 0$. Then (2.6) follows from (2.20) and (2.21). This completes the proof.

Corollary 8 *Let* $F(t; ., y, z)$ *be continuously differentiable on* $\mathbb{R}^n$ *for any* $t \in \mathbb{R}$ *and any* $y, z \in \mathbb{R}^n$ *and let* $B(t, s) = B_0(s), t \in \mathbb{R}, s \in [-h, 0]$. *Suppose there exist* $\delta > 0$ *and* $A_0, A_1, A_2 \in R^{n\times n}$ *,such that* (2.19) *holds and*

$$|F(t; 0, y, z)| \leq A_1|y| + A_2|z|, \forall t \in \mathbb{R}; \forall y, z \in B_\delta. \qquad (2.22)$$

If $A_0 + A_1 + A_2 \int_{-h}^{0} |B_0(s)|\, ds$ is Hurwitz stable then the zero solution of (2.1) *is ES. In addition, if*(2.19) *holds for any $t \in \mathbb{R}$ and any $x, y, z \in \mathbb{R}^n$ and*(2.22) *holds for any $t \in \mathbb{R}$ and any $y, z \in \mathbb{R}^n$ then the zero solution of* (2.1) *is globally exponentially stable.*

2.2 Systems of the second form

Let now consider a nonlinear differential system with time-varying delays of the form

$$\dot{x}(t) = A(t)\,x(t) + F(t; x(t), x(t-\tau_1(t)), ..., x(t-\tau_m(t)), \int B(s)x(t+s)ds), \quad (3.1)$$

where $t \geq \sigma \geq 0$ and

(i) $h_k(.)$, $h(.) : \mathbb{R}_+ \to \mathbb{R}_+$, $k \in m$, are given continuous functions such that $0 < h_k(t) \leq h_k$,$0 < h(t) \leq h, h \geq h_k \ \forall k \in m$ for some positive numbers h, h_k, $k \in \underline{m}$;

(ii) $A(.) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ *and* $B(.) : [-h, 0] \to \mathbb{R}^{n\times n}$ are given continuous functions.

(iii)$F(.;.,...,.) : \mathbb{R}_+ \times \overbrace{\mathbb{R}^n \times ... \times \mathbb{R}^n}^{(m+2)\ times} \to \mathbb{R}^n$,is a given continuous function such that

$F(t; 0, ..., 0) = 0, \ \forall t \geq 0$ *and* $F(t; u_1, u_2, ..., u_{m+2})$ is (locally) Lipschitz continuous with respect to $u_1, u_2, ..., u_{m+2}$ on each compact subset of $\mathbb{R}_+ \times \overbrace{\mathbb{R}^n \times ... \times \mathbb{R}^n}^{(m+2)\ times}$.

Let the system (3.1) satisfying (i), (ii) ,(iii) Then that for a fixed $\sigma \geq 0$ and a given $\varphi \in C$, there exists aunique local solution of (3.1) satisfying the initial condition

$$x(\sigma + s) = \varphi(s), \ s \in [-h, 0]. \quad (3.2)$$

This solution is defined and continuous on $[\sigma - h, \gamma)$ for some $\gamma > \sigma$and satisfies (3.1) for every $t \in [\sigma, \gamma)$. It is denoted by $x(.;\sigma, \varphi)$. Furthermore, if the interval $[\sigma - h, \gamma)$ is the maximum interval of existence of the solution $x(.;\sigma, \varphi)$ then $x(.;\sigma, \varphi)$ is said to be noncontinuable.

The existence of a noncontinuable solution follows from Zorn's lemma and the maximum interval of existence must be open.

2.2.1 Criteria for exponential stability.

Let $A(t) := (a_{ij}(t))$, $t \geq 0$. Suppose there exist $A_0 := \left(a_{ij}^{(0)}\right) \in \mathbb{R}^{n\times n}$and $A_k := \left(a_{ij}^{(k)}\right) \in \mathbb{R}_+^{n\times n}, k \in \underline{m+2}$ so that

$$a_{ii}(t) \leq a_{ii}(0), \forall t \geq 0, i \in n; \qquad |a_{ij}(t)| \leq a_{ij}^0, \quad \forall t \geq 0, \forall i \neq j, i, j \in \underline{n} \tag{3.3}$$

and

$$|F(t; u_1, u_2, ..., u_{m+2})| \leq \sum A_k |u_k|, \quad \forall t \geq 0, \forall u_1, u_2, ..., u_{m+2} \in \mathbb{R}^n. \tag{3.4}$$

If

$$M := A_0 + \sum A_k + \int A_{m+2} |B(s)| \, ds$$

satisfies one of the equivalent conditions$(i)-$ $(v)$of Theorem of properties of Metzler matrices , Then (3.1)is locally exponentially stable.

In addition, if the function$F$ is positive homogeneous of degree one with respect to $u_1, u_2, ..., u_{m+2}$, that is,$F(t; \alpha u_1, ..., \alpha u_{m+2}) = \alpha F(t; u_1, , ..., u_{m+2})$, for any $\alpha \geq 0, t \geq 0,$

$u_1, u_2, ..., u_{m+2} \in \mathbb{R}^n$, Then(3.1) is globally exponentially stable.

Proof. Since M is a Metzler matrix, any two of $(i) - (v)$ of Theoremof properties of Metzler matrices are equivalent. We first show that (3.1) is locally exponentially stable provided(iv) of this Theorem ■

holds.

Let $\varphi \in \zeta$ be given and let $x(t) := x(t; \sigma, \varphi), t \in [\sigma - h, \varphi)$ be a noncontinuablesolution of $(3.1) - (3.2)$. We divide the proof into two steps.

Step I.

There exists $\beta > 0$ such that for any $\sigma \geq 0$ and any $r > 0$ and any $\varphi \in \zeta_r$,we have

$$\|x(t; \sigma, \varphi)\| \leq K e^{-\beta(t-\sigma)}, \forall t \in [\sigma, \gamma), \tag{3.7}$$

where K depends on β, r.

By (iv) of Theorem of properties of Metzler matrices, there exists $p \in \mathbb{R}^n_+$ such that

$$\left(A_0 + \sum_{k=1}^{m+1} A_k + \int_{-h}^{0} A_{m+2} \left| B(s) \right| ds \right) p \ll 0. \quad (3.8)$$

By continuity, (3.8) also holds for some $p := (\alpha_1, \alpha_2, ..., \alpha_n) T, \alpha_i > 0, \forall i \in n$.Furthermore, (3.8) implies that

$$\left(A_0 + A_1 + \sum_{k=1}^{m+1} e^{\beta h} A_k + \int_{-h}^{0} e^{-\beta s} A_{m+2} \left| B(s) \right| ds \right) p \ll -\beta \left(\alpha_1, \alpha_2, ..., \alpha_n \right)^T, \quad (3.9)$$

for some sufficiently small $\beta > 0$. Fix $r > 0$ and choose $K > 0$ such that$|\varphi(t)| \ll K e^{-\beta t p}$ for any $t \in [-h, 0]$ and for any $\varphi \in \zeta_r$.

Define $u(t) := K e^{-\beta(t-\sigma)p}, t \in [\sigma - h, \infty)$.Set $x(t) := x(t; \sigma, \varphi), t \in [\sigma - h, \gamma)$.Then, we have $|x(t)| \ll u(t), \forall t \in [\sigma - h, \sigma]$.

We claim that $|x(t)| \leq u(t)$ for any $t \in [\sigma, \gamma)$.

Assume on the contrary that there exists $t_0 > \sigma$ such that $|x(t_0)| \nleq u(t_0)$.

Set $t_1 := \inf \{ t \in (\sigma, \gamma) : |x(t_0)| \nleq u(t_0) \}$. By continuity, $t_1 > \sigma$ and there is $i_0 \in n$ such that $|x(t)| \leq u(t), \forall t \in [\sigma, t_1)$;

$$|x_{i_0}(t_1)| = u_{i_0}(t_1), |x_{i_0}(t)| > u_{i_0}(t) \, \forall t \in (t_1, t_1 + \varepsilon), \quad (3.10)$$

for some $\varepsilon \succ 0$.For $i \in \underline{n}$,we have

$$\left| \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} \right| = sign \left(\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{pmatrix} \right) \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix}$$

$$\leq \begin{pmatrix} a_{11}\left|x_1(t)\right| + a_{12}\left|x_2(t)\right| + a_{13}\left|x_3(t)\right| + \cdots + a_{1n}\left|x_n(t)\right| \\ a_{21}\left|x_1(t)\right| + a_{22}\left|x_2(t)\right| + a_{23}\left|x_3(t)\right| + \cdots + a_{2n}\left|x_n(t)\right| \\ a_{31}\left|x_1(t)\right| + a_{32}\left|x_2(t)\right| + a_{33}\left|x_3(t)\right| + \cdots + a_{3n}\left|x_n(t)\right| \\ \vdots \\ a_{n1}\left|x_n(t)\right| + a_{n2}\left|x_n(t)\right| + a_{nn}\left|x_n(t)\right| + \cdots + a_{nn}\left|x_n(t)\right| \end{pmatrix} +$$

$$\begin{pmatrix} \left|F_1(t; x_1(t), x_1(t-\tau_1(t)), ..., x_1(t-\tau_m(t)), \int B(s)x_1(t+s)ds),\right| \\ \left|F_2(t; x_2(t), x_2(t-\tau_1(t)), ..., x_2(t-\tau_m(t)), \int B(s)x_2(t+s)ds),\right| \\ \left|F_3(t; x_3(t), x_3(t-\tau_1(t)), ..., x_3(t-\tau_m(t)), \int B(s)x_3(t+s)ds),\right| \\ \vdots \\ \left|F_n(t; x_n(t), x_n(t-\tau_1(t)), ..., x_n(t-\tau_m(t)), \int B(s)x_n(t+s)ds),\right| \end{pmatrix}$$

$$\preceq$$

$$\begin{pmatrix} a_{11}\left|x_1(t)\right| & 0 & 0 & \cdots & 0 \\ 0 & a_{22}\left|x_2(t)\right| & 0 & \cdots & 0 \\ 0 & 0 & a_{33}\left|x_3(t)\right| & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}\left|x_n(t)\right| \end{pmatrix}$$

$$+$$

$$\begin{pmatrix} a_{12}\left|x_2(t)\right| + a_{13}\left|x_3(t)\right| + \cdots + a_{1n}\left|x_n(t)\right| \\ a_{21}\left|x_1(t)\right| + a_{23}\left|x_3(t)\right| + \cdots + a_{2n}\left|x_n(t)\right| \\ a_{31}\left|x_1(t)\right| + a_{32}\left|x_2(t)\right| + \cdots + a_{3n}\left|x_n(t)\right| \\ \vdots \\ a_{n1}\left|x_1(t)\right| + a_{n2}\left|x_2(t)\right| + a_{n3}\left|x_3(t)\right| + \cdots \end{pmatrix} +$$

$$\begin{pmatrix} \left|F_1(t; x_1(t), x_1(t-\tau_1(t)), ..., x_1(t-\tau_m(t)), \int B(s)x_1(t+s)ds),\right| \\ \left|F_2(t; x_2(t), x_2(t-\tau_1(t)), ..., x_2(t-\tau_m(t)), \int B(s)x_2(t+s)ds),\right| \\ \left|F_3(t; x_3(t), x_3(t-\tau_1(t)), ..., x_3(t-\tau_m(t)), \int B(s)x_3(t+s)ds),\right| \\ \vdots \\ \left|F_n(t; x_n(t), x_n(t-\tau_1(t)), ..., x_n(t-\tau_m(t)), \int B(s)x_n(t+s)ds),\right| \end{pmatrix}$$

it follow

$$\frac{d}{dt}\left|x_i\left(t\right)\right| = sgn\left(x_i\left(t\right)\right)\dot{x_i}\left(t\right) \leq a_{ii}\left(t\right)\left|x_i\left(t\right)\right|$$

$$+\sum_{j=1,j\neq i}^{n}\left|a_{ij}\left(t\right)\right|\left|x_j\left(t\right)\right| + \left|F_i(t;x\left(t\right),x(t-h_1\left(t\right)),...,x\left(t-h_m\left(t\right)\right),\int B\left(s\right)x\left(t+s\right)ds)\right|$$

.

for almost any $t\in\left[\sigma,\gamma\right)$.Then (3.3) implies

$$\begin{aligned}\frac{d}{dt}\left|x_i\left(t\right)\right| &\leq a_{ii}^{(0)}\left(t\right)\left|x_i\left(t\right)\right|\\ &+\sum_{j=1,j\neq i}^{n}a_{ij}^{(0)}\left|x_j\left(t\right)\right| + \left|F_i(t;x\left(t\right),x(t-h_1\left(t\right)),...,x\left(t-h_m\left(t\right)\right),\int B\left(s\right)x\left(t+s\right)ds)\right|,\end{aligned}$$

for almost any $t\in[\sigma,\gamma)$. It follows that for any $t\in[\sigma,\gamma)$

$$D^+\left|x_i\left(t\right)\right| := \lim\sup_{h\to 0^+}\frac{|x_i(t+h)|-|x_i\left(t\right)|}{h} = \lim\sup_{h\to 0^+}\frac{1}{h}\int_t^{t+h}\frac{d}{ds}\left|x_i\left(s\right)\right|ds \leq a_{ii}^{(0)}\left|x_i\left(t\right)\right|$$

$$+\sum_{j=1,j\neq i}^{n}a_{ij}^{(0)}\left|x_j\left(t\right)\right| + \left|F_i(t;x\left(t\right),x(t-h_1\left(t\right)),...,x\left(t-h_m\left(t\right)\right),\int B\left(s\right)x\left(t+s\right)ds)\right|,$$

where D^+ denotes the Dini upper-right derivative. Let $A_{m+2}\left|B\left(s\right)\right| := \left(c_{ij}\left(s\right)\right), s\in\left[-h,0\right].$ Taking(3.4)into account, we have for any $t\in[\sigma,\gamma)$,

$$\begin{aligned} D^+ |x_i(t)| \;&\leq\; a_{ii}^{(0)} |x_i(t)| + \sum_{j=1, j\neq i}^{n} a_{ij}^{(0)} |x_j(t)| + \sum_{j=1}^{n} a_{ij}^{(1)} |x_j(t)| + \sum_{k=2}^{m+1} \sum_{j=1}^{n} a_{ij}^{(k)} |x_j(t - h_k(t))| \\ &+ \sum_{j=1}^{n} \int_{-h(t)}^{0} c_{ij}(s) |x_j(t+s)| \, ds. \end{aligned}$$

In particular, it follows from (3.9) and (3.10)that

$$D^+ |x_{i_0}(t_1)| \leq a_{i_0 i_0}^{(0)} K e^{-\beta(t_1-\sigma)} \alpha_{i_0} + \sum_{j=1, j\neq i}^{n} a_{i_0 j}^{(0)} K e^{-\beta(t_1-\sigma)} \alpha_j +$$

$$\sum_{j=1}^{n} a_{i_0 j}^{(1)} K e^{-\beta(t_1-\sigma)} \alpha_j + \sum_{k=2}^{m+1} \sum_{j=1}^{n} a_{i_0 j}^{(k)} K e^{-\beta(t_1-\sigma)} e^{\beta h} \alpha_j + \sum_{j=1}^{n} \int_{-h(t)}^{0} c_{i_0 j} e^{-\beta s} \alpha_j ds$$

$$= K e^{-\beta(t_1-\sigma)} \left(\sum_{j=1}^{n} a_{i_0 j}^{(1)} \alpha_j + \sum_{j=1}^{n} a_{i_0 j}^{(1)} \alpha_j + \sum_{k=2}^{m+1} \sum_{j=1}^{n} a_{i_0 j}^{(k)} e^{\beta h} \alpha_j + \sum_{j=1}^{n} \int_{-h}^{0} c_{i_0 j} e^{-\beta s} \alpha_j ds \right)$$

$$\leq -\beta K e^{-\beta(t_1-\sigma)} \alpha_{i_0} = D^+ u_{i_0}(t_1)$$

However, this conflicts with(3.10). Therefore

$$|x(t; \sigma, \varphi)| \leq u(t) = K e^{-\beta(t-\sigma)} p, \forall \sigma \geq 0; \forall \varphi \in \zeta_r; \forall t \in [\sigma, \gamma).$$

By the monotonicity of vector norms, this yields

$$\|x(t; \sigma, \varphi)\| \leq K_1 e^{-\beta(t-\sigma)}, \forall \sigma \geq 0; \forall \varphi \in \zeta_r; \forall t \in [\sigma, \gamma), for some K_1 > 0.$$

Step II.

We claim that $\gamma = \infty$ and so(3.1) is locally exponentially stable.

Seeking a contradiction, we assume that $\gamma < \infty$. Then it follows from (3.7) that $x(.; \sigma, \varphi)$ is bounded on $[\sigma, \gamma)$.

Furthermore, this together with(3.1) and(3.4) imply that $\dot{x}(.)$is bounded on$[\sigma,\gamma)$. Thus $x(.)$ is uniformly continuous on$[\sigma,\gamma)$. Therefore,$\lim_{t\to\gamma}-x(t)$exists and $x(.)$ can be extended to a continuous function on $[\sigma,\gamma)$. Moreover, the closure of $\{x_t : t\in[\sigma,\gamma)\}$ is a compact set in C,by Arzéla–Ascoli theorem.

Note that $\{(t,x_t) : t\in[\sigma,\gamma)\}\subset[\sigma,\gamma)\times$the closure of$\{x_t : t\in[\sigma,\gamma)\}$.

Thus, the closure of $\{(t,x_t) : t\in[\sigma,\gamma)\}$ is a compact set in $\mathbb{R}_+\times C$. Since (γ,x_γ) belongs to this compact set, one can find a solution of(3.1) through this point to the right of γ .

This contradicts thenoncontinuability hypothesis on $x(.)$. Thus γ must be equal to ∞.

Finally, we show that (3.1) is globally exponentially stable provided F is positive homogeneous of degree one with respect to $u_1,u_2,...,u_{m+2}$. Let $\varphi\in C$ be given.

Since F is positive homogeneous of degree one with respect to $u_1,u_2,...,u_{m+2}$,it follows that $\frac{1}{\|\varphi\|}x(.;\sigma,\varphi)$ is the solution of(3.1)satisfying the initial condition

$$x(t+\sigma)=\frac{1}{\|\varphi\|}\varphi(t),t\in[-h,0]$$

Since $\frac{1}{\|\varphi\|}\varphi\in\zeta_1$, we have

$$\left\|\frac{1}{\|\varphi\|}x(t;\sigma,\varphi)\right\|\geq Ke^{-\beta(t-\sigma)},\forall t\geq\sigma,$$

or equivalently, $\|x(t;\sigma,\varphi)\|\leq Ke^{-\beta(t-\sigma)}\|\varphi\|,\forall t\geq\sigma$. Here K,β are independent with φ and thus (3.1) is globally exponentially stable. This completes the proof.

Remark 1 *It is important to note that if $F(t;u_1,u_2,...,u_{m+2})$ is (globally) Lipschitz continuous with respect to $u_1,u_2,...,u_{m+2}$ on $\mathbb{R}_+\times\overbrace{\mathbb{R}^n\times...\times\mathbb{R}^n}^{(m+2)\ times}$ and*

$F(t;0,0,...,0)$,$\forall t\geq 0$, then (3.4) holds automatically for some $A_k\in\mathbb{R}_+^{n\times n},k\in\underline{m+2}$

In particular, when $A(.)$ is a constant matrix-valued function, we get the following.

Theorem 9 *Let $A(t)\equiv A:=(a_{ij})$, $\forall t\geq 0$. Suppose there exist $A_k:=\left(a_{ij}^{(k)}\right)\in\mathbb{R}_+^{n\times n},k\in m+2$*

so that (4) holds. If

$$M := diag\,(a_{11}, a_{22}, ..., a_{nn}) + |A - diag\,(a_{11}, a_{22}, ..., a_{nn})| + \sum_{k=1}^{m+1} A_k + \int_{-h}^{0} A_{m+2}\,|B\,(s)|\,ds$$

satisfies one of the equivalent conditions (i)–(v) of of properties of Metzler matrices then(3.1) is locally exponentially stable. In addition, if the function F is positive homogeneous of degree one with respect to $u_1, u_2, ..., u_{m+2}$, then(3.1)is globally exponentially stable.

We now consider a differential system with delays of the form

$$\dot{x}(t) = A\,(t)\,x\,(t) + \sum_{k=1}^{m} A_k\,(t)\,x\,(t - h_k\,(t)) + G\left(t; \int_{-h(t)}^{0} B\,(s)\,x\,(t+s)\,ds\right), \tag{4.1}$$

where

(i) $h_k\,(.)\,, h\,(.) : \mathbb{R}_+ \to \mathbb{R}_+, k \in \underline{m}$ are given continuous functions such that $0 < h_k\,(t) \leq h_k, 0 < h\,(t) \leq h, h \geq h_k, \forall k \in \underline{m}$,for some positive numbers $h, h_k, k \in \underline{m}$;

(ii) $A\,(.)\,, A_k\,(.) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}, k \in \underline{m}$ and $B\,(.) : [-h, 0] \to \mathbb{R}^{n\times n}$ are given continuous functions.

(iii) $G\,(.;.) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$,is a given continuous function satisfying $G\,(t; 0) = 0, \forall t \geq 0$ and is Lipschitz continuous with respect to the second argument on each compact subset of $\mathbb{R}_+ \times \mathbb{R}^n$.

Theorem 10 *Let* $A\,(t) := (a_{ij}\,(t))$, $t \geq 0$. *Suppose there exist* $A_0 := \left(a_{ij}^{(0)}\right) \in \mathbb{R}^{n\times n}$ *and* $A_k \in \mathbb{R}_+^{n\times n}, k \in m+1$ *so that*

$$a_{ii}\,(t) \leq a_{ii}^{(0)}, \forall t \geq 0; \quad |a_{ij}\,(t)| \leq a_{ij}^{(0)}, \quad \forall t \geq 0, \forall i \neq j, \quad i, j \in \underline{n}$$

,

and

$$A_k\left(t\right) \le A_k, \forall t \ge 0, k \in m; \quad \left|G\left(t;u\right)\right| \le A_{m+1}\left|u\right|, \ \forall t \ge 0, \quad \forall u \in \mathbb{R}^n$$

.

If

$$M := A_0 + \sum_{k=1}^{m} A_k + \int_{-h}^{0} A_{m+1}\left|B\left(s\right)\right| ds$$

satisfies one of the equivalent conditions $(i) - (v)$ ofof properties of Metzler matrices . Then (4.1) is locally exponentially stable.

In addition, if the function G is positive homogeneous of degree one with respect to the second argument then (4.1)is globally exponentially stable.

Remark 2 *The assumption* $\left|A_k\left(t\right)\right| \le A_k, \ \forall t \ge 0, k \in m$ *can be relaxed by* $\sum_{k=1}^{m}\left|A_k\left(t\right)\right| \le M_0, \forall t \ge 0,$ *for some* $M_0 \in \mathbb{R}_+^{n\times n}$ *. In this case, the matrix* M *is defined by*

$$M := A_0 + M_0 + \int_{-h}^{0} A_{m+1}\left|B\left(s\right)\right| ds.$$

Chapter 3

Application

Example 4 *Consider the differential equation with time-varying delay*

$$\dot{x}(t) = (-2+\sin t)\, x(t) + \sqrt{ae^{-t}x(t)^2 + b\left(\cos\sqrt{t}\right)^2 x(t-h(t))^2 + c\left(\int\limits_{-h(t)}^{0} e^s x(t+s)\, ds\right)^2} \tag{5.1}$$

where $a, b, c \geq 0$ are parameters and $h(.) : \mathbb{R}_+ \to \mathbb{R}_+$ is a given continuous function satisfying $0 < h(t) \leq h, \forall t \geq 0$ for some $h > 0$.

we have (5.1) is of the form (2.1) with

$a(t) := -2+\sin t, t \geq 0;$ and$F(t; u_1, u_2, u_3) := \sqrt{ae^{-t}u_1^2 + b\left(\cos\sqrt{t}\right)^2 u_2^2 + cu_3^2}, t \geq 0, u_1, u_2, u_3 \in \mathbb{R}$.

Furthermore, $a(t) := -2\sin t \leq -1, \forall t \geq 0$ and F is positive homogeneous of degree one with respect to u_1, u_2, u_3 and satisfies

$$F(t; u_1, u_2, u_3) := \sqrt{ae^{-t}u_1^2 + b\left(\cos\sqrt{t}\right)^2 u_1^2 + cu_2^2} \leq \sqrt{a}\,|u_1| + \sqrt{b}\,|u_2| + \sqrt{c}\,|u_3|$$

and

$$|F(t; u_1, u_2, u_3) - F(t; v_1, v_2, v_3)| \leq \sqrt{a}\,|u_1 - v_1| + \sqrt{b}\,|u_2 - v_2| + \sqrt{C}\,|u_3 - v_3|,$$

for any $t \geq 0$, and any $u_i, v_i \in \mathbb{R}, i \in \{1, 2, 3\}$. Therefore all hypotheses of Theorem 2.2 hold.

Thus, (5.1) is globally exponentially stable if

$$-1+\sqrt{a}+\sqrt{b}+\sqrt{C}\int_{-h}^{0} e^{s}ds<0,$$

or equivalently

$$-1+\sqrt{a}+\sqrt{b}+\sqrt{C}\left(1-e^{-h}\right)<0.$$

Example 5 *Consider the following nonlinear differential equation with constant delay*

$$\dot{x}=(-2+sint)x^{\gamma}(t)+a^{2}sin^{2}tx^{2\gamma}(t)+b^{2}cos^{2}t\frac{x^{2\gamma}(t-\tau)}{1+x^{2}(t)}+c\int_{-\tau}^{0} e^{s}x^{\tau}(t+s)ds, \tag{6.1}$$

where $t\geq 0, a, b, c\geq 0, \tau>0$ and $0<\gamma\leq 1$ are constants.

- When $\gamma=1$,

$$|F(t,u_0,u1,u2)|=\left|\sqrt{a^{2}sin^{2}tu_0^{2}+b^{2}cos^{2}t\frac{u_1^{2}}{1+u_0^{2}}}+cu_2\right|\leq a|u_0|+b|u_1|+c|u_2|.$$

we have Equation. (6.1) is globally exponentially stable if

$$-1+a+b+c(1-e^{-\tau})<0. \quad (6.2)$$

- When $0<\gamma<1$,

$$|G(t,u_0,u_1,u_2)|=\sqrt{a^{2}sin^{2}tu_0^{2\gamma}+b^{2}cos^{2}t\frac{u_1^{2\gamma}}{1+u_0^{2}}}+cu_2\leq a|u_0|^{\gamma}+b|u_1|^{\gamma}+c|u_2|.$$

we have that all solutions of Equation. (6.1) exponentially converge to zero if (6.2) holds.

Example 6 *Consider the nonlinear delay differential equation*

$$\dot{x}(t) = -\sin x(t) + \alpha \sin\left(x\left(t - \frac{\pi}{2}\right)\right), \tag{7.1}$$

where $\alpha \in \mathbb{R}$ is a parameter.Clearly, (7.1) is of the form

$$\dot{x}(t) = f\left(x(t), x(t - \frac{\pi}{2})\right),$$

where

$$f(x, y) := -\sin x + \alpha \sin y, \qquad x, y \in \mathbb{R}.$$

Fix $n \in \mathbb{N}, n \geq 3$.Then we have

$$\frac{\partial f}{\partial x}(x, y) = -\cos x \leq -\cos\left(\frac{\pi}{n}\right) < 0, x \in \left[\frac{-\pi}{n}, \frac{\pi}{n}\right],$$

and

$$|\ f(0, y)| \leq |\alpha|\,|y|, \qquad y \in \mathbb{R}.$$

By the Theorem , the zero solution of (39) is ES, if there exists $\beta > 0$ such that

$$-\cos\left(\frac{\pi}{n}\right) + |a|\, e^{\frac{\beta\pi}{2}} < -\beta.$$

By continuity, this holds if

$$-\cos\left(\frac{\pi}{n}\right) + |a| < 0.$$

Thus, the zero solution of (7.1) is Exponetially stale, if $|a| < \cos\left(\frac{\pi}{n}\right)$ .Since $n \geq 3$, is arbitrary, it follows that

the zero solution of (7.1) is Exponetially stale, if $|a| < 1$.

Conclusion

Condition of global exponential stability of nonlinear differential systems with time varying delay is extended to a more general nonlinear differential systems with time varying delay.

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