Abbes Laghrour university 24/05/2025 1h30 2nd year mathematics
Faculty of sciences and technology algebra 4 exam

Exercise 1 (6pts)

Let f_1^*, f_2^* be two linear forms on $V = \mathbb{R}^2$, defined by:

$$f_1^*(x, y) = x - y,\ f_2^*(x, y) = 3x + 2y$$

1. Show that (f_1^*, f_2^*) forms a basis for V^* (the dual space of V).
2. Express $h(x, y) = -7x - 8y$ in terms of the basis (f_1^*, f_2^*).
3. Let (f_1, f_2) be the **Predual** (antedual) basis of (f_1^*, f_2^*). Deduce (without calculations) the values of:

$$h(f_1), h(f_2).$$

Exercise 2 (4pts)

Determine the real value a for which the following quadratic form defines an inner (scalar) product:

$$q(X) = x_1^2 + (1+a)x_2^2 + (1+a+a^2)x_3^2 + 2x_1x_2 - 2ax_2x_3.$$

Exercise 3 (10pts)

Consider the quadratic form defined on $E = \mathbb{R}^3$ by:

$$q(X) = 2x_1^2 + 2x_1x_2 + x_2^2 - 2x_2x_3 + 2x_3^2.$$

1. Determine the polar form (the bilinear form associated with q) and the associated matrix A of q.
2. Using Gaussian elimination, diagonalize and deduce its rank and signature.
3. Find an orthogonal basis of $E = \mathbb{R}^3$ with respect to q.
4. Determine an invertible matrix P and a diagonal matrix D such that $D = {}^tPAP$.

exercise 2 =

* Express q in orthogonal form =

Reduction of x_1 =

$$x_1^2 + \cancel{(1+a+a^2)}\, 2x_1x_2 = \underbrace{(x_1 + x_2)}_{\ell_1}^2 - x_2^2$$

then $q(x) = \ell_1^2 - x_2^2 + (1+a)x_2^2 + (1+a+a^2)x_3^2$
$- 2ax_2x_3$

$= \ell_1^2 + ax_2^2 + (1+a+a^2)x_3^2 - 2ax_2x_3$

Reduction of x_2:

$$ax_2^2 - 2ax_2x_3 = a(x_2^2 - 2x_2x_3)$$

$$= a\left[(x_2 - x_3)^2 - x_3^2\right] = a\underbrace{(x_2 - x_3)}_{\ell_2}^2 - ax_3^2$$

then

$q(x) = \ell_1^2 + a\ell_2^2 - ax_3^2 + (1+a+a^2)x_3^2$

$= \ell_1^2 + a\ell_2^2 + (1+a^2)x_3^2$

* A scalar product is a symetric, positive definite bilinear form. Therefore, a must be strictly greater than zero $(a > 0)$

Exo 3 =

(1) $\frac{1}{2}\frac{\partial q}{\partial x_1} = \frac{1}{2}\left[4x_1 + 2x_2\right] = 2x_1 + x_2$

$\frac{1}{2}\frac{\partial q}{\partial x_2} = \frac{1}{2}\left[2x_1 + 2x_2 - 2x_3\right] = x_1 + x_2 - x_3$

$\frac{1}{2}\frac{\partial q}{\partial x_3} = \frac{1}{2}\left[-2x_2 + 4x_3\right] = -x_2 + 2x_3$

Then $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

$f_q(X,Y) = {}^tX\,A\,Y$.

(2) Reductions of x_1!

$2x_1^2 + 2x_1x_2 = 2(x_1^2 + x_1x_2) = 2\left[\left(x_1 + \frac{x_2}{2}\right)^2 - \frac{x_2^2}{4}\right]$

$= 2\underbrace{\left(x_1 + \frac{x_2}{2}\right)}_{l_1^*}{}^2 - \frac{x_2^2}{4}$

Then $q(x) = 2l_1^{*2} - \frac{x_2^2}{2} + x_2^2 - 2x_2x_3 + 2x_3^2$

$= 2l_1^{*2} + \frac{1}{2}x_2^2 - 2x_2x_3 + 2x_3^2$

Reductions of x_2 =

$\frac{1}{2}x_2^2 - 2x_2x_3 = \frac{1}{2}\left[x_2^2 - 4x_2x_3\right]$

$= \frac{1}{2}\left[(x_2 - 2x_3)^2 - 4x_3^2\right]$

$= \frac{1}{2}\underbrace{(x_2 - 2x_3)}_{l_2^*}{}^2 - 2x_3^2$

$q(x) = 2l_1^{*2} + \frac{1}{2}l_2^{*2} \cancel{- 2x_3^2 + 2x_3^2}$

$= 2l_1^{*2} + \frac{1}{2}l_2^{*2}$

$\mathrm{Rank}(q) = 2$

$\mathrm{Sign}(q) = (2,0)$

(3) $\quad \mathcal{B} = P_{(e_i^*) \to \ell_i^*} =$

$$P = P_{(e_i) \to \ell_i} = {}^t \mathcal{B}^{-1}$$

$$\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \quad \ell_3^*(x) = x_3 .$$

Then $P = \begin{pmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

finally $\ell_1 = (1,0,0)$, $\ell_2 = (-\frac{1}{2},1,0)$, $\ell_3 = (1,-2,1)$

(4) $\quad P = \begin{pmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{Rank}(q) = 2$

$\mathrm{Sign}(q) = (2,0)$

(3) $\quad Q = P_{(e_i^*) \to l_i^*} =$

$P = P_{(e_i) \to l_i} = {}^t Q^{-1}$

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \quad l_3^*(x) = x_3 .$$

Then $P = \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & 1 & +2 \\ 0 & 0 & 1 \end{pmatrix}$

finally $l_1 = (1,0,0)$, $l_2 = (-\frac{1}{2}, 1, 0)$, $l_3 = (-1, +2, 1)$

(4) $\quad P = \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & 1 & +2 \\ 0 & 0 & 1 \end{pmatrix}$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$