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بسم الله الرحمن الرحيم

Acknowledgment

It is often said that the journey is as important as the destination. These five years of training allowed us to fully understand the meaning of this simple phrase. This journey, in fact, has not been achieved without challenges and without raising many questions. for which the answers require long hours of work. First of all, we thank Allah who gave us the courage and the will to accomplish this modest work.

Our warmest thanks also go to the supervisor in this work **Ms I.Nasrallah** for her continuous help, her valuable advice and her patience with me. I also thank the members of the jury **Dr A.Bragdi** and **Ms A.Achichi** for the honor they have done by agreeing to sit on our defense and review my work.

Dedication

First of all, I would like to thank "**Allah**" for guiding me during all these years and making my dreams come true.

I dedicate this thesis

To my dear parents my mother and my father.

For their patience, love, support and encouragement.

To my friends and comrades.

Without forgetting all the teachers whatsoever

primary, middle, secondary or higher education.

Souakria.A

Contents

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NOTATION

- $\mathbb{R}, \mathbb{C}$: Set of real numbers (resp. complex).

- $\mathbb{N}$: Set of natural numbers.

- $Re(.)$: Real part.

- $C([a,b])$: Space of continuous functions on $[a,b]$.

- $C^n([a,b])$: Space of n times continuously differentiable functions.

- $I^{\alpha}_{a^+}$: Intégrale fractionnaire of Riemann-Liouville the right side ordre α.

- $I^{\alpha}_{b^-}$: Riemann Liouville fractional integral on the left side ordre α.

- ${}^{RL}D^{\alpha}_{a}$: Riemann Liouville fractional derivative ordre α.

- ${}^{C}D^{\alpha}_{a}$: Caputo fractional derivative ordre α.

- $\Gamma(.)$: The Gamma function.

- $\beta(.)$: The Beta function.

- $E_{\alpha,\beta}(.)$: The Mittag-Leffler function of two parameter.

- $E_\alpha(.)$: The Mittag-Leffler function of one parameter.

- $\mathcal{L}$: Laplace transform.

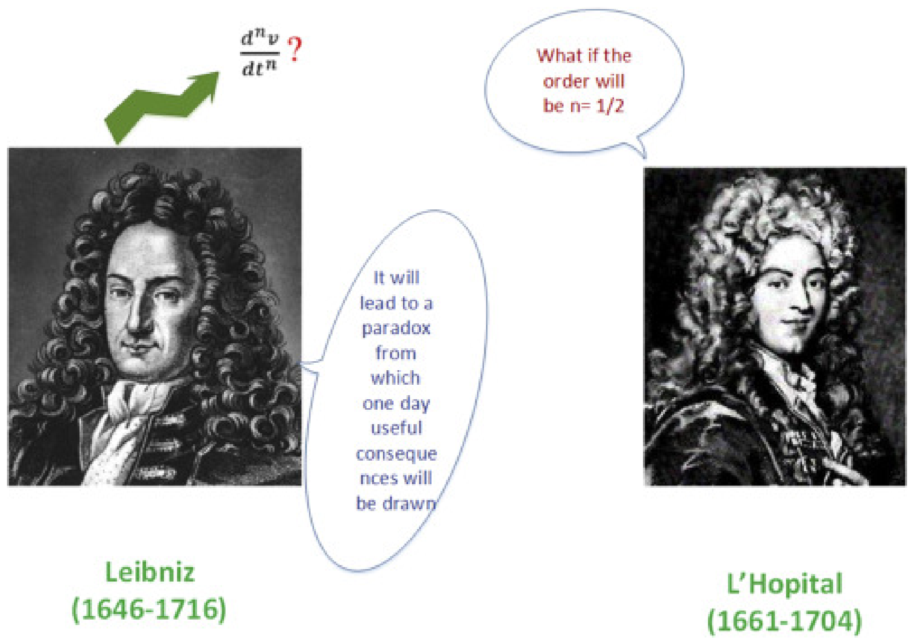
- $\mathcal{L}^{-1}$: Inverse Laplace transform.

- $[\alpha]$: Integer part of α.

- $L^P([a,b])$: The space of function p integrable on]a,b[.

Introduction

Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals of non-integer orders (called fractional derivatives and integrals, briefly differintegrals). In particular, this discipline involves the notion and methods of solving of differential equations involving fractional derivatives of the unknown function (called fractional differential equations). The history of fractional calculus started almost at the same time when classical calculus was established. It was first mentioned in Leibnizs letter to lHospital in 1695, where the idea of semiderivative was suggested. During time fractional calculus was built on formal foundations by many famous mathematicians, e.g. Liouville, GrĹunwald, Riemann, Euler, Lagrange, Heaviside, Fourier, Abel etc. A lot of them proposed original approaches, which can be found chronologically in [11][10].



The fact, that the differintegral is an operator which includes both integer-order derivatives and integrals as special cases, is the reason why in present fractional calculus becomes very popular and many applications arise. The fractional integral may be used e.g. for better describing the cumulation of some quantity, when the order of integration is unknown, it can be determined as a parameter of a regression model as Podlubnт́y presents in [1]. Analogously the fractional derivative is sometimes used for describing damping.

Other applications occur in the following fields: fluid flow, viscoelasticity, control theory of dynamical systems, diffusive transport akin to diffusion, electrical networks, probability and statistics, dynamical processes in self-similar and porous structures, electrochemistry of corrosion, optics and signal processing, rheology etc.

The main subject of this thesis is differential equation of fractional order and its applications this thesis is organized in *three chapter*:

Chapter 1 This chapter presents some preliminary the most important special functions involved in Fractional Calculus. A special attention is devoted to the Gamma function, used in Fractional Calculus calculations. Other special functions such as the Euler, Beta, and Mittag-Leffler functions are also introduced and some definition and properties of Laplace transform.

Chapter 2 This chapter introduces the fractional integral and Fractional derivative, in the sense of Riemann-Liouville and Caputo. The properties of these fractional operators are discussed and some examples.

Chapter 3 This chapter is devoted to the use of the Laplace Transform in Franctional Calculs, because the Riemann-Liouville Fractional integral and Fractional derivative allow the derivation of closed-form solutions with the aid of the Laplace Transform method also to solve Fractional Calculus equation.

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Preliminary

At the beginning of this chapter we remind two facts from elementary mathematical analysis, e.g. the change of order of integration in two dimensions and the derivative of integrals depending on a parameter. Let us point out that we will use the Lebesgue integral in whole thesis. Then we will introduce some important functions which are used in connection with fractional calculus such as the Gamma function which plays the role of the generalized factorial, the Beta function and the Mittag-Leffler, more information about these functions can be found in [1],[2],[10] or[13].

In the last part we are going to present some basic facts about Laplace transform and its properties. More details can be found, in[2].

1.1 Some requisites from ordinary calculus

In this section we recall two procedures which are very useful and important to keep in mind during reading. In particular, the second one is, in some sense, fundamental for fractional calculus as we will see later.

Change of order of integration

Change of order of integration is a trick which we will use e.g. during the calculation of the fractional integral of the power function. We point out that this process does not impose any new condition for the integrated function, it is only a different view at the area we integrate over.

There are known more general versions (e.g. Fubini theorem), but for us the case of triangular areas is sufficient. The following formula (1.1) holds for all functions $f(t, \tau, \xi)$ integrable with regard to τ and ξ

$$\int_a^t \int_a^\tau f(t,\tau,\xi)d\xi d\tau = \int_a^t \int_\xi^t f(t,\tau,\xi)d\tau d\xi. \tag{1.1}$$

The geometrical idea of this formula will become clear from figure 1.1.

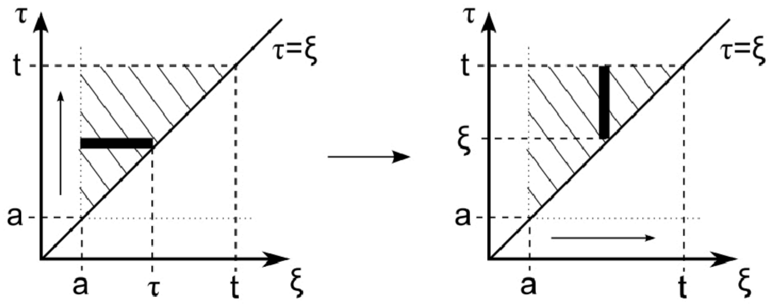

Figure 1.1: Geometrical illustration - change of order of integration.

Derivative of Integrals depending on a parameter

As we will see later, differintegrals are mostly given in a form of an integral depending on a parameter which is equal to the upper limit of integration. Hence it is very important to know the rule for the derivative with regard to this parameter. It can be proven that the following formula holds when the integrated function $g(t,\tau)$ is integrable with regard to the second variable, its derivative $\frac{\partial}{\partial t}g(t,\tau)$ is continuous and $g(t,\tau)$ is defined in all points (t,t).

$$\frac{d}{dt}\int_a^t g(t,\tau)d\tau = g(t,t) + \int_a^t \frac{\partial}{\partial t}g(t,\tau)d\tau.$$

In fractional calculus we often work with functions of the type $g(t,\tau) = (t-\tau)^r f(\tau)$ for some $r \geq 0$, thus let us look at the result in such situation. The case $r = 0$ is quite trivial (we simply obtain $f(t)$, otherwise we get the formula (1.2) bellow since in this case $g(t,t) = 0$ for all t.

$$\frac{d}{dt}\int_a^t (t-\tau)^r f(\tau)d\tau = r\int_a^t (t-\tau)^{r-1} f(\tau)d\tau. \tag{1.2}$$

1.2 Special function

We need to present functions that play an important role in the theory of fractional calculus, these are Euler's Gamma function, the Beta function and the Mittag-leffler function.[7][6][3][14]

1.2.1 Gamma function

In the integer-order calculus the factorial plays an important role because it is one of the most fundamental combinatorial tools. The Gamma function has the same importance in the fractional-order calculus.

Definition 1.2.1. *Gamma function is defined from the right half of the complex plan with* $(\alpha \in \mathbb{C}, Re(\alpha) > 0)$

Gamma function (Γ) is defined as:

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx. \tag{1.3}$$

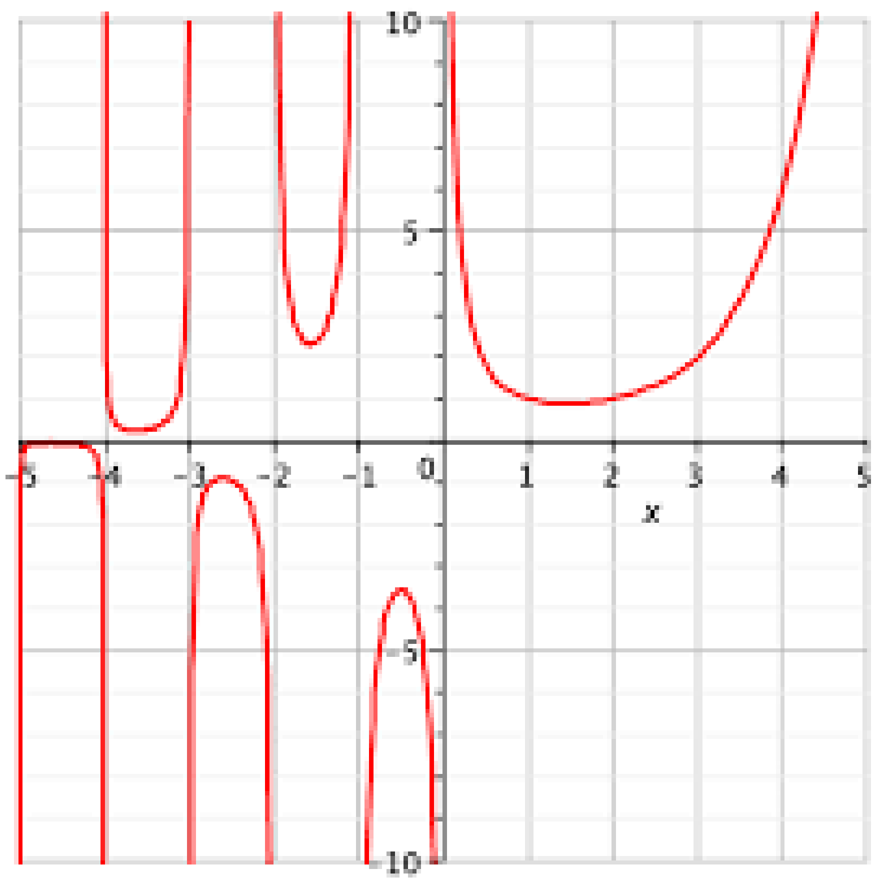

Figure 1.2: Gamma Function.

Properties of the Gamma function

The function $\Gamma(\alpha)$ obeys the property:

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha), \tag{1.4}$$

which can be easily proved by integrating by parts:

$$\begin{aligned}\Gamma(\alpha+1) &= \int_0^{+\infty} t^{\alpha}e^{-t}dt\\ &= \left[-t^{\alpha}e^{-t}\right]_0^{+\infty}+\alpha\int_0^{+\infty} t^{\alpha-1}e^{-t}dt\\ &= \alpha\Gamma(\alpha).\end{aligned}$$

The other property of Gamma function is that

$$\Gamma(n+1)=n!.$$

According to (1.3) we have

$$\Gamma(n)=\int_0^{+\infty} e^{-t}t^{n-1}dt$$

we will apply an integration by parts (n) time

$$\begin{aligned}\Gamma(n+1) &= n!\int_0^{+\infty} e^{(-t)}dt\\ &= n![-e^{-t}]_0^{+\infty}\\ &= n!.\end{aligned}$$

Also Another important property of the gamma function is that it has simple poles at the points $\alpha=-n,(n=0,1,2,\ldots)$. We can write the definition (1.4) in the form

$$\Gamma(\alpha)=\int_0^{1} e^{-t}t^{\alpha-1}dt+\int_1^{+\infty} e^{-t}t^{\alpha-1}dt. \tag{1.5}$$

We give some particular values of $\Gamma(\alpha)$. For $\alpha=\frac{1}{2}$, changing variables $t=u^2$ gives

$$\Gamma\left(\frac{1}{2}\right)=\int_0^{+\infty}\frac{e^{-t}}{\sqrt{t}}dt=2\int_0^{+\infty}e^{-u^2}du=\sqrt{\pi}. \tag{1.6}$$

The functional equation (1.4) entails for positive integers n

$$\begin{aligned}\Gamma\left(\alpha+\frac{1}{2}\right) &= \frac{1.3.5\ldots(2n-1)}{2^n}\Gamma\left(\frac{1}{2}\right).\\ \Gamma\left(\alpha+\frac{1}{3}\right) &= \frac{1.4.7\ldots(3n-2)}{3^n}\Gamma\left(\frac{1}{3}\right).\\ \Gamma\left(\alpha+\frac{1}{4}\right) &= \frac{1.5.9\ldots(4n-3)}{4^n}\Gamma\left(\frac{1}{4}\right).\end{aligned}$$

And for negative values,

$$\Gamma\left(-n+\frac{1}{2}\right)=\frac{(-1)^n2^n}{1.3.5.7\ldots(2n-1)}\sqrt{\pi}.$$

1.2.2 Beta function

The Beta function is very important for the computation of the fractional derivatives of the power function. It is defined by the two-parameter integral

$$B(\alpha,\beta)=\int_0^1 \tau^{\alpha-1}(1-\tau)^{\beta-1}\,\mathrm{d}\tau,$$

for α,β satisfying $Re(\alpha)>0$ and $Re(\beta)>0$. If we use the Laplace transform for convolutions, we get a relation between the Beta function and the Gamma function which implies

$$B(\alpha,\beta)=B(\beta,\alpha),$$

and

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

1.2.3 Mittag-Leffler function

The exponential function e^z is very important in the theory of integer-order differential equations. We can write it in a form of series:

$$e^z=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)}.$$

The generalizations of this function, so called functions of the Mittag-Leffler type, play an important role in the theory of fractional differential equations. First we introduce a two-parameter Mittag-Leffler function defined by formula,[14]

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad (z,\beta\in\mathbb{C},\quad Re(\alpha)>0). \tag{1.7}$$

when $\beta=1$ its equal to the exponential function which called Mittag-Leffler of one parameter(figure1.4)

For special choices of the values of the parameters α,β we obtain well-known classical functions, e.g.:

$$\begin{aligned}
E_{1,1}(z) &= \sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z.\\
E_{0,1}(z) &= \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1)}=\sum_{k=0}^{\infty}z^k.\\
E_{1,0}(z) &= \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k)}=ze^z.\\
E_{1,2}(z) &= \frac{e^z-1}{z}.
\end{aligned}$$

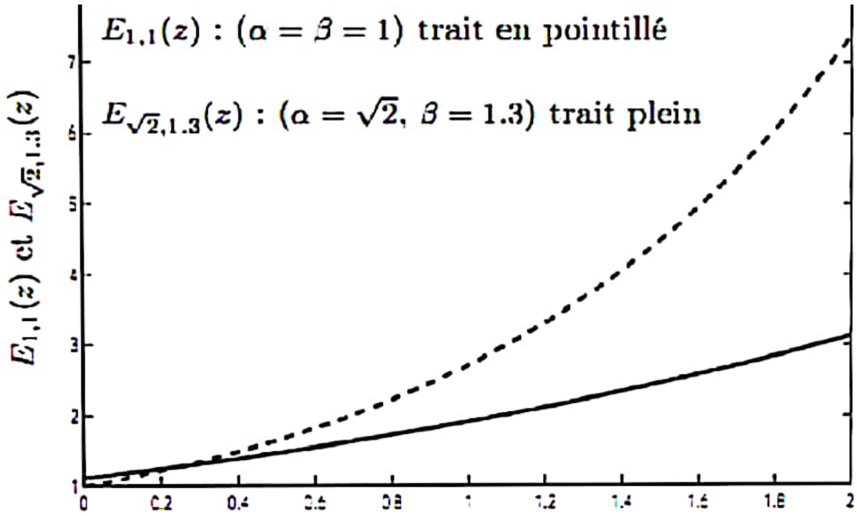

Figure 1.3: Mittag-Leffler of two parameter.

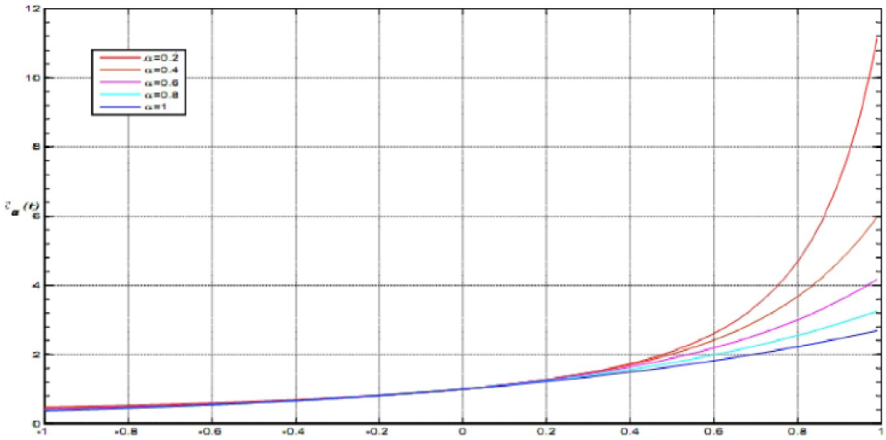

Figure 1.4: Mittag-Leffler of one parameter.

As we will see later, classical derivatives of the Mittag-Leffler function appear in solution of fractional diffrential equations. Since the series (1.7) is uniformly convergent we may differentiate term by term and obtain

$$E_{\alpha,\beta}^{(m)}(z) = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{z^k}{\Gamma(\alpha k + \alpha m + \beta)}.$$

1.3 Laplace transform

The idea of transforming a "difficult" problem into an "easier" problem is one that is used widely in mathematics.

There are many types of transforms available to mathematicians, engineers and scientists.

We are going to examine one such transformation, the Laplace transform, which can be used to solve certain types of differential equations and also has applications in control theory.

Definition 1.3.1. *Given a function $f(t), t \geq 0$, its Laplace transform is defined as*

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt = \lim_{A\to\infty} \int_0^{A} e^{-st} f(t)dt. \tag{1.8}$$

We say the transform converges if the limit exists, and diverges if not. Next we will give examples on computing the Laplace transform of given functions by definition.[2][7]

Note: Laplace transforms are only concerned with functions where $t \geq 0$.

Properties All these properties follow directly from the formula (1.8) ,we can find thier proof directly in[13]

• Generalized linearity (if the series $\sum_{k=0}^{\infty} a_k f_k(t)$ is uniformly convergent)

$$\mathcal{L}\left\{\sum_{k=0}^{\infty} a_k f_k(t)\right\} = \sum_{k=0}^{\infty} a_k F_k(s).$$

• The image of derivatives

$$\mathcal{L}\left\{\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\right\} = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0). \tag{1.9}$$

• The image of integrals

$$\mathcal{L}\left\{\int_0^t f(x)\mathrm{d}x\right\} = \frac{F(s)}{s}.$$

• Differentiation of a Transform

We have

$$F^{(n)}(s) = \mathcal{L}\left\{(-t)^n f(t)\right\}. \tag{1.10}$$

It can be proven by induction. For n=1, we have, successively:

$$F'(s) = \mathcal{L}\left\{-tf(t)\right\}$$

$$\begin{aligned} \frac{dF(s)}{ds} &= \frac{d}{ds}\int_0^{+\infty} e^{-st} f(t)dt \\ &= -\int_0^{+\infty} e^{-st} t f(t)dt \\ &= \mathcal{L}\left\{-tf(t)\right\}, \end{aligned}$$

Finally

$$F^{(n)}(s) = \frac{d}{ds}[F^{(n-1)}(s)].$$

- The image of convolutions

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s). \tag{1.11}$$

with

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} \\ G(s) &= \mathcal{L}\{g(t)\} \end{aligned}$$

here the convolution is defined by

$$(f * g)(t) = \int_0^t f(x)g(t-x)dx.$$

Its obvious that the convolution is commutative, associative and distributive.

Exemple 1.3.1. lets find Laplace Transform of the function $f(t) = e^{at}$.

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} \\ &= \lim_{A\to\infty} \int_0^A e^{-st}e^{at}dt \\ &= \lim_{A\to\infty} \int_0^A e^{-(s-a)t}dt \\ &= \lim_{A\to\infty} -\frac{1}{s-a}e^{-(s-a)t}\Big|_0^A \\ &= \lim_{A\to\infty} -\frac{1}{s-a}\left(e^{-(s-a)A} - 1\right) \\ &= \frac{1}{s-a}, \quad (s > a). \end{aligned}$$

Exemple 1.3.2. lets find the Laplace Transform of the function$f(t) = t^n$, for $n \geq 1$ integer.

$$\begin{aligned} F(s) &= \lim_{A\to\infty} \int_0^A e^{-st}t^n dt \\ &= \lim_{A\to\infty} \left\{ t^n \frac{e^{-st}}{-s}\Big|_0^A - \int_0^A \frac{nt^{n-1}e^{-st}}{-s}dt \right\} \\ &= 0 + \frac{n}{s}\lim_{A\to\infty} \int_0^A e^{-st}t^{n-1}dt \\ &= \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\} \end{aligned}$$

So we get a recursive relation

$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}, \quad \forall n \geq 1$$

which means

$$\begin{aligned}\mathcal{L}\left\{t^{n-1}\right\} &= \frac{n-1}{s}\mathcal{L}\left\{t^{n-2}\right\} \\ \mathcal{L}\left\{t^{n-2}\right\} &= \frac{n-2}{s}\mathcal{L}\left\{t^{n-3}\right\}, \\ &\cdots\end{aligned}$$

By induction, we get

$$\begin{aligned}\mathcal{L}\left\{t^{n}\right\} &= \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\} \\ &= \frac{n}{s}\frac{(n-1)}{s}\mathcal{L}\left\{t^{n-2}\right\} \\ &= \frac{n}{s}\frac{(n-1)}{s}\frac{(n-2)}{s}\mathcal{L}\left\{t^{n-3}\right\} \\ &= \cdots \\ &= \frac{n}{s}\frac{(n-1)}{s}\frac{(n-2)}{s}\cdots\frac{1}{s}\mathcal{L}\{1\} \\ &= \frac{n!}{s^n}\frac{1}{s} \\ &= \frac{n!}{s^{n+1}}, \quad (s>0)\end{aligned}$$

Table of Laplace Transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$\delta(t)$	1	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
1	$1/s$	$\sin\omega t$	$\frac{\omega}{s^2+w^2}$
t	$1/s^2$	$\cosh at$	$\frac{s}{s^2-a^2}$
t^2	$2/s^3$	$\sinh at$	$\frac{a}{s^2-a^2}$
t^n	$\frac{n!}{s^{n+1}}$	$e^{at}\cos\omega t$	$\frac{x-a}{(s-a)^2+\omega^2}$
e^{at}	$\frac{1}{s-a}$	$e^{at}\sin\omega t$	$\frac{w}{(s-a)^2+\omega^2}$
te^{-at}	$\frac{1}{(x+a)^2}$	$t^{n-1}e^{-at}$	$\frac{(n-1)}{(s+a)^n}$

1.3.1 Inverse Laplace transform

Definition 1.3.2. *The corresponding inverse Laplace transform is*

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\lim_{A\to\infty}\int_{k-it}^{k+it} F(s)e^{st}dt = f(t)$$

if

$$F(s) = \mathcal{L}\{f(t)\}.$$

where $i=\sqrt{-1}$ *and* $k\in\mathbb{R}$.

Propertie

The following formula is valid:

$$\mathcal{L}^{-1}\left[\frac{s^{-(\alpha-\beta)}}{s^{\beta}-a}\right]=t^{\alpha-1}E_{\beta,\alpha}\left(at^{\beta}\right),\alpha,\beta>0,s^{\alpha}>|a|,$$

when β=1, we get

$$\mathcal{L}^{-1}\left[\frac{s^{-(\alpha-1)}}{s-a}\right]=t^{\alpha-1}E_{1,\alpha}(at).$$

The proof of this identity is :

$$\begin{aligned}
\mathcal{L}\left[t^{\alpha-1}E_{\beta,\alpha}\left(at^{\beta}\right)\right] &= \int_0^{\infty} e^{-st}t^{\alpha-1}E_{\beta,\alpha}\left(at^{\beta}\right)dt \\
&= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\beta k+\alpha)}\int_0^{\infty} e^{-st}t^{\beta k+\alpha-1}dt \\
&= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\beta k+\alpha)}L\left[t^{\beta k+\alpha-1}\right] \\
&= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\beta k+\alpha)}\frac{\Gamma(\beta k+\alpha)}{s^{\beta k+\alpha}} \\
&= \frac{1}{s^{\alpha}}\sum_{k=0}^{\infty}\left(\frac{a}{s^{\beta}}\right)^k \\
&= \frac{s^{-(\alpha-\beta)}}{s^{\beta}-a}.
\end{aligned}$$

Technique: find the way back.

Lets see some simple examples:

Exemple 1.3.3. lets find the inverse Laplace transform of the function $\frac{3}{s^2+4}$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\} &= \mathcal{L}^{-1}\left\{\frac{3}{2}\cdot\frac{2}{s^2+2^2}\right\} \\
&= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} \\
&= \frac{3}{2}\sin 2t.
\end{aligned}$$

Exemple 1.3.4. lets find the inverse Laplace transform of the function $\frac{s+1}{s^2+4}$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\} &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\
&= \cos 2t+\frac{1}{2}\sin 2t.
\end{aligned}$$

Table of Inverse Laplace Transform

$\mathbf{F}(s)$	$f(t)=\mathcal{L}^{-1}\{\mathbf{F}(s)$
$\frac{1}{s^{\alpha}}$	$\frac{t^{\alpha-1}}{\Gamma(\alpha)}$
$\frac{1}{(s+a)^{\alpha}}$	$\frac{t^{\alpha-1}}{\Gamma(\alpha)}\exp(-\alpha t)$
$\frac{1}{s^{\alpha}-a}$	$t^{\alpha-1}E_{\alpha,\alpha}\left(at^{\alpha}\right)$
$\frac{s^{\alpha-1}}{(s+a)^{\alpha}}$	$E_{\alpha}\left(-at^{\alpha}\right)$
$\frac{a}{s(s^{\alpha}+a)}$	$1-E_{\alpha}\left(-at^{\alpha}\right)$
$\frac{1}{s^{\alpha}(s-a)}$	$t^{\alpha}E_{1,\alpha+1}(at)$
$\frac{s^{\alpha-\beta}}{s^{\alpha}-a}$	$t^{\beta-1}E_{\alpha,\beta}\left(at^{\alpha}\right)$

2 Fractional calculus

Today there exist many diffrent forms of fractional integral operators, ranging from divided-diffrence types to infinite-sum types, but the Riemman-Liouville operator is still the most frequently used when fractional integration is performed.

Riemann's modified form of Liouville's fractional integral operator is a direct generalization of Cauchy's formula for an n-fold integral:

$$\int_a^x dx_1 \int_a^{x_1} dx_2 ..... \int_a^{x_{n-1}} f(x_n) dx_n = \frac{1}{(n-1)!} \int_a^x f(t)(x-t)^{n-1} dt. \tag{2.1}$$

and since $(n-1)! = \Gamma(n)$, Riemann realized that the right-hand side of (2.1) might have meaning even when n takes non-integer values.

2.1 Riemann-Liouville fractional integral

Definition 2.1.1. *Let $f \in L^1(]a,b[)$ be the fractional Riemann-Liouville integral of the function f of order ($\alpha \in \mathbb{C}$, $Re(\alpha) > 0$) is defined in the right side by:*

$$\left(I_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \quad (Re(\alpha) > 0, x > a).$$

Alternivity, it can be defined also the left Fractional Integral as:

$$\left(I_{b^-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} f(\tau) d\tau, \quad (Re(\alpha) > 0, x < b).$$

With $\Gamma(\alpha)$ is the Gamma function.

Exemple 2.1.1. Let $f(t) = C$, $C \in \mathbb{R}$.

Then:

$$\begin{aligned}
I_{a^+}^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{t} (t-s)^{\alpha-1} f(s) ds \\
&= \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{t} (t-s)^{\alpha-1} C ds \\
&= \frac{C}{\Gamma(\alpha)} \int_{a}^{t} (t-s)^{\alpha-1} ds \\
&= \frac{C}{\alpha\Gamma(\alpha)} \left[-(t-s)^{\alpha}\right]_a^t \\
&= \frac{C}{\alpha\Gamma(\alpha)} (t-a)^{\alpha} \\
I_{a^+}^{a} f(t) &= \frac{C(t-a)^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

Theorem 2.1.1. [13] Let $f \in L^1(]a,b[)$ and $\text{Re}(\alpha) > 0$. Then, $\left(I_{a^+}^{\alpha} f\right)(x)$ exists for all $x \in]a,b[$, and we have

$$I_{a^+}^{\alpha} f \in L^1(]a,b[).$$

Proof Let $f \in L^1(]a,b[)$ and $\text{Re}(\alpha) > 0, b > x > a$

$$\begin{aligned}
\left|\left(I_{a^+}^{\alpha} f\right)(x)\right| &= \left|\frac{1}{\Gamma(\alpha)} \int_{\alpha}^{x} (x-\tau)^{\alpha-1} f(\tau) d\tau\right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{x} \left|(x-\tau)^{\alpha-1}\right| |f(\tau)| d\tau \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{x} |(x-\tau)|^{\alpha-1} |f(\tau)| d\tau \\
&\leq \frac{1}{\Gamma(\alpha)} \sup_{x \in]a,b]} \left|(x-\tau)^{\alpha-1}\right| \int_{\alpha}^{x} |f(\tau)| d\tau \\
&< \infty.
\end{aligned}$$

2.2 Fractional derivative

2.2.1 Fractional derivatives in the sense of Riemman-Liouville

Definition 2.2.1. *Let $\alpha \geq 0$ with $\alpha \in \mathbb{R}$ and $n = [\alpha] + 1$ a positive integer such that $n - 1 \leq \alpha < n$, f is a locally integrable function defined on $[a, x]$, the derivative of order α of f is defined by the following formula:*

$${}^{RL}D_a^{\alpha} f(x) = D^n I_a^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-\tau)^{n-\alpha-1} f(\tau) d\tau.$$

Especially, if $\alpha = 0$ then:

$$^{RL}D_a^0 f(x) = I_a^0 f(x) = f(x),$$

if $\alpha = n$, then:

$$^{RL}D_a^\alpha f(x) = f^{(n)}(x),$$

if moreover $0 \leq \alpha < 1$, then $n = 1$ hence

$$^{RL}D_a^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-\tau)^{-\alpha} f(\tau) d\tau.$$

If f is of class $\mathbf{C}^k$ then by doing integrations by parts and repeated diferentiations we obtain:

$$^{RL}D_a^\alpha f(x) = \sum_{j=0}^{k-1} \frac{f^{(j)}(a)(x-a)^{j-\alpha}}{\Gamma(j-\alpha+1)} + \frac{1}{\Gamma(k-\alpha)} \int_a^x (x-\tau)^{k-\alpha-1} f^{(k)}(\tau) d\tau.$$

2.2.2 Fractional derivative in the caputo sense

Definition 2.2.2. *Let $\alpha \in \mathbb{R}$ with $(\alpha > 0)$ and $n = [\alpha] + 1$ such that: $n - 1 \leq \alpha < n$ and $f \in C^n([a, b])$ then the fractional derivative of order α in the sense of Caputo of the function f is defined by:*

$$\begin{aligned} ^{C}D_a^\alpha f(x) &= I_a^{(n-\alpha)} D^n f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau. \end{aligned}$$

Proposition 2.1. *If $n - 1 < \alpha < n$, where $n \in \mathbb{N}^*$, and $\alpha \in \mathbb{R}$, then:*

$$\begin{aligned} \lim_{\alpha \to n} {}^{C}D_0^\alpha f(x) &= f^{(n)}(x). \\ \lim_{\alpha \to n-1} {}^{C}D^{\alpha 0} f(x) &= f^{(n-1)}(x) - f^{(n-1)}(0). \end{aligned}$$

In the formula

$$^{C}D_0^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(\tau) d\tau}{(x-\tau)^{\alpha+1-n}},$$

we will use the integration by parts, obtaining:

$$\begin{aligned} \int_0^x u(\tau) v'(\tau) d\tau = u(\tau) v(\tau)|_0^x - \int_0^x u'(\tau) v(\tau) d\tau \\ u(\tau) = f^{(n)}(\tau), \quad v'(\tau) = (x-\tau)^{n-\alpha-1} \\ u'(\tau) = f^{(n+1)}(\tau), \quad v(\tau) = \frac{-(x-\tau)^{n-\alpha}}{n-\alpha}, \end{aligned}$$

it results:

$$\begin{aligned} {}^{C}D_0^{\alpha}f(x) &= \frac{1}{\Gamma(n-\alpha)}\left[f^{(n)}(\tau)\frac{(x-\tau)^{n-\alpha}}{n-\alpha}\Big|_0^x\right. \\ &+ \left.\frac{1}{n-\alpha}\int_0^x (x-\tau)^{n-\alpha}f^{(n+1)}(\tau)d\tau\,\right]. \end{aligned}$$

Using the property of Γ function $\Gamma(n-\alpha+1) = (n-\alpha)\Gamma(n-\alpha)$, it results:

$$ {}^{C}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha+1)}\left[f^{(n)}(0)x^{n-\alpha} + \int_0^x f^{(n+1)}(\tau)(x-\tau)^{n-\alpha}d\tau\right].$$

$$\begin{aligned} \lim_{\alpha\to n} {}^{C}D_0^{\alpha}f(t) &= \left[f^{(n)}(0) + \int_0^x f^{(n+1)}(\tau)d\tau\right] \\ &= f^{(n)}(0) + f^{(n)}(\tau)\Big|_0^x \\ &= f^{(n)}(x). \\ \lim_{\alpha\to n-1} {}^{C}D_0^{\alpha}f(x) &= \left[f^{(n)}(0) + \int_0^x f^{(n+1)}(\tau)(x-\tau)d\tau\right] \\ &= f^{(n)}(0) + (x-\tau)f^{(n)}(\tau)\Big|_0^x \\ &= f^{(n-1)}(x) - f^{(n-1)}(0). \end{aligned}$$

Exemple 2.2.1. Let us calculate the fractional derivative for $\alpha > 0, n-1 < \alpha < n, \beta > n-1$ of the function $f(t) = t^{\beta}$ using the definitions, for the case:

1. For the Riemann-Liouville derivative, we can write:

$$ {}^{RL}D_0^{\alpha}t^{\beta} = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t u^{\beta}(t-u)^{n-\alpha-1}du,$$

and we take: $u = vt, \quad du = tdv$ It follows:

$$\begin{aligned} {}^{RL}D_0^{\alpha}t^{\beta} &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^1 (vt)^{\beta}[(1-v)t]^{n-\alpha-1}tdv \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^1 (1-v)^{n-\alpha-1}v^{\beta}t^{n-\alpha+\beta}dv \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^1 (1-v)^{n-\alpha-1}v^{\beta}\frac{d^n}{dt^n}t^{n-\alpha+\beta}dv \end{aligned}$$

but

$$\frac{d^n}{dt^n}t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)}t^{\lambda-n},$$

so that it results:

$$ {}^{RL}D_0^{\alpha}t^{\beta} = \frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha+\beta+1)}{\Gamma(-\alpha+\beta+1)}t^{-\alpha+\beta}\int_0^1 (1-v)^{n-\alpha-1}v^{\beta}dv$$

$$\int_0^1 (1-v)^{n-\alpha-1} v^\beta dv = B(n-\alpha, \beta+1) = \frac{\Gamma(n-\alpha)\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)}$$

$$^{RL}D_0^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(-\alpha+\beta+1)} t^{\beta-\alpha}$$

2. In this case we apply the definition of the Caputo derivative of t^β :

$$\begin{aligned} ^{C}D_0^\alpha t^\beta &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\left(u^\beta\right)^{(n)}}{(t-u)^{\alpha+1-\beta}} du \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)} u^{\beta-n} (t-u)^{n-\alpha-1} du. \end{aligned}$$

We use the change of variable $u = vt$, resulting after calculations:

$$^{C}D_0^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)} \int_0^1 (uv)^{\beta-n} \left[(t-v)^{n-\alpha-1}\right] tdv.$$

Finally, we obtain:

$$\begin{aligned} ^{C}D_0^\alpha t^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)} B(\beta-n+1, n-\alpha) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}. \end{aligned}$$

Exemple 2.2.2. The Caputo fractional derivative for the constant function is equal to zero :

$$^{c}D^\alpha c = 0, \quad c = \text{ const }.$$

Let $n-1 \leq \alpha \leq n, n \in \mathbb{N}$, $n \geq 1$, we apply the definition of the Caputo derivative

$$^{c}D^\alpha f(t) = I^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} f^{(n)}(x) dx, \quad x > 0$$

and since the n-time derivative $c^{(n)}(n \in \mathbb{N}, n \geq 1)$ is equal to 0:

$$^{c}D_0^\alpha c = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{c^{(n)}}{(t-x)^{\alpha+1-n}} dx = 0.$$

2.2.3 Properties of fractional derivatives

In this part will find out whether some basic properties, such as linearity, rule of Leibniz, rule of Zero and composition, always apply to integrals and fractional derivatives.[7]

Linearity property

$$I_{a+}^{\alpha}[C_1 f(t) + C_2 g(t)] = C_1 I_{a+}^{\alpha} f(t) + C_2 I_{a+}^{\alpha} g(t).$$

where: C_1 and C_2 are constants and $f(t)$ and $g(t)$ are two arbitrary functions.

$$\begin{aligned} I_{a+}^{\alpha}\left[C_1 f(t) + C_2 g(t)\right] &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \left[C_1 f(\tau) + C_2 g(\tau)\right] d\tau \\ &= C_1 \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau + C_2 \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} g(\tau) d\tau \\ &= C_1 I_{a+}^{\alpha} f(t) + C_2 I_{a+}^{\alpha} g(t). \end{aligned}$$

A similar proof can be given for the Riemann-Liouville derivative.

Rule of Zero

We can prove that if $f(t)$ is continuous for $t \geq a$ then we have [7]

$$\lim_{\alpha \longmapsto 0} D_a^{\alpha} f(t) = f(t).$$

The proof is on p.65-67[7].Therefore, we define:

$$D_a^0 f(t) = f(t).$$

The product rule and Leibniz's rule

If f and g are functions whose derivative of their product that we know is given by the rule of the product [7]

$$(f(t).g(t))' = f'.g + g'.f,$$

this can be generalized to

$$(fg)^{(n)} = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} f^{(k)} g^{(n-k)},$$

which is also known as Leibniz's rule. In the last expression f and g are n-times differentiable functions. If $f(x)$ and $g(x)$ and their derivatives are continuous in $[a, t]$, it can be proved that Leibniz's rule for fractional derivatives is given by the following expression:

$$D_a^{\alpha}(f(t)g(t)) = \sum_{k=0}^{m} \begin{pmatrix} \alpha \\ k \end{pmatrix} f^{(k)}(t) D_a^{\alpha-k} g(t).$$

where the binomial coefficient is given by

$$\begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{\alpha!}{(\alpha-k)!k!},$$

and $m \in \mathbb{N}$ satisfies $(m \leq \alpha \leq m+1)$. The proof is quite long so it will not be given here, but can be found in p.91-97/[7]. If we know the fractional derivative of a function, say $g(t)$ and we want to determine the fractional derivative of a function which is a product of $g(t)$ and another function,say $f(t)$, Leibniz's rule is very useful.

The composition (fractional integration of a fractional integral)

The Riemann-Liouville fractional integral has the following important property

$$I_{a^+}^{\alpha}\left(I_{a^+}^{\beta} f(t)\right) = I_{a}^{\alpha+\beta} f(t), \quad \text{for}\, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$$

So we have for $(Re(\alpha > 0), Re(\beta > 0))$, it results:

$$I_{a^+}^{\alpha} I_{a^+}^{\beta} f(t) = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^t (t-\tau)^{\alpha-1} \int_a^{\tau} (\tau-\xi)^{\beta-1} f(\xi) d\tau d\xi,$$

if we apply the Dirichlet equality

$$\int_a^t \int_a^{\tau} f(\xi) d\tau d\xi = \int_a^t \int_{\xi}^t f(\xi) d\tau d\xi$$

we obtain:

$$I_{a^+}^{\alpha}\left(I_{a^+}^{\beta} f(t)\right) = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^t \int_{\xi}^t (t-\tau)^{\alpha-1} (\tau-\xi)^{\beta-1} f(\xi) d\tau d\xi,$$

variable changing

$$\tau = \xi + z(t-\xi), \quad d\tau = (t-\xi)dz, \quad t-\tau = (1-z)(t-\xi),$$

we get

$$I_{a^+}^{\alpha}\left(I_{a^+}^{\beta} f(t)\right) = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^t (t-\xi)^{\alpha+\beta-1} f(\xi) \int_0^1 (1-z)^{\alpha-1} z^{\beta-1} dz d\xi,$$

but:

$$\begin{aligned}
\int_0^1 (1-z)^{\alpha-1} z^{\beta-1} dz &= B(\alpha-1, \beta-1) \\
&= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.
\end{aligned}$$

Finally, it results:

$$\begin{aligned}
I_{a^+}^{\alpha}\left(I_{a^+}^{\beta} f(t)\right) &= \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-\xi)^{\alpha+\beta-1} f(\xi) d\xi \\
&= I_{a^+}^{\alpha+\beta} f(t).
\end{aligned}$$

Another important property of the Riemann-Liouville fractional derivative

$$ {}^{RL}D_a^{\alpha}\left[I_{a^+}^{\beta}f(t)\right] = {}^{RL}D_a^{\alpha-\beta}f(t). $$

So by using the definition

$$ \begin{aligned} {}^{RL}D_a^{\alpha}\left[I_{a^+}^{\beta}f(t)\right] &= \frac{d^n}{dt^n}\left[I_{a^+}^{n-\alpha}\left[I_{a^+}^{\beta}f(t)\right]\right] \\ &= \frac{d^n}{dt^n}\left[I_{a^+}^{n-(\alpha-\beta)}f(t)\right] \\ &= {}^{RL}D_a^{\alpha-\beta}f(t), \end{aligned} $$

if $\alpha = \beta$ we get

$$ {}^{RL}D_a^{\alpha}\left[I_{a^+}^{\beta}f(t)\right] = f(t). $$

• There are another possibilities when we deal with the composition of differint-integrals, which are not needed in this thesis.

2.2.4 Examples

This section covers some examples of fractional derivatives and integrals. First of all, we'll look at the power function.

The power function

The power function is one of the most important functions because many functions can be defined by an infinite power series. First we calculate its fractional integral of Riemann-Liouville of order $\alpha > 0$

$$ \mathrm{I}_{a^+}^{\alpha}(t-a)^{\beta} = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}(\tau-a)^{\beta}\mathrm{d}\tau, $$

variable changing $\frac{\tau-a}{t-a} = \xi \quad \mathrm{d}\tau = (t-a)\mathrm{d}\xi, \quad \xi : 0 \to 1$

$$ \mathrm{I}_{a^+}^{\alpha}(t-a)^{\beta} = \frac{(t-a)^{\beta+\alpha}}{\Gamma(\alpha)}\int_0^1 (1-\xi)^{\alpha-1}\xi^{\beta}\mathrm{d}\xi \tag{2.2} $$

$$ = \frac{(t-a)^{\beta+\alpha}}{\Gamma(\alpha)}B(\alpha,\beta+1) $$

$$ = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}, \quad \beta > -1. \tag{2.3} $$

The condition $\beta > -1$ arises naturally as a product of the required integrability. Now we can calculate the Riemann-Liouville derivative of the power function by using the formula (2.3) and just derived relation (2.3), where as usual $\alpha > 0$,

$$\begin{aligned}
{}^{RL}\mathrm{D}_a^{\alpha}(t-a)^{\beta} &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathrm{I}_{a^+}^{(n-\alpha)}(t-a)^{\beta} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}(t-a)^{\beta+n-\alpha} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \quad \beta > -1. \qquad (2.4)
\end{aligned}$$

We see that the results of (2.4) and (2.3) are formally the same, the condition on β is also unchanged, so we may write generally

$${}^{RL}\mathrm{D}_a^{\alpha}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \quad \alpha \in \mathbb{R}, \beta > -1. \qquad (2.5)$$

The following two examples can clarify this using concrete numbers. First of all, we want to calculate the half-derivative of the function $f(x) = x$, so in the last expression we define $t = x$, $a = 0$, $\beta = 1$ and $\alpha = \frac{1}{2}$. We then obtain

$$\begin{aligned}
{}^{RL}D_0^{\frac{1}{2}}(x-0)^1 &= \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)}(x-0)^{1-\frac{1}{2}} \\
{}^{RL}D_0^{\frac{1}{2}}x &= \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}} \\
{}^{RL}D_0^{\frac{1}{2}}x &= 2\sqrt{\frac{x}{\pi}}. \qquad (2.6)
\end{aligned}$$

In the following example, we want to calculate the derivative of order $\frac{3}{4}$ of the function $f(x) = x^2$, so again in formula (2.5) we define $t = x$,$a = 0$, but now $\beta = 2$ and $\alpha = \frac{3}{4}$.
This gives us

$$\begin{aligned}
D_a^{\frac{3}{4}}(x-0)^2 &= \frac{\Gamma(2+1)}{\Gamma(2-\frac{3}{4}+1)}(x-0)^{2-\frac{3}{4}} \qquad (2.7) \\
D_a^{\frac{3}{4}}x^2 &= \frac{\Gamma(3)}{\Gamma(\frac{9}{4})}x^{\frac{5}{4}}.
\end{aligned}$$

Proposition 2.2. *The Caputo fractional derivative of order $\alpha > 0$ with $n-1 < \alpha < n$ of a power function $f(t) = t^{\beta}$ for $\beta \geq 0$ is defined by:*

$${}^{c}D^{\alpha}t^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{-\alpha} = D^{\alpha}t^{\beta} & (\beta > n-1). \\ 0 & (\beta \leq n-1). \end{cases}$$

For more examples and more details we can see in [7] such as the exponential function and the trigonometric functions.

2.2.5 Relation with the Riemann-Liouville derivative and Caputo derivatives

Let $\alpha > 0$ with $n-1 < \alpha < n, (n \in N^*)$, suppose that f is a function such that ${}^{C}D_a^{\alpha} f(t)$ and ${}^{RL}D_a^{\alpha} f(t)$ exist,then

$$ {}^{C}D_a^{\alpha} f(x) = {}^{RL}D_a^{\alpha} f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}. $$

We deduce that if $f^{(k)}(a) = 0, (k = 0, 1, 2, \ldots, n-1)$

$$ {}^{C}D_a^{\alpha} f(x) = {}^{RL}D_a^{\alpha} f(x). $$

3 Fractional differential equations

Fractional differential equations are a generalization of ordinary differential equations.They can be solved by several methods, of which the Laplace transform is the most used method. We will explore this method, but first we give some basic properties of the Laplace transform fractional derivative, which are necessary to understand this chapter.

3.1 Laplace transform of fractional derivative

We will first explore the Laplace transform of the RiemannLiouville fractional integral. Using the definition and setting the lower bound a equal to zero, we get [14]

$$I_0^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \tag{3.1}$$

using the definition of the convolution, we write (3.1) as:

$$I_0^\alpha f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} * f(x).$$

We have Laplace transform of function $x^{\alpha-1}$

$$\mathcal{L}\left\{x^{\alpha-1}\right\} = \frac{\Gamma(\alpha)}{s^\alpha}.$$

Using the form of the Laplace convolution transformation we obtained the transformation of integration of Riemann-Liouville :

$$\begin{aligned}
\mathcal{L}\left\{I_0^\alpha f(x)\right\} &= \frac{1}{\Gamma(\alpha)} \mathcal{L}\left\{x^{\alpha-1} * f(x)\right\} \\
&= \frac{1}{\Gamma(\alpha)} \mathcal{L}\left\{x^{\alpha-1}\right\} F(s) \\
&= s^{-\alpha} F(s).
\end{aligned}$$

So we get that

$$\mathcal{L}\left\{I_0^\alpha f(x)\right\} = s^{-\alpha} F(s). \tag{3.2}$$

3.1.1 Laplace transform of Riemann-Liouville derivative

We will explore the Laplace transform of the Riemann-Liouville fractional derivative. As suggested in [7] we will write this fractional derivative in the following form

$$\mathcal{L}\left\{{}^{RL}D_0^\alpha f(x)\right\} = \mathcal{L}\left\{\frac{d^n}{dx^n} I_0^{n-\alpha} f(x)\right\},$$

using the laplace transform forme (1.9)

$$\begin{aligned}\mathcal{L}\left\{\frac{d^n}{dx^n} I_0^{n-\alpha} f(x)\right\} &= s^n \mathcal{L}\left\{I_0^{n-\alpha} f(x)\right\} - \sum_{k=0}^{n-1} s^k \frac{d^{n-k-1}}{n-k-1} I_0^{n-\alpha} f(0) \\ &= s^n (s^{-(n-\alpha)} F(s)) - \sum_{k=0}^{n-1} s^k \frac{d^{n-k-1}}{n-k-1} I_0^{n-\alpha} D_0^{\alpha-n} f(0).\end{aligned}$$

We find the final forme of the Laplace Transform of Riemann Liouville ordre $\alpha > 0$

$$\mathcal{L}\left\{{}^{RL}D_0^\alpha f(x)\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k [D_0^{\alpha-k-1} f(x)]_{x=0}, \qquad (n-1 \leq \alpha < n).$$

3.1.2 Laplace transform of caputo fractional derivative

The formula of the Laplace transform of the fractional Caputo derivative defined in the form:

$${}^{C}D_0^\alpha f(x) = I_0^{n-\alpha} g(x), \quad g(x) = f^{(n)}(x), \quad (n-1 < \alpha \leq n)$$

using the formula (3.2) of the Laplace Transform of the fractional integral of Riemann Liouville, We will have :

$$\mathcal{L}\left\{{}^{C}D_0^\alpha f(x)\right\} = s^{-(n-\alpha)} G(s), \tag{3.3}$$

or, thanks to (1.9)

$$\begin{aligned}G(s) &= s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) \\ &= s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0).\end{aligned} \tag{3.4}$$

Entering (3.3),(3.4) we come to the form of the laplace transform for the Caputo fractional derivative:

$$\mathcal{L}\left\{{}^{C}D_0^\alpha f(x)\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0).$$

We will see that these special cases are useful for solving some simple fractionals differential equations, which will be covered in the examples at the end of this chapter.

Exemple 3.1.1. Lets calculate the Caputo $\mathcal{L}\Big\{{}^{C}D^{\alpha}\Big\}$,
$for \quad \alpha = \dfrac{1}{2}$

$$\begin{aligned}\mathcal{L}\left\{{}^{C}D^{\frac{1}{2}}t^{2}\right\} &= \frac{1}{s^{-\frac{1}{2}}}\mathcal{L}\left\{t^{2}\right\}\\ \mathcal{L}\left\{{}^{C}D^{\frac{1}{2}}t^{2}\right\} &= \frac{2}{s^{\frac{5}{2}}}.\end{aligned}$$

So if we want to find ${}^{C}D^{\frac{1}{2}}t^{2}$ we get

$${}^{C}D^{\frac{1}{2}}t^{2} = \mathcal{L}^{-1}\left\{\frac{2}{s^{\frac{5}{2}}}\right\} = \frac{2t^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} = \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}}.$$

3.2 Laplace transform method

Before continuing, we also need the Laplace transform of a very important function for linear fractional differential equations consisting of two terms. We need to explore the Laplace transform of the following function [14]

$$\mathcal{L}\left\{t^{\alpha m+\beta-1}E^{(m)}_{\alpha,\beta}(at^{\alpha})\right\}. \tag{3.5}$$

If we look closer, we can see that this function is a combination of the power function and the m time derivative of Mittag-Leffler function, the evaluation of this Mittag-Leffler function at at^{α} gives

$$E^{(m)}_{\alpha,\beta}(at^{\alpha}) = \sum_{k=0}^{\infty}\frac{(k+m)!}{k!}\frac{(at^{\alpha})^{k}}{\Gamma(\alpha k+\alpha m+\beta)} = \sum_{k=0}^{\infty}\frac{(k+m)!}{k!}\frac{a^{k}(t^{\alpha})^{k}}{\Gamma(\alpha k+\alpha m+\beta)}, \tag{3.6}$$

substituting this expression into (3.5) gives

$$\mathcal{L}\left\{t^{\alpha m+\beta-1}E^{(m)}_{\alpha,\beta}\left(at^{\alpha}\right)\right\} = \mathcal{L}\left\{t^{\alpha m+\beta-1}\sum_{k=0}^{\infty}\frac{(k+m)!}{k!}\frac{a^{k}t^{\alpha k}}{\Gamma(\alpha k+\alpha m+\beta)}\right\}, \tag{3.7}$$

using the linearity of the Laplace transform, we can rewrite the last expression as

$$\mathcal{L}\left\{t^{am+\beta-1}E^{(m)}_{\alpha,\beta}\left(at^{\alpha}\right)\right\} = \sum_{k=0}^{\infty}\frac{(k+m)!a^{k}}{k!\Gamma(\alpha k+\alpha m+\beta)}\mathcal{L}\left\{t^{\alpha k+\alpha m+\beta-1}\right\}. \tag{3.8}$$

Now we want to inspect $\mathcal{L}\left\{t^{\alpha k+am+\beta-1}\right\}$ from the last equation. We have already determined the Laplace transform of the power function

$$\mathcal{L}\left\{t^{\alpha-1}\right\} = \Gamma(\alpha)s^{-\alpha}, \tag{3.9}$$

so in this case we have

$$\begin{aligned}\mathcal{L}\left\{t^{ak+\alpha m+\beta-1}\right\} &= \Gamma(\alpha k+\alpha m+\beta)s^{-(ak+\alpha m+\beta)} \\ &= \frac{\Gamma(\alpha k+\alpha m+\beta)}{s^{ak+am+\beta}} \end{aligned} \tag{3.10}$$

Substituting this into (3.8) gives us

$$\begin{aligned}\mathcal{L}\left\{t^{\alpha m+\beta-1}E^{(m)}_{\alpha,\beta}\left(at^{\alpha}\right)\right\} &= \sum_{k=0}^{\infty}\frac{(k+m)!a^k}{k!\Gamma(\alpha k+\alpha m+\beta)}\frac{\Gamma(\alpha k+\alpha m+\beta)}{s^{ak+\alpha m+\beta}} \\ &= \sum_{k=0}^{\infty}\frac{(k+m)!a^k}{k!s^{ak+\alpha m+\beta}} \\ &= \sum_{k=0}^{\infty}\frac{(k+m)!}{k!}\frac{a^k}{s^{ak+\alpha m+\beta}} \\ &= s^{-\alpha m-\beta}\sum_{k=0}^{\infty}\frac{(k+m)!}{k!}\left(\frac{a}{s^{a}}\right)^k.\end{aligned}$$

To further rewrite the last expression, we look at the series

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(k+m)!}{k!}\left(\frac{a}{s^{\alpha}}\right)^k &= \sum_{k=0}^{\infty}(k+m)(k+m-1)\cdots(k+1)\left(\frac{a}{s^{\alpha}}\right)^k \\ &= \sum_{k=m}^{\infty}k(k-1)\cdots(k-m+1)\left(\frac{a}{s^{a}}\right)^{k-m} \\ &= \frac{\mathrm{d}^m}{\mathrm{d}t^m}\sum_{k=0}^{\infty}\left(\frac{a}{s^{\alpha}}\right)^k.\end{aligned}$$

Since the first terms m disappear after the differentiation, we can rewrite the last expression in the form

$$\sum_{k=0}^{\infty}\frac{(k+m)!}{k!}\left(\frac{a}{s^{\alpha}}\right)^k = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\sum_{k=0}^{\infty}\left(\frac{a}{s^{\alpha}}\right)^k = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\frac{1}{1-\frac{a}{s^{a}}} = \frac{m!}{\left(1-\frac{a}{s^{\alpha}}\right)^{m+1}}.$$

So substituting this, we finally get [14]

$$L\left\{t^{\alpha m+\beta-1}E^{(m)}_{\alpha,\beta}\left(at^{\alpha}\right)\right\} = s^{-\alpha m-\beta}\frac{m!}{\left(1-\frac{a}{s^{\alpha}}\right)^{m+1}} = \frac{m!s^{\alpha-\beta}}{(s^{\alpha}-a)^{m+1}}.$$

3.2.1 Application on the solution of fractional differential equations

Exemple 3.2.1. We would like to solve the fractional differential equation given by

$${}^{RL}D_0^{\frac{1}{3}}f(t) = c_1 f(t), \quad {}^{RL}D_0^{-\frac{2}{3}}f(0) = c_2. \tag{3.11}$$

where c_1 is a constant. Since $0 \leq \alpha = \frac{1}{3} < 1$ we will use the Laplace transform of the Riemann-Liouville fractional derivative for $n = 1$ and take the Laplace transform on both sides of the last equation. If we also use the linearity of the Laplace transform, this gives

$$\mathcal{L}\left\{ {}^{RL}D_0^{\frac{1}{3}} f(t) \right\} = \mathcal{L}\left\{ c_1 f(t) \right\} = c_1 \mathcal{L}\{f(t)\} \tag{3.12}$$

$$\begin{aligned} s^{\frac{1}{3}}F(s) - {}^{RL}D_0^{\frac{1}{3}-1} f(0) &= c_1 F(s) \\ s^{\frac{1}{3}}F(s) - {}^{RL}D_0^{-\frac{2}{3}} f(0) &= c_1 F(s). \end{aligned}$$

We see that ${}^{RL}D_0^{-\frac{2}{3}} f(0)$ is the value of ${}^{RL}D_0^{-\frac{2}{3}} f(t)$ evaluated to $t = 0$. If we assume that this value exists, we can set ${}^{RL}D_0^{-\frac{2}{3}} f(0)$ equal to c_2 to get

$$s^{\frac{1}{3}}F(s) - c_2 = c_1 F(s). \tag{3.13}$$

If we solve this for $F(s)$ we get

$$F(s) = \frac{c_2}{s^{\frac{1}{3}} - c_1}$$

Finelly, thanks to table, we find inverse Laplace transform of $F(s)$, then we conclud that :

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{ \frac{c_2}{s^{\frac{1}{3}} - c_1} \right\} \\ &= c_2 t^{\frac{1}{3}-1} E_{\frac{1}{3},\frac{1}{3}}\left(c_1 t^{\frac{1}{3}} \right) \\ &= c_2 t^{-\frac{2}{3}} E_{\frac{1}{3},\frac{1}{3}}\left(c_1 t^{\frac{1}{3}} \right). \end{aligned}$$

Exemple 3.2.2. Let be the fractional differential equation :

$${}^{C}D_0^{\alpha} y(x) + a y(x) = 0, \quad x > 0, \quad 0 < \alpha < 1$$

where a is real constant and $y(0) = C$.

Using Laplace transfom of Caputo fractional derivative, if we also use the linearity of the Laplace transform, this gives:

$$\mathcal{L}\left\{ {}^{C}D_0^{\alpha} y(x) \right\} + a\mathcal{L}\left\{ y(x) \right\} = 0,$$

what gives us

$$(s^{\alpha}Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} y^{(k)}(0)) + aY(s) = 0,$$

since, 0< α<1 we get for n=1

$$s^{\alpha}Y(s) - s^{\alpha-1}y(0) + aY(s) = 0,$$

after that

$$(s^{\alpha} + a)Y(s) - Cs^{\alpha-1} = 0,$$

we get

$$Y(s) = C\frac{s^{\alpha-1}}{s^{\alpha}+a},$$

applying inverse Laplace transform

$$y(x) = C\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+a}\right\},$$

using :

$$\mathcal{L}^{-1}\left\{\frac{m!s^{\alpha-\beta}}{(s^{\alpha}-a)^{m+1}}\right\} = x^{\alpha m+\beta-1}E^{(m)}_{\alpha,\beta}(x^{\alpha}),$$

with β=1 , m=0

$$y(x) = CE_{\alpha,1}(-ax^{\alpha})$$

Exemple 3.2.3. Now we would like to solve the fractional differential equation given by

$$^{RL}D_0^{\frac{19}{12}}f(x) = 0, \quad ^{RL}D_0^{\frac{7}{12}} = c_3, \quad ^{RL}D_0^{-\frac{5}{12}} = c_4.$$

Since $1 \leq p = \frac{19}{12} < 2$, we will use the Laplace transform of the Riemann-Liouville fractional derivative for $n = 2$ and take the Laplace transform on both sides.
This gives

$$\begin{aligned}
\mathcal{L}\left\{^{RL}D_0^{\frac{19}{12}}f(t)\right\} &= 0 \\
s^{\frac{19}{12}}F(s) - {}^{RL}D_t^{\frac{19}{12}-1}f(0) - s^{RL}D_0^{\frac{19}{12}-2}f(0) &= 0 \\
s^{\frac{19}{12}}F(s) - {}^{RL}D_t^{\frac{7}{12}}f(0) - s^{RL}D_0^{-\frac{5}{12}}f(0) &= 0.
\end{aligned}$$

Following the same steps as in example 1, we get the following solution

$$\begin{aligned}
f(x) &= L^{-1}\left\{\frac{c_3}{s^{\frac{19}{12}}}\right\} + L^{-1}\left\{\frac{c_4 s}{s^{\frac{19}{12}}}\right\} \\
&= L^{-1}\left\{\frac{c_3}{s^{\frac{19}{12}}}\right\} + L^{-1}\left\{\frac{c_4}{s^{\frac{7}{12}}}\right\} \\
&= \frac{c_3 t^{\frac{7}{12}}}{\Gamma\left(\frac{19}{12}\right)} + \frac{c_4 t^{-\frac{5}{12}}}{\Gamma\left(\frac{7}{12}\right)}.
\end{aligned}$$

3.3 Applications of fractional calculus in engineering

Fractional calculus is used in many fields, for example in engineering, physics, economy, biological processes, etc. Many models can be represented by fractional differential equations and are therefore increasingly used in these branches. It brings new possibilities, namely that fractional

derivatives can describe memory effects, it is therefore possible to assess the influence of the past on the behavior of the system at present.
One of the first scientists to use fractional calculus to solve a problem was Norwegian mathematician Niels Henrik Abel. In 1823, he applied it in the formulation of his solution to the tautochrone problem. The idea of this problem is to find the curve of a wire without friction that lies in the plane (x, y) such that the time required for a particle slides to the lowest point of the curve is independent of where the particle is placed [7]

3.3.1 Fractional kinetics of charge carriers in supercapacitors

Fractional order kinetic equations provide a unifying tool for the description the behavior of charge carriers in disordered media

Distribution of relaxation rates

A typical electrochemical (ES) supercapacitor consists of two separate porous electrodes by an ion-permeable separator and an electrolyte ionically connecting electrodes (Figure 3.1(a)). Approximately, the porous electrode can be modeled by a network of individual pores (Figure 3.1(b)) arranged in parallel (Figure 3.1(c))[14]

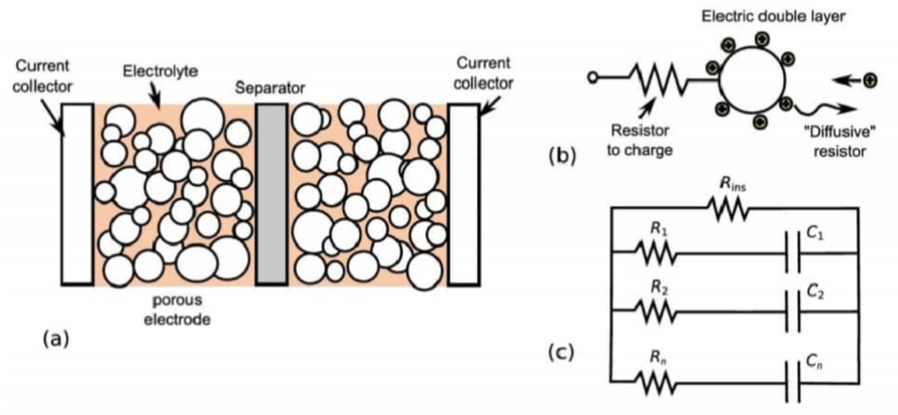

Figure 3.1: Diagram of an electric double-layer supercapacitor (EDLC) (a), of the component elementary circuit (b) and the simplest equivalent circuit of EDLC (c)

The parallel arrangement of individual pores is responsible for the distribution of relaxation times associated with "elementary capacitors". Each elementary component discharges according to the simplest relaxation equation: $f(t) = \mu f(t), \quad (\mu > 0)$, having a solution under the form of an exponential a function. Considering a dichotomous stochastic process in which the relaxation between two states is not given by a single rate m, but by a distribution $r(\mu)$, the authors of [14] represented the relaxation function by means of superposition

$$f(t) = \int_0^\infty \rho(\mu) exp(-\mu) d\mu \tag{3.14}$$

Moreover, by considering a form of scale of this distribution, they arrive at the equation of fractional differential relaxation

$$D_t^\alpha f(t) = -\mu_0 f(t) \delta_\alpha(t) f_0 \quad \delta_\alpha(t) = \frac{t_+^{-\alpha}}{\Gamma(1-\alpha)}, \tag{3.15}$$

whose solution is expressed in terms of the **Mittag-Leffler** function $E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1-\alpha k)}$

$$f(t) = f_0 E_\alpha(-\mu t^\alpha) \tag{3.16}$$

Diffusion-limited charge transfer

Due to the fact that EDLCs with carbon electrodes have low pseudocapacitance due of the Faraday reaction of surface functional groups, the general case of the transfer of charge limited by diffusion at the surface of the electrodes is considered in [14]. Oldham notes in [14] that electrochemistry "was not the first scientific discipline to benefit from computation fractional, but she was certainly one of the first to reap a sustained harvest of useful concepts and methodologies".

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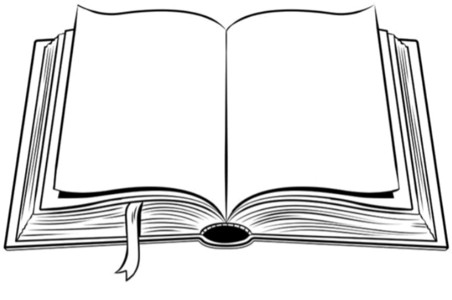
Abstract

In this thesis, we have dealt with fractional derivatives and fractional integrals, shortly differ integrals. After a short introduction and some preliminaries and basic function like Gamma, Beta, Mettag-Leffler function then different approaches for defining a differ-integral have been explored. Then some basic properties of differ-integrals, such as linearity, the Leibniz rule and composition, have been presented. There after the definitions of the differ integrals have been applied to a few examples. Also, linear fractional-order differential equations and one method for solving them have been discussed.

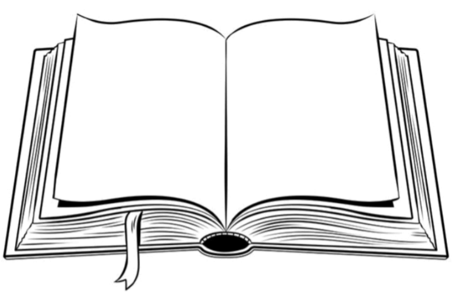
Keywords: fractional derivatives and fractional integrals, fractional calculus, fractional differential equations. Riemann-Liouville integral, Riemann-Liouville derivative, Caputo derivative.



Résumé

Dans ce mémoire, nous avons traité les dérivées et les intégrales d’ordre fractionnaire, en bref les différo-intégrales. Après une brève introduction et quelques préliminaires, différentes approches pour définir une différo-intégrale ont été explorées. Ensuite, certaines propriétés de base des différo-intégrales, telles que la linéarité, la règle de Leibniz et la composition, ont été présentées. Par la suite, les définitions des différo-intégrales ont été appliquées à quelques exemples. En outre, des équations différentielles d’ordre fractionnaire linéaires et l’une méthode pour les résoudre ont été discutées.

Mots clés : Dérivées et intégrales fractionnaires, calcul fractionnaire, équations différentielles fractionnaires. intégrale et dérivée fractionnaire de Riemann-Liouville. dérivée fractionnaire de Caputo, la transformation de Laplace .



ملخص

في هذه المذكّرة ، تعاملنا مع المشتقات والتكاملات الكسرية، والتي يمكن أن تسمّى اختصاراً ب"المشتكاملات" . بعد مقدمة قصيرة و بعض التمهيدات ، تم استكشاف مختلف المقاربات المعتمدة لتعريف هذا النوع من المشتقات و التكاملات، ثم تم تقديم بعض الخصائص الأساسية للمشتكاملات مثل خاصيّة الخطيّة، قاعدة ليبنز و خاصيّة التركيب. بعد ذلك تم تطبيق هذه المشتكاملات على بعض الأمثلة، كما تمت مناقشة المعادلات التفاضلية الخطية ذات الرّتب الكسرية و تمّ مناقشة احدى طرق حلها.

الكلمات المفتاحية: المشتقات الكسرية والتكاملات الجزئية ، حساب ريمان- التفاضل والتكامل الكسري ، المعادلات التفاضلية الكسرية. تكامل ليوفيل ، مشتق ريمان-ليوفيل ، مشتق كابوتو.