الجمهورية الجزائرية الديمقراطية الشعبية

PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

وزارة التعليم العالي و البحث العلمي

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

جامعة عباس لغرور خنشلة

ABBES LAGHROUR- KHENCHELA UNIVERSITY

Faculty of Sciences and Technology

Department of Mathematics and Computer Science

N° de série :……

Mémoire de fin d'études

Pour l'obtention du diplôme de **Master**

Filière: **Mathématiques**

Spécialité: **Mathématiques Appliquées**

Intitulé

On some inverse problems for degenerate parabolic equations

Réalisé par : **BOUZOU Aya**

MAAROUK Abir

Dirigé par : **Dr. RAMOUL Hichem**

Présenté le **13/09/2020**

Membres de jury :

M. BAHRI Boubakeur	**MAA**	**Président**
Mme. NASRALLAH Ilhem	**MAA**	**Examinateur**

2019-2020

ملخص

هذه المذكرة تقتصرعلى دراسة مسألة عكسية ذات معادلة تكافئية منحلة الشكل في فضاء ذو البعد الواحد.

حتى نكون أكثر دقة، لقد تمكننا من برهنة معادلة الأستقرار من نوع ليبشيتز لتعيين المنبع باستعمال متراجحة كارلومان كما تمكننا ايضا من استرجاع الشرط الابتدائي بواسطة نظرية التحدب اللوغاريتمي.

Contents

DÉDICACE

Personnellement je dédie ce travail aux être qui me sont les plus chers, je cite:

Les parents les plus chers au monde, papa et maman, aucune dédicace ne saurait exprimer mon respect et mon amour, que dieu les garde et les protège.

la mémoire de mon seul et grand frère Rami,disparu trop tôt,qui m'a toujours poussé et motivé dans mes études, les mots ne suffisent guère pour exprimer l'amour que je porte pour vous, j'aurait tant aimé que vous soyez present, j'espère que du monde qui est sien maintenant, il apprécie cet humble geste comme preuve de reconnaissance de la part d'une soeur qui a toujours priée pour le salut de son âme. puisse Dieu,le tout puissant, l'avoir en sa sainte miséricorde.

Ma seule et petite soeur Meriem , je vous souhaite une vie pleine de bonheur et de succès.

Une spéciale dedicace a ma meilleure amie Wissam qui m'a assisté dans les moments difficiles merci d'être toujour là pour moi.

Aya chère amie avant d'être binôme avec laquelle j'ai pris beaucoup de plaisir a travailler.

Toutes mes amies sans exception.

Ceux qui m'aime et qui attendent avec impatience ma réussite.

J'espère être a la hauteur de vos attentes.

Abir

C'est avec une très grande émotion et un immense plaisir que je dédie ce modeste travail

A mes très chers parents qui m'ont soutenu toute la durée de mes études.

A mes frères et mes soeurs.

A tous ceux qui m'aiment et j'aime.

AYA

Thanks

First of all we would like to thank ALLAH who gave us the strength and the patience to accomplish this modest work.
Secondly we would like to underline the excellent thanks to our supervisor Mr Ramoul Hichem for his honour in proposing this theme and for all his help and advice , and even more for all the time that he spent to follow us during the writing of this work, despite his many obligations.
Our sincere thanks also go to the members of the jury for their interest in research by agreeing to examine our work and enrich it with their proposals.
Our thanks also extend to all our teachers during the years of study .
At the end we would like to thank all the people who have participated directly or indirectly in the realisation of this work.

Abstract

This memoir is devoted to study an inverse problem for one-dimensional degenerate parabolic equation. More precisely, we have established a Lipschitz stability for the identification of the source term using Carleman inequality and we have recovered also the initial condition through the logarithmic convexity theory.

Introduction

In this memoir, we determine the source term g and the initial condition μ in the following initial boundary value problem:

$$\begin{cases} u_t - (x^\alpha u_x)_x = g(t,x), & (t,x) \in Q_T, \\ u(t,1) = 0, & t \in (0,T), \\ \text{and} \begin{array}{l} u(t,0) = 0, \\ (x^\alpha u_x)(t,0) = 0, \end{array} & \begin{array}{l} \text{for } 0 \le \alpha < 1 \\ \text{for } 1 \le \alpha < 2 \end{array}, t \in (0,T), \\ u(0,x) = \varphi(x), & x \in (0,1) \end{cases} \tag{1}$$

where $T > 0$, $Q_T = (0,T) \times (0,1)$.

More precisely, our problem can be stated as follows: It is a possible to determine the term source g and the initial condition μ from the following measurement:

$$u_t|_{\omega_T^{t_0} = (t_0,T)\times\omega} \text{ and } \left(x^\alpha u_x\right)_x (T',.)|_{(0,1)}$$

where $\omega = (a,b)$ is a subdomain of $(0,1)$ and $T' = \dfrac{t_0 + T}{2}$ with $t_0 \in (0,T)$?

The above problem arising in climatology. This climate model aims at understanding the effect of many parameters on the ice covering of the earth surface. It takes the form of some one-dimensional nonlinear parabolic problem with degenerate diffusion.

The key ingredient to prove our results is based on a Carleman estimate for a locally distributed observation.

The memoir is organized as follows. In chapter 1, we study the functional framework and the well-posedness of our problem. The chapter 2 is devoted to establish the Carleman estimate corresponding to our study case. In chapter 3, we prove a Lipschitz stability and uniqueness result concerning the identification of the term source g in the problem (1). The chapter 4 is essentially devoted to determine the initial condition μ in (1). We end our memoir with a conclusion, prospects and an appendix in which we give the proof of Hardy inequality-type.

CHAPTER

1 Preliminaries and settings

Abstract

THIS deals with the well-posedness of the problem (1). The validity of the direct problem is given using specific tools in functional analysis and suitable arguments in PDE's.

1.1 Weighted Sobolev spaces

In this memoir we study the following problem:

$$\begin{cases} u_t - (x^\alpha u_x)_x = g(t,x), & (t,x) \in Q_T, \\ u(t,1) = 0, & t \in (0,T), \\ \text{and } \begin{array}{ll} u(t,0) = 0, & \text{for } 0 \le \alpha < 1 \\ (x^\alpha u_x)(t,0) = 0, & \text{for } 1 \le \alpha < 2 \end{array}, & t \in (0,T), \\ u(0,x) = \varphi(x), & x \in (0,1) \end{cases} \tag{1.1}$$

where $T > 0$, $Q_T = (0,T) \times (0,1)$.

Definition 1.1.1. *For $0 \le \alpha < 2$, we define the Hilbert space (weighted Sobolev space) $H^1_\alpha(0,1)$ by*

$$H^1_\alpha(0,1) := \left\{ u \in L^2(0,1); \quad x^{\frac{\alpha}{2}} u_x \in L^2(0,1) \right\}$$

endowed with the following inner product

$$(u,v)_{H^1_\alpha(0,1)} := (u,v)_{L^2(0,1)} + \left(x^{\frac{\alpha}{2}} u_x, x^{\frac{\alpha}{2}} v_x \right)_{L^2(0,1)}$$

and the associated norm

$$\|u\|_{H^1_\alpha(0,1)} = \sqrt{(u,u)_{H^1_\alpha(0,1)}}$$

It is easy to see that $H^1_\alpha(0,1) \subset H^1_{loc}(0,1)$ and thus the functions belonging to $H^1_\alpha(0,1)$ are continuous in $(0,1)$. In this case $0 \le \alpha < 1$, we can define the following space

$$H^1_{\alpha,0}(0,1) := \{ u \in H^1_\alpha(0,1), \quad u(0) = u(1) = 0 \}$$

and $H^1_{\alpha,0}(0,1)$ is equipped with the inner product of $H^1_\alpha(0,1)$

However, in the case $1 \le \alpha < 2$, the boundary value at $x = 0$ for an element of $H^1_\alpha(0,1)$ does not exist any more. That is why, the definition is changed into

$$H^1_{\alpha,0}(0,1) := \{ u \in H^1_\alpha(0,1), \quad u(1) = 0 \}.$$

We can now define the continuous bilinear form $a \in \mathscr{L}\left(H^1_{\alpha,0}(0,1), H^1_{\alpha,0}(0,1)\right)$ by

$$a : \begin{cases} H^1_{\alpha,0} \times H^1_{\alpha,0} \to \mathbb{R} \\ (v,w) \longmapsto \int_0^1 x^{\frac{\alpha}{2}} v_x x^{\frac{\alpha}{2}} w_x dx. \end{cases}$$

The form a is $H^1_{\alpha,0}(0,1) - L^2(0,1)$ coercive:

$$\exists \kappa > 0, \exists \beta \in \mathbb{R}, \forall v \in H^1_{\alpha,0}(0,1), \quad a(v,w) + \beta \|v\|^2_{L^2(0,1)} \ge \kappa \|v\|^2_{H^1_\alpha(0,1)}.$$

Then, we define the unbounded operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ as follows

$$D(A) = \left\{ v \in H^1_{\alpha,0}(0,1), \quad \exists C > 0, \ a(v,w) \le C \|w\|^2_{L^2(0,1)}, \quad \forall w \in H^1_{\alpha,0}(0,1) \right\},$$

$$\forall v \in D(A), \ \forall w \in H^1_{\alpha,0}(0,1), \qquad (Av, w)_{L^2(0,1)} = a(v,w).$$

Hence,

$$H^1_{\alpha,0}(0,1) \subset L^2(0,1) \subset \left(H^1_{\alpha,0}(0,1)\right)',$$

where is the dual of $H^1_{\alpha,0}(0,1)$.

We can characterize the operator $(A, D(A))$ as follows in order to make the analysis of A easier (see [CTY:10])

Lemma 1.1.1. *For $0 \le \alpha < 2$, $D(A)$ is given as follows*

$$D(A) = \left\{u \in H^1_{\alpha,0}(0,1), \quad x^\alpha u_x \in H^1(0,1)\right\}$$

and for all $u \in D(A)$, $Au := (x^\alpha u_x)_x$.

1.2 Well-posedness of the degenerate problem

Since the standard theory of parabolic equations does not work, we give here some regularity results corresponding to the our degenerate problem.

Lemma 1.2.1. *$(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $L^2(0,1)$. Moreover, this semigroup is analytic (see [T:79]).*

Thanks to the above lemma we have the following results: (see [CMV:05], [CMV:08],[CTY:10])

Theorem 1.2.1. *For all $u_0 \in D(A)$, for all $g \in H^1\left(0,T;L^2(0,1)\right)$, the problem has a unique solution*

$$u \in C([0,T]; D(A)) \cap C^1([0,T]; L^2(0,1)).$$

Theorem 1.2.2. *For all $u_0 \in D(A)$, for all $g \in L^2\left(0,T;L^2(0,1)\right)$, the problem has a unique solution satisfying, for all $\varepsilon > 0$,*

$$u \in C([\varepsilon,T]; D(A)) \cap H^1([\varepsilon,T]; L^2(0,1)).$$

If moreover $g \in H^1\left(0,T;L^2(0,1)\right)$, then for all $\varepsilon > 0$

$$u \in C([\varepsilon,T]; D(A)) \cap C^1([\varepsilon,T]; L^2(0,1)).$$

Chapter 2

Carleman estimate for a distributed observation

Abstract

THIS chapter is devoted to prove a global Carleman estimate in order to establish our stability result. the key ingredient is based on a suitable weight function defined on subdomain of $[0, 1]$. The statement of the global Carleman result estimate is enunciated in the paper [CTY:10] without proof. Following the same steps as those used in the case of the boundary case (see [CTY:10]), we give a detailed proof of the desired result. Compared to the boundary case, many calculations are completely different in our case.

2.1 Weight function

Fix $T>0$, $t_0\in(t_0,T)$ and define $T'=\dfrac{T+t_0}{2}$. Here, we consider an nonempty open subinterval $\omega:=(a,b)$ and we define a suitable weight function corresponding to the distributed observation.

For all $t\in(t_0,T)$, we define

$$\theta(t)=\frac{1}{(t-t_0)^4(T-t)^4} \tag{2.1}$$

and

$$p(x)=G_0-\int_0^x\frac{\phi_-(s)}{s^\alpha}e^{S(\phi_+(s))^2}ds \quad \forall x\in[0,1], \tag{2.2}$$

where G_0 and S are positive constants (to be fixed later) and ϕ_- and ϕ_+ are the two functions defined below. Let $\tilde{\omega}:=(L_1,L_2)$ where $0<\alpha<L_1<L_2<b<1$ and let ϕ_1 and ϕ_2 be two smooth cut-off functions $\phi_+,\phi_-:\mathbb{R}\to\mathbb{R}$ such that

$$\begin{cases} 0\le\phi_1(x)\le1,\ 0\le\phi_2(x)\le1, & \text{for all } x\in[0,1],\\ \phi_1(x)=1,\ \phi_2(x)=0, & \text{for all } x\in[0,L_1],\\ \phi_1(x)=0,\ \phi_2(x)=1, & \text{for all } x\in[L_2,1],\\ \phi_1(x)+\phi_2(x)>0, & \text{for all } x\in[0,1],\end{cases} \tag{2.3}$$

Next, we define, for all $x\in[0,1]$

$$\begin{aligned}\phi_+(x)&:=x\phi_1(x)+(1-x)\phi_2(x),\\ \phi_-(x)&:=x\phi_1(x)-(1-x)\phi_2(x).\end{aligned} \tag{2.4}$$

It is easy to see that for some constant $C>0$ and all $x\in[0,1]$, we have

$$|\phi_-(x)|\le Cx(1-x). \tag{2.5}$$

We have the following lemma (the proof is given in [MV:06])

Lemma 2.1.1. *If G_0 and S are large enough, then p is a well-defined strictly positive function on $[0,1]$ and the following properties hold:*

1. *there exists $m_0>0$ such that*

$$\forall x\in(0,L_1)\times(L_2,1),\quad 2x^\alpha p_{xx}+\alpha x^{\alpha-1}p_x\le-m_0;$$

2. *there exists $M_0>0$ such that*

$$\forall x\in(0,1),\quad \left|\left(x^\alpha\left(x^\alpha p_x\right)_{xx}\right)_x\right|2x^\alpha p_{xx}+\alpha x^{\alpha-1}p_x\le M_0.$$

Now, we define for $(t,x)\in(t_0,T)\times(0,1)$,

$$\sigma(t,x)=\theta(t)\,p(x), \tag{2.6}$$

and

$$\beta(t)=T+t_0-2t.$$

2.2 A global Carleman estimate

In this section, we prove a global Carleman estimate for the following problem

$$\begin{cases} z_t-(x^\alpha z_x)_x=h(t,x), & (t,x)\in Q_T^{t_0}, \\ z(t,1)=0, & t\in(t_0,T), \\ \text{and } \begin{array}{l} z(t,0)=0, \\ (x^\alpha z_x)(t,0)=0, \end{array} & \begin{array}{l}\text{for } 0\le\alpha<1 \\ \text{for } 1\le\alpha<2\end{array}, t\in(t_0,T) \end{cases} \tag{2.8}$$

where $h\in L^2\left(t_0,T;L^2(0,1)\right)$ and $Q_T^{t_0}=(t_0,T)\times(0,1)$.

The Carleman estimate is given as follows:

Proposition 2.2.1. *Let $\alpha\in[0,2[$. Then there exist $C_1=C_1(T,t_0,\omega,\alpha)>0$ and $R_0=R_0(T,t_0,\omega,\alpha)>0$ such that $\forall R\ge R_0$*

$$\begin{aligned} &\int\int_{Q_T^{t_0}}\left(R^3\theta^3x^{2-\alpha}(1-x)^2z^2+R\theta^{\frac{3}{2}}|\beta|pz^2+R\theta x^\alpha z_x^2+\frac{1}{R\theta}z_t^2\right)e^{-2R\sigma}dxdt \\ &\le C_1\left(\int\int_{Q_T^{t_0}}h^2e^{-2R\sigma}dxdt+\int\int_{\omega_T^{t_0}}R^3\theta^3z^2e^{-2R\sigma}dxdt\right), \end{aligned} \tag{2.9}$$

where $\omega_T^{t_0}=(t_0,T)\times\omega$ for all weak solutions $z\in L^2(t_0,T;D(A))\cap H^1(t_0,T;L^2(0,1))$ of (2.8).

Proof. Given a solution and a positive $R>0$, we define $w(t,x):=e^{-R\sigma(t,x)}z(t,x)$ for $a.a\ (t,x)\in Q_T^{t_0}$. By a routine calculations, we observe that w satisfies $P_R^+w+P_R^-w=he^{-R\sigma}$ where

$$\begin{aligned} P_R^+w&=R\sigma_tw+R^2x^\alpha\sigma_x^2w+(x^\alpha w_x)_x, \\ P_R^-w&=w_t+R(x^\alpha\sigma_x)_xw+2Rx^\alpha\sigma_xw_x. \end{aligned}$$

Analogously to the boundary case (see [CTY:10]), we can obtain that there exist $C_1=C_1(T,t_0,\omega,\alpha)>0$ and $R_1=R_1(T,t_0,\omega,\alpha)>0$ such that $\forall R\ge R_1$, the following inequality holds

$$\begin{aligned} &\|P_R^+w\|^2_{L^2\left(Q_T^{t_0}\right)}+\|P_R^-w\|^2_{L^2\left(Q_T^{t_0}\right)}+\int\int_{Q_T^{t_0}}R^3\theta^3x^{2-\alpha}(1-x)^2w^2dxdt \\ &+\int\int_{Q_T^{t_0}}R\theta x^\alpha w_x^2dxdt \\ \le\ & C_1\left(\int\int_{Q_T^{t_0}}h^2e^{-2R\sigma}dxdt+\int\int_{\omega_T^{t_0}}R^3\theta^3z^2e^{-2R\sigma}dxdt\right). \end{aligned} \tag{2.10}$$

Now, we devide the proof into some steps:

Step 1: Upper bounds of $\int\int_{Q_T^{t_0}}R^3\theta^3x^{2-\alpha}(1-x)^2z^2e^{-2R\sigma}dxdt$ and $\int\int_{Q_T^{t_0}}R\theta x^\alpha z_x^2e^{-2R\sigma}dxdt$.

Substituting w by $ze^{-R\sigma}$, we deduce, from (2.10), that

$$\int\int_{Q_T^{t_0}}R^3\theta^3x^{2-\alpha}(1-x)^2w^2dxdt\le C_1\left(\int\int_{Q_T^{t_0}}h^2e^{-2R\sigma}dxdt+\int\int_{\omega_T^{t_0}}R^3\theta^3z^2e^{-2R\sigma}dxdt\right).$$

In addition, $w_x = -R\sigma_x z e^{-R\sigma} + z_x e^{-R\sigma}$. Thus,

$$\int\int_{Q_T^{t_0}} R\theta x^\alpha z_x^2 e^{-2R\sigma} dxdt \le 2\int\int_{Q_T^{t_0}} R^3\theta^3 x^\alpha p_x^2 z^2 e^{-2R\sigma} dxdt + 2\int\int_{Q_T^{t_0}} R\theta x^\alpha w_x^2 dxdt.$$

Using (2.5),

$$p_x(x) = -\frac{\phi_-(x)}{x^\alpha} e^{S(\phi_+(x))^2}$$

for all $x \in [0,1]$ and the fact that $e^{S(\phi_+(x))}$ is a bounded function on $[0,1]$, one can get, there exists $C = C(T, t_0, \omega, \alpha) > 0$ such that

$$\int\int_{Q_T^{t_0}} R^3\theta^3 x^\alpha p_x^2 z^2 e^{-2R\sigma} dxdt \le C\int\int_{Q_T^{t_0}} R^3\theta^3 x^{2-\alpha}(1-x)^2 z^2 e^{-2R\sigma} dxdt.$$

Then, in view of the above inequalities, we obtain

$$\int\int_{Q_T^{t_0}} R\theta x^\alpha z^2 e^{-2R\sigma} dxdt \le C\left(\int\int_{Q_T^{t_0}} h^2 e^{-2R\sigma} dxdt + \int\int_{\omega_T^{t_0}} R^3\theta^3 z^2 e^{-2R\sigma} dxdt\right).$$

Step 2: Upper bound of $\int\int_{Q_T^{t_0}} R\theta^{\frac{3}{2}} |\beta| pz^2 e^{-2R\sigma} dxdt.$

First, we consider the following identity:

$$P_R^+ w = -4R\beta\theta^{\frac{5}{4}} pw + R^2 x^\alpha \sigma_x^2 w + \left(x^\alpha w_x\right)_x.$$

We also recall that

$$\beta(t) \ge 0 \quad t \in \left[t_0, T'\right] \text{ and } \beta(t) \le 0 \quad t \in \left[T', T\right].$$

Consequently, we can define the following mapping

$$\begin{aligned} \gamma \quad &: \quad [t_0, T] \to \mathbb{R} \\ t \quad &\mapsto \quad \begin{cases} 1, & t \in [t_0, T'), \\ -1, & t \in \left[T', T\right]. \end{cases} \end{aligned}$$

Then, for all $t \in [t_0, T]$, $\gamma(t)\beta(t) = |\beta(t)|$ and $|\gamma(t)| \le 1$. Therefore, we obtain

$$\gamma\theta^{\frac{1}{4}} P_R^+ w w = -4R\beta\theta^{\frac{3}{2}} pw^2 + \gamma\theta^{\frac{1}{4}} R^2 x^\alpha \sigma_x^2 w^2 + \gamma\theta^{\frac{1}{4}} \left(x^\alpha w_x\right)_x w.$$

Integrating by parts with respect to x, we have:

$$\begin{aligned} \int\int_{Q_T^{t_0}} R\theta^{\frac{3}{2}} |\beta| pw^2 dxdt \quad &= \quad -\frac{1}{4}\int\int_{Q_T^{t_0}} \gamma P_R^+ w\theta^{\frac{1}{4}} w dxdt + \frac{1}{4}\int\int_{Q_T^{t_0}} \gamma\theta^{\frac{1}{4}} R^2 x^\alpha \sigma_x^2 w^2 dxdt \\ &\quad -\frac{1}{4}\int\int_{Q_T^{t_0}} \gamma\theta^{\frac{1}{4}} x^\alpha w_x^2 dxdt. \end{aligned}$$

Since $|\gamma(t)| \le 1$ for all $t \in [t_0, T]$, it follows that

$$\begin{aligned} \int\int_{Q_T^{t_0}} R\theta^{\frac{3}{2}} |\beta| pw^2 dxdt \quad &\le \quad \frac{1}{4}\int\int_{Q_T^{t_0}} \left|P_R^+ w\theta^{\frac{1}{4}} w\right| dxdt + \frac{1}{4}{}_{Q_T^{t_0}} \theta^{\frac{1}{4}} R^2 x^\alpha \sigma_x^2 w^2 dxdt \\ &\quad + \frac{1}{4}\int\int_{Q_T^{t_0}} \theta^{\frac{1}{4}} x^\alpha w_x^2 dxdt. \end{aligned}$$

On the other hand, we have

$$\frac{1}{4}\int\int_{Q_T^{t_0}}\left|P_R^+ w\theta^{\frac{1}{4}}w\right|dxdt \leq \underbrace{\frac{1}{8}\left\|P_R^+ w\right\|^2_{L^2\left(Q_T^{t_0}\right)}}_{J_1}+\underbrace{\frac{1}{8}\int\int_{Q_T^{t_0}}\theta^{\frac{1}{2}}w^2dxdt}_{J_2}.$$

Moreover,

$$\begin{aligned}\frac{1}{4}\int\int_{Q_T^{t_0}}\theta^{\frac{1}{4}}R^2x^\alpha\sigma_x^2w^2dxdt &= \frac{1}{4}\int\int_{Q_T^{t_0}}\theta^{\frac{9}{4}}R^2x^\alpha p_x^2w^2dxdt\\ &= \frac{1}{4}\int\int_{Q_T^{t_0}}\theta^{\frac{9}{4}}R^2x^\alpha\left(-\frac{\phi_-(x)}{x^\alpha}e^{S(\phi_+(x))^2}\right)^2w^2dxdt\\ &\leq \frac{C}{4}\int\int_{Q_T^{t_0}}\theta^{\frac{9}{4}}R^2x^{2-\alpha}(1-x)^2w^2dxdt.\end{aligned}$$

Consequently,

$$\begin{aligned}\int\int_{Q_T^{t_0}}R\theta^{\frac{3}{2}}|\beta|pw^2dxdt &\leq J_1+J_2+\frac{C}{4}\int\int_{Q_T^{t_0}}\theta^{\frac{9}{4}}R^2x^{2-\alpha}(1-x)^2w^2dxdt\\ &\quad+\frac{1}{4}\int\int_{Q_T^{t_0}}\theta^{\frac{1}{4}}x^\alpha w_x^2dxdt.\end{aligned}$$

Using the definition of θ, we deduce that there exists a constant $C=C(t_0,T)>0$ such that for all $t\in(t_0,T)$, $\theta^{\frac{9}{4}}(t)\leq C\theta^3(t)$ and $\theta^{\frac{1}{4}}(t)\leq C\theta(t)$. Consequently, for R large enough and utilizing (2.10), we get

$$\begin{aligned}&J_1+\frac{C}{4}\int\int_{Q_T^{t_0}}\theta^{\frac{9}{4}}R^2x^{2-\alpha}(1-x)^2w^2dxdt+\frac{1}{4}\int\int_{Q_T^{t_0}}\theta^{\frac{1}{4}}x^\alpha w_x^2dxdt\\ \leq\; & C\left(\int\int_{Q_T^{t_0}}h^2e^{-2R\sigma}dxdt+\int\int_{\omega_T^{t_0}}R^3\theta^3z^2e^{-2R\sigma}dxdt\right).\end{aligned}$$

We deal now with the term J_2. Let us consider $\alpha^*\in(\max(1,\alpha),2)$. By using the fact that $x\in(0,1)$, $\dfrac{x^{\alpha^*}}{x^2(1-x)^2}\geq 1$ and Hardy inequality (see Appendix A), we obtain

$$\begin{aligned}\int\int_{Q_T^{t_0}}\theta^{\frac{1}{2}}w^2dxdt &\leq \int\int_{Q_T^{t_0}}\theta^{\frac{1}{2}}\frac{x^{\alpha^*}}{x^2(1-x)^2}w^2dxdt\\ &\leq C(\alpha^*)\int\int_{Q_T^{t_0}}\theta^{\frac{1}{2}}x^{\alpha^*}w_x^2dxdt\\ &\leq C\left(t_0,T,\alpha^*\right)\int\int_{Q_T^{t_0}}\theta x^\alpha w_x^2dxdt,\end{aligned}$$

where we have used, in the last inequality, $\theta^{\frac{1}{2}}(t)\leq C\left(t_0,T\right)\theta(t)$ and $x^{\alpha^*}\leq x^\alpha$ for all $x\in[0,1]$.

Then, by(2.10) and R large enough, we get

$$\int\int_{Q_T^{t_0}} R\theta^{\frac{3}{2}}|\beta| pw^2 dxdt \leq C(t_0, T, \alpha)\left(\int\int_{Q_T^{t_0}} h^2 e^{-2R\sigma} dxdt + \int\int_{\omega_T^{t_0}} R^3\theta^3 z^2 e^{-2R\sigma} dxdt\right),$$

or, equivalently,

$$\int\int_{Q_T^{t_0}} R\theta^{\frac{3}{2}}|\beta| pz^2 e^{-2R\sigma} dxdt \leq C(t_0, T, \alpha)\left(\int\int_{Q_T^{t_0}} h^2 e^{-2R\sigma} dxdt + \int\int_{\omega_T^{t_0}} R^3\theta^3 z^2 e^{-2R\sigma} dxdt\right).$$

Step 3: Upper bound of $\int\int_{Q_T^{t_0}} \frac{1}{R\theta} w_t^2 dxdt.$

Using the definition of $P_R^- w$, we write

$$\begin{aligned} P_R^- w &= w_t + R(x^\alpha \sigma_x)_x w + 2Rx^\alpha \sigma_x w_x \\ &= w_t + R\theta(x^\alpha p_x)_x w + 2\theta R x^\alpha p_x w_x. \end{aligned}$$

Let us introduce the following function: $\kappa(x) := x^\alpha p_x(x)$, for all $x \in (0,1)$. Then,

$$P_R^- w = w_t + R\theta\kappa_x w + 2\theta R x^\alpha p_x w_x.$$

Hence,

$$\frac{1}{\sqrt{R\theta}} w_t = -\sqrt{R\theta}\kappa_x w - 2\sqrt{R\theta} x^\alpha p_x w_x + \frac{P_R^- w}{\sqrt{R\theta}}.$$

It easy to see, through (2.2) that pour tout $x \in (0,1$,

$$\kappa(x) = x^\alpha p_x(x) = -\phi_-(x) e^{S(\phi_+(x))^2}.$$

This implies that κ_x is bounded on $[0,1]$. Since $\frac{1}{\sqrt{\theta}}$ is also bounded on $[t_0, T]$. It follows that, there exists $C = C(t_0, T, \alpha) > 0$ such that R large enough,

$$\begin{aligned} \int\int_{Q_T^{t_0}} \frac{1}{R\theta} w_t^2 dxdt &\leq C(t_0, T, \alpha,)\left(\int\int_{Q_T^{t_0}} R\theta w^2 dxdt + \int\int_{Q_T^{t_0}} R\theta x^{2\alpha} p_x^2 w_x^2 + \|P_R^- w\|^2_{L^2(Q_T^{t_0})}\right) \\ &= C(t_0, T, \alpha)\left(\int\int_{Q_T^{t_0}} R\theta w^2 dxdt + \int\int_{Q_T^{t_0}} R\theta x^\alpha (x^\alpha p_x^2) w_x^2 + \|P_R^- w\|^2_{L^2(Q_T^{t_0})}\right). \end{aligned}$$

Since $x \in (0,1)$ and $0 \leq \alpha < 2$ and using 2.5, there exists $C > 0$ such that

$$x^\alpha p_x^2 \leq C x^{2-\alpha}(1-x)^2 \leq C.$$

Hence,

$$\int\int_{Q_T^{t_0}} \frac{1}{R\theta} w_t^2 dxdt \leq C(t_0, T, \alpha)\left(\int\int_{Q_T^{t_0}} R\theta w^2 dxdt + \int\int_{Q_T^{t_0}} R\theta x^\alpha w_x^2 + \|P_R^- w\|^2_{L^2(Q_T^{t_0})}\right)$$

Using again Hardy inequality for $\alpha^* \in (\max(1,\alpha),2)$, one can get

$$\begin{aligned}\int\int_{Q_T^{t_0}} R\theta w^2 dxdt &\leq \int\int_{Q_T^{t_0}} R\theta \frac{x^{\alpha^*}}{x^2(1-x)^2} w^2 dxdt \\ &\leq C(\alpha^*)\int\int_{Q_T^{t_0}} R\theta x^{\alpha^*} w_x^2 dxdt \leq C(t_0,T,\alpha^*)\int\int_{Q_T^{t_0}} R\theta x^{\alpha} w_x^2 dxdt.\end{aligned}$$

Consequently,

$$\int\int_{Q_T^{t_0}} R\theta w^2 dxdt \leq C(t_0,T,\alpha)\left(\int\int_{Q_T^{t_0}} R\theta x^{\alpha} w_x^2 dxdt + \int\int_{Q_T^{t_0}} R\theta x^{\alpha} w_x^2 + \|P_R^- w\|^2_{L^2(Q_T^{t_0})}\right).$$

Taking R large enough, (2.10) allows us to obtain

$$\int\int_{Q_T^{t_0}} \frac{1}{R\theta} w_t^2 dxdt \leq C(t_0,T,\alpha)\left(\int\int_{Q_T^{t_0}} h^2 e^{-2R\sigma} dxdt + \int\int_{\omega_T^{t_0}} R^3\theta^3 z^2 e^{-2R\sigma} dxdt\right).$$

Step 4: Upper bound of $\int\int_{Q_T^{t_0}} \frac{1}{R\theta} z_t^2 e^{-2R\sigma} dxdt$.

We have $w_t = z_t e^{-R\sigma} - R\sigma_t w$. Therefore,

$$\int\int_{Q_T^{t_0}} \frac{1}{R\theta} z_t^2 e^{-2R\sigma} dxdt \leq 2\left(\int\int_{Q_T^{t_0}} \frac{1}{R\theta} w_t^2 dxdt + \int\int_{Q_T^{t_0}} \frac{R^2\sigma_t^2}{R\theta} w^2 dxdt\right).$$

For all $(t,x) \in Q_T^{t_0}$, we have:

$$R\sigma_t(t,x) = R\theta_t(t)\, p(x) = -4R\beta(t)\theta^{\frac{5}{4}}(t)\, p(x).$$

Then,

$$\begin{aligned}\int\int_{Q_T^{t_0}} \frac{R^2\sigma_t^2}{R\theta} w^2 dxdt &= 16\int\int_{Q_T^{t_0}} R\theta^{\frac{3}{2}}\beta^2 p^2 w^2 dxdt \\ &\leq C(t_0,T,\alpha)\int\int_{Q_T^{t_0}} R\theta^{\frac{3}{2}}|\beta|\, p w^2 dxdt\end{aligned}$$

where $C(t_0,T,\alpha) = \sup_{(t,x)\in[0,1]\times[t_0,T]} |\beta(t)|\, p(x)$. It follows that

$$\int\int_{Q_T^{t_0}} \frac{1}{R\theta} z_t^2 e^{-2R\sigma} dxdt \leq C(t_0,T,\alpha)\left(\int\int_{Q_T^{t_0}} h^2 e^{-2R\sigma} dxdt + \int\int_{\omega_T^{t_0}} R^3\theta^3 z^2 e^{-2R\sigma} dxdt\right).$$

Combining all the four steps, we obtain the desired Carleman estimate, which conclude the proof. ■

Chapter

3 Inverse source problem for a degenerate PDE

Abstract

THIS chapter deals with the determination of source term for one-dimensional degenerate parabolic equation using distributed measurements. We establish a Lipschitz stability and uniqueness result for this inverse problem using the Carleman estimate proved in the previous chapter. It is worth mentioning that this chapter gives detailed and adapted proofs with respect to those given in [CTY:10].

3.1 Statement of the inverse problem

We recall the problem (1.1):

$$\begin{cases} u_t - (x^\alpha u_x)_x = g(t,x), & (t,x) \in Q_T, \\ u(t,1) = 0, & t \in (0,T), \\ \text{and } \begin{array}{l} u(t,0) = 0, \\ (x^\alpha u_x)(t,0) = 0, \end{array} & \begin{array}{l} \text{for } 0 \le \alpha < 1 \\ \text{for } 1 \le \alpha < 2 \end{array}, t \in (0,T), \\ u(0,x) = \varphi(x), & x \in (0,1) \end{cases}$$

where $T > 0$, $Q_T = (0,T) \times (0,1)$.

Let $C_0 > 0$. Let us introduce the following condition on source term:

$$\left| \frac{\partial g}{\partial t}(t,x) \right| \le C_0 \left| g(T',x) \right| \quad \text{for almost all (a.a.)} (t,x) \in (0,T) \times (0,1). \tag{3.1}$$

Let now

$$\mathcal{G}(C_0) = \left\{ g \in H^1\left(0,T; L^2(0,1)\right), \quad g \text{ satisfies (3.1)} \right\}$$

Let $r : [0,T] \times [0,1] \to \mathbb{R}$ be a given function of class $C^1([0,T] \times [0,1])$ and $d > 0$ be a positive constant such that

$$\forall x \in [0,1], \quad r(T',x) > d. \tag{3.2}$$

We have the following lemma

Lemma 3.1.1. *The space $\mathcal{E} := \{ rf \mid f \in L^2(0,1)$ is included in $\mathcal{G}(C_0)$ with*

$$C_0 := \frac{\sup\limits_{(t,x) \in \overline{Q_T}} \left| \dfrac{\partial r}{\partial t}(t,x) \right|}{d}.$$

Proof. Let $f \in L^2(0,1)$ and $g = rf \in \mathcal{E}$. Obviously, we have $g \in H^1(0,T; L^2(0,1))$. Then for all $t \in [0,T]$ and for almost all $x \in (0,1)$,

$$\begin{aligned} \left| \frac{\partial g}{\partial t}(t,x) \right| = \left| \frac{\partial r}{\partial t}(t,x) f(x) \right| &\le \sup_{(t,y) \in \overline{Q_T}} \left| \frac{\partial r}{\partial t}(t,y) \right| \left| f(x) \right| \\ &\le \frac{\sup_{(t,y) \in \overline{Q_T}} \left| \frac{\partial r}{\partial t}(t,y) \right|}{d} r(T',x) \left| f(x) \right| = C_0 \left| g(T',x) \right|. \end{aligned}$$

■

In the sequel, we prove a Lipschitz stability and uniqueness result for the coefficient f in the following problem

$$\begin{cases} u_t - (x^\alpha u_x)_x = r(t,x) f(x), & (t,x) \in Q_T, \\ u(t,1) = 0, & t \in (0,T), \\ \text{and } \begin{array}{l} u(t,0) = 0, \\ (x^\alpha u_x)(t,0) = 0, \end{array} & \begin{array}{l} \text{for } 0 \le \alpha < 1 \\ \text{for } 1 \le \alpha < 2 \end{array}, t \in (0,T), \\ u(0,x) = \varphi(x), & x \in (0,1) \end{cases} \tag{3.3}$$

where $r \in C^1([0,T] \times [0,1])$ and satisfies (3.2).

3.2 Main result

Theorem 3.2.1. *Let $\alpha \in [0,2)$, $u_0 \in L^2(0,1)$ and let $\omega := (a,b)$ with $0<a<b<1$. Then, there exists $C = C(T, t_0, C_0, \omega, \alpha) > 0$ such that for all $f_1 \in L^2(0,1)$ and for all $f_2 \in L^2(0,1)$, the weak solutions u_1 and u_2 of (3.3) satisfy*

$$\|f_1 - f_2\|^2_{L^2(0,1)} \leq C\left(\left\|\left(x^\alpha u_{1,x}\right)_x(T',.) - \left(x^\alpha u_{2,x}\right)_x(T',.)\right\|^2_{L^2(0,1)} + \|u_{1,t} - u_{2,t}\|^2_{L^2\left(\omega_T^{t_0}\right)}\right).$$

Proof. Let $u_0 \in L^2(0,1)$ and $f_1, f_2 \in L^2(0,1)$. Let us consider the weak solutions u_1 and u_2 of problem (3.3). According to the regularity of r, we have $u_1, u_2 \in C([\varepsilon, T]; D(A)) \cap C^1\left([\varepsilon, T]; L^2(0,1)\right)$ for all $\varepsilon > 0$. Consequently, $w = u_1 - u_2 \in C([\varepsilon, T]; D(A)) \cap C^1\left([\varepsilon, T]; L^2(0,1)\right)$ is the weak solution of the following problem

$$\begin{cases} w_t - (x^\alpha w_x)_x = r(t,x)\left(f_1(x) - f_1(x)\right), & (t,x) \in Q_T, \\ w(t,1) = 0, & t \in (0,T), \\ \text{and } \begin{matrix} w(t,0) = 0, \\ (x^\alpha w_x)(t,0) = 0, \end{matrix} & \begin{matrix} \text{for } 0 \leq \alpha < 1 \\ \text{for } 1 \leq \alpha < 2 \end{matrix}, t \in (0,T), \\ w(0,x) = 0, & x \in (0,1) \end{cases} \tag{3.4}$$

If we define $z = w_t$, z is a solution of the following problem:

$$\begin{cases} z_t - (x^\alpha z_x)_x = \dfrac{\partial r(t,x)}{\partial t}\left(f_1(x) - f_2(x)\right), & (t,x) \in Q_T, \\ z(t,1) = 0, & t \in (0,T), \\ \text{and } \begin{matrix} z(t,0) = 0, \\ (x^\alpha z_x)(t,0) = 0, \end{matrix} & \begin{matrix} \text{for } 0 \leq \alpha < 1 \\ \text{for } 1 \leq \alpha < 2 \end{matrix}, t \in (0,T), \\ z(0,x) = r(0,x)\left(f_1(x) - f_1(x)\right), & x \in (0,1) \end{cases} \tag{3.5}$$

where $r \in C^1([0,T] \times [0,1])$ and satisfies (3.2).

Using the results of the chapter 1, we have

$$z \in L^2(\varepsilon, T; D(A)) \cap H^1\left(\varepsilon, T; L^2(0,1)\right), \quad \forall \varepsilon > 0.$$

In particular, $z \in L^2(t_0, T; D(A)) \cap H^1\left(t_0, T; L^2(0,1)\right)$ is thus a solution of (2.8) with

$$h(t,x) = \frac{\partial r(t,x)}{\partial t}\left(f_1(x) - f_2(x)\right).$$

Hence, we can apply Carleman estimate (2.9) and we obtain, for all $R \geq R_0$,

$$\begin{aligned} I_0 &= \int\int_{Q_T^{t_0}} (R^3\theta^3 x^{2-\alpha}(1-x)^2 z^2 + R\theta^{\frac{3}{2}} |\beta| p z^2 + R\theta x^\alpha z_x^2 + \frac{1}{R\theta} z_t^2) e^{-2R\sigma}\, dx\, dt \\ &\leq C_1 \underbrace{\left(\int\int_{Q_T^{t_0}} (\frac{\partial r}{\partial t}(f_1 - f_2))^2 e^{-2R\sigma}\, dx\, dt + \int\int_{\omega_T^{t_0}} R^3\theta^3 z^2 e^{-2R\sigma}\, dx\, dt\right)}_{=I_1} \end{aligned} \tag{3.6}$$

We divide the proof in three steps :

(a) There exists a constant $C = C(T, t_0, C_0, \omega) > 0$ such that

$$I_1 \leq C\left[\frac{1}{\sqrt{R}}\int_0^1 r^2\left(T', x\right)\left(f_1 - f_2\right)^2(x)\, e^{-2R\sigma(T',x)}dx + \left\|u_{1,t} - u_{2,t}\right\|^2_{L^2\left(\omega_T^{t_0}\right)}\right].$$

(b) There exists a constant $C = C(T, t_0, C_0, \omega, \alpha) > 0$ such that

$$\int_0^1 z^2(T', x)e^{-2R\sigma(T',x)}dx \leq C I_0.$$

(c) Conclusion.

Proof of step (a):

Lemma 3.2.1. *There exists a constant $C = C(T, t_0, \omega) > 0$ such that*

$$\int\int_{\omega_T^{t_0}} R^3\theta^3 z^2 e^{-2R\sigma} dx dt \leq C\left\|u_{1,t} - u_{2,t}\right\|^2_{L^2\left(\omega_T^{t_0}\right)}.$$

Proof. Since p is a positive continuous function on $[0,1]$, there exists $p_0 := \inf_{x\in[0,1]} p(x) > 0$. Then, we have for all $(t, x) \in (t_0, T) \times (0,1)$,

$$R^3\theta^3(t)e^{-2R\sigma(t,x)} \leq R^3\theta^3(t)e^{-2p_0R\theta(t)} \leq L = \max_{x\geq 0}\{x^3 e^{-2p_0 x}\}.$$

Consequently,

$$\int\int_{\omega_T^{t_0}} R^3\theta^3 z^2 e^{-2R\sigma(t,x)} dx dt \leq L\int\int_{\omega_T^{t_0}} z^2 dx dt = L\parallel w_t\parallel^2_{L^2(\omega_T^{t_0})}.$$

On the other hand, $u_1, u_2 \in C([\varepsilon, T]; D(A)) \cap C^1([\varepsilon, T]; L^2(0,1))$, then $w_t = u_{1,t} - u_{2,t}$ is valid on $\omega_T^{t_0}$ and the proof of lemma is finished. ■

Since $g = r\left(f_1 - f_2\right) \in \mathcal{G}(C_0)$, we have for almost all $(t, x) \in Q_T^{t_0}$, $\left|g_t(t, x)\right| \leq C_0\left|g(T', x)\right|$. Thus, taking into account the above lemma, we obtain

$$I_1 \leq C_0^2\int\int_{Q_T^{t_0}} r^2(T', x)\left(f_1 - f_2\right)^2(x)\, e^{-2R\sigma(t,x)}dx dt + C\parallel u_{1,t} - u_{2,t}\parallel^2_{L^2(\omega_T^{t_0})}.$$

Lemma 3.2.2. *There exists $C(t_0, T) > 0$ such that*

$$\int\int_{Q_T^{t_0}} r^2(T', x)\left(f_1 - f_2\right)^2(x)\, e^{-2R\sigma(t,x)}dx dt \leq C(t_0, T)\frac{1}{\sqrt{R}}\int_0^1 r^2(T', x)\left(f_1 - f_2\right)^2(x)\, e^{-2R\sigma(T',x)}dx.$$

Proof. Let $l(t) = (t - t_0)(T - t)$ $\forall t \in [t_0, T]$, then for all $(t, x) \in (t_0, T) \times [0,1]$, we can write $\sigma(t, x) = \dfrac{p(x)}{l(t)^4}$. Thus we have

$$\frac{\partial\sigma(t, x)}{\partial t} = -4p(x)\frac{l'(t)}{l(t)^5},$$

with $l'(t) = T + t_0 - 2t$. Since

$$l'(T') = l'(\frac{t_0 + T}{2}) = 0,$$

we get

$$\frac{\partial \sigma(T', x)}{\partial t} = 0.$$

On the other hand, for all $t \in (t_0, T)$, we have

$$\frac{\partial^2 \sigma(t,x)}{\partial t^2} = \rho(x) \frac{20 l'(t)^2 - 4 l''(t) l(t)}{l(t)^6}.$$

As $l''(t) = -2$, $\forall t \in [t_0, T]$, it follows that

$$\frac{\partial^2 \sigma(t,x)}{\partial t} = p(x) \frac{20 l'(t)^2 + 8 l(t)}{l(t)^6} \tag{3.7}$$

By applying Taylor's expansion to the function $t \to \sigma(t,x)$ on the interval $[T', t]$ there exists a function $\theta(t,x)$ with $T' < \theta(t,x) < t$ such that

$$\sigma(t,x) = \sigma(T',x) + \frac{\partial \sigma(T',x)}{\partial t}(t - T') + \frac{1}{2!} \frac{\partial^2 \sigma}{\partial t^2}(\theta(t,x), x)(t - T')^2,$$

which yields

$$\sigma(t,x) = \sigma(T',x) + \frac{1}{2} \frac{\partial^2 \sigma}{\partial t^2}(\theta(t,x), x)(t, T')^2. \tag{3.8}$$

Using the fact that $T' = \frac{t_0 + T}{2}$ is the minimum of the function $\theta = \frac{1}{l(t)^4}$, we get $\theta(t) \geq \theta(T'), \forall t \in (t_0, T)$. In addition, $p(x) \geq p_0 > 0$, $\forall x \in [0.1]$, then we obtain

$$\frac{\partial^2 \sigma(t,x)}{\partial t^2} \geq p_0 \frac{8}{l(t)^5} = 8 p_0 \frac{1}{l(t)^4} \times \frac{1}{l(t)} = 8 p_0 \theta(t) \times \theta(t)^{\frac{1}{4}} \tag{3.9}$$

$$= 8 p_0 \theta(t)^{\frac{5}{4}} \geq 8 p_0 \theta(T')^{\frac{5}{4}} := \mu_0(t_0, T) = \mu_0 \tag{3.10}$$

Combining (3.7) and (3.9), we get

$$\sigma(t,x) \geq \sigma(T',x) + \frac{(t - T')^2}{2} \mu_0 \Rightarrow -\sigma(t,x) \leq -\sigma(T',x) - \frac{(t - T')^2}{2} \mu_0.$$

Consequently, for all $x \in [0.1]$,

$$\begin{aligned}
\int_{t_0}^{T} e^{-2R\sigma(t,x)} dt &\leq e^{-2R\sigma(T',x)} \int_{t_0}^{T} e^{-\mu_0 R (t - T')^2} dt \\
&\leq e^{-2R\sigma(T',x)} \int_{-\infty}^{+\infty} e^{-\mu_0 R (t - T')^2} dt \\
&= e^{-2R\sigma(T',x)} \frac{1}{\sqrt{\mu_0 R}} \int_{-\infty}^{+\infty} e^{-u^2} du \\
&= C(t_0, T) \frac{1}{\sqrt{R}} e^{-2R\sigma(T',x)}
\end{aligned}$$

where

$$C(t_0,T)=\frac{1}{\mu_0}\int_{-\infty}^{+\infty}e^{-u^2}du=\sqrt{\frac{\pi}{\mu_0}}.$$

Multiplying the above inequality by $r^2(T',x)\left(f_1-f_2\right)^2(x)$ and integrating over (0.1), we obtain the desired resut and the proof of the lemma is achieved. ■

Proof of step (b): Since for $x\in(0,1)$,

$$\lim_{t\to t_0}z^2(t,x)e^{-2R\sigma(T',x)}=0,$$

we have

$$\int_0^1 z^2(T',x)e^{-2R\sigma(T',x)}dx=\int_{t_0}^{T'}\frac{\partial}{\partial t}(\int_0^1 z^2(t,x)e^{-2R\sigma(t,x)}dxdt) \tag{3.11}$$
$$=\underbrace{\int_{t_0}^{T'}\int_0^1[2zz_t-2R\sigma_t z^2]e^{-2R\sigma(t,x)}dxdt.}_{=J}$$

We start to find an upper bound of J.

$$\begin{aligned}\int_{t_0}^{T'}\int_0^1 2zz_te^{-2R\sigma(t,x)}dxdt&=\int_{t_0}^{T'}\int_0^1 2\sqrt{R\theta}ze^{-R\sigma(t,x)}\frac{1}{\sqrt{R\theta}}z_te^{-R\sigma(t,x)}dxdt\\&\le\int_{t_0}^{T'}\int_0^1[R\theta z^2e^{-2R\sigma(t,x)}+\frac{z_t^2}{R\theta}e^{-2R\sigma(t,x)}]dxdt\\&\le\int_{t_0}^{T}\int_0^1[R\theta z^2e^{-2R\sigma(t,x)}+\frac{z_t^2}{R\theta}e^{-2R\sigma(t,x)}]dxdt.\end{aligned} \tag{3.12}$$

since $T'\le T$.

The following lemma is useful before applying Hardy inequality (see Appendix A).

Lemma 3.2.3. *Let* $\alpha^*\in(\max(1,\alpha),2)$. *Then, for a.a.* $t\in(t_0,T)$, $x\to z(t,x)\,e^{-R\sigma(t,x)}$ *belongs to* $H^1_{\alpha^*,0}(0,1)$.

Proof. First of all, for *a.a.* $t\in(t_0,T)$, we have

$$\lim_{x\to1^-}z(t,x)e^{-R\sigma(T',x)}=0$$

since for a.a. $t\in(t_0,T)$, $z(t,.)\in D(A)$. We must show now that $\int_0^1 x^{\alpha^*}\left(\left(ze^{-R\sigma}\right)_x\right)^2dx<\infty$. We have

$$\left(ze^{-R\sigma}\right)_x=z_xe^{-R\sigma}-R\sigma_xze^{-R\sigma}.$$

Hence,

$$\begin{aligned}\left(\left(ze^{-R\sigma}\right)_x\right)^2&\le 2z_x^2e^{-2R\sigma}+2R^2\sigma_x^2z^2e^{-2R\sigma}\\&\le 2z_x^2+2R^2\sigma_x^2z^2.\end{aligned}$$

Moreover, $\int_0^1 x^\alpha z_x^2 dx < \infty$, for almost all $t \in (t_0, T)$. Then

$$\int_0^1 x^{\alpha^*} z_x^2 dx \le \int_0^1 x^\alpha z_x^2 dx < \infty,$$

since $\alpha^* \le \alpha$. In addition, we have

$$\sigma_x = \theta p_x = -\theta \frac{\phi^-}{x^\alpha} e^{S(\phi^+)^2}.$$

This yields

$$\begin{aligned}\int_0^1 x^{\alpha^*} R^2 \sigma_x^2 z^2 dx &\le (\theta R)^2 C \int_0^1 x^{\alpha^*} x^{2-2\alpha} (1-x)^2 z^2 dx\\ &\le (\theta R)^2 C \int_0^1 z^2 dx,\end{aligned}$$

where we have used the fact that $2-\alpha+\alpha^*-\alpha > 0$. Since for all $t \in (t_0, T)$, $z(t,.) \in L^2(0,1)$, the proof of the lemma follows. ∎

Let $\alpha^* \in (\max(1,\alpha), 2)$. Through the above lemma, we can apply Hardy inequality to the function $f = z(t,.)e^{-R\sigma(t,.)}$ and we obtain

$$\begin{aligned}\int_0^1 \frac{x^{\alpha^*}}{x^2(1-x)^2} z^2 e^{-2R\sigma} dx &\le C(\alpha^*) \int_0^1 x^{\alpha^*} ((ze^{-R\sigma})_x)^2 dx.\\ &\le C(\alpha^*) \int_0^1 x^{\alpha^*} [z_x^2 e^{-2R\sigma} + R^2\theta^2 p_x^2 z^2 e^{-2R\sigma}] dx.\end{aligned}$$

Since $\dfrac{x^{\alpha^*}}{x^2(1-x)^2} \ge 1$, we also obtain

$$\int_{t_0}^T \int_0^1 R\theta z^2 e^{-2R\sigma} dxdt \le \int_{t_0}^T R\theta \int_0^1 \frac{x^{\alpha^*}}{x^2(1-x)^2} z^2 e^{-2R\sigma} dxdt,$$

Hence

$$\int_{t_0}^T \int_0^1 R\theta z^2 e^{-2R\sigma} dxdt \le C(\alpha^*)[\int_{t_0}^T \int_0^1 R\theta x^{\alpha^*} z_x^2 e^{-2R\sigma} dxdt + \int_{t_0}^T \int_0^1 R^3\theta^3 p_x^2 x^{\alpha^*} z^2 e^{-2R\sigma} dxdt],$$

On the other hand, since $x^{\alpha^*} \le x^\alpha$, we get

$$\int_{t_0}^T \int_0^1 R\theta z^2 e^{-2R\sigma} dxdt \le C(\alpha^*)[\int_{t_0}^T \int_0^1 R\theta x^{\alpha} z_x^2 e^{-2R\sigma} dxdt + \int_{t_0}^T \int_0^1 R^3\theta^3 p_x^2 x^{\alpha} z^2 e^{-2R\sigma} dxdt].$$

Using (2.5), we have

$$| p_x(x) | = | -\frac{\phi_-(x)}{x^\alpha} e^{s(\phi_+(x))^2} | \le C \frac{x(1-x)}{x^\alpha}$$

and we obtain

$$\int_{t_0}^T \int_0^1 R^3\theta^3 \rho_x^2 x^\alpha z^2 e^{-2R\sigma} dxdt \le C \int_{t_0}^T \int_0^1 R^3\theta^3 x^{2-\alpha} (1-x)^2 z^2 e^{-2R\sigma} dxdt.$$

Combining the above inequalities, we conclude that

$$\int_{t_0}^{T'}\int_0^1 2zz_t e^{-2R\sigma}dxdt \leq C(\int_{t_0}^{T}\int_0^1 R\theta x^{\alpha} z_x^2 e^{-2R\sigma}dxdt+\int_{t_0}^{T}\int_0^1 R^3\theta^3 x^{2-\alpha}(1-x)^2 z^2 e^{-2R\sigma}dxdt)$$

Which give an upper bound for the first term of J.

It remains now to find un upper bound for the second term of J. Recalling that $\theta_t = -4\beta\theta^{\frac{5}{4}}$, we get

$$\begin{aligned}\int_0^{T'}\int_0^1 -2R\sigma_t z^2 e^{-2R\sigma}dxdt &= \int_{t_0}^{T'}\int_0^1 8R\theta^{\frac{5}{4}}\beta p z^2 e^{-2R\sigma}dxdt\\ &\leq \int_{t_0}^{T'}\int_0^1 8R\theta^{\frac{5}{4}}|\beta| p z^2 e^{-2R\sigma}dxdt\\ &\leq C(t_0,T)\int_0^{T}\int_0^1 R\theta^{\frac{3}{2}}|\beta| p z^2 e^{-2R\sigma},\end{aligned}$$

since $\theta^{\frac{5}{4}} \leq c(t_0,T)\theta^{\frac{3}{2}}$, thanks to the definition of I_0, the proof of step (b) is complete.

Proof of step (c): conclusion

Owing the definition of I_0, steps (a) and (b), we deduce that there exists a constant $C = C(T,t_0,\alpha) > 0$ such that

$$\int_0^1 z^2(T',x)e^{-2R\sigma(T',x)}dx \leq C[\frac{1}{\sqrt{R}}\int_0^1 r^2(T',x)\left(f_1-f_2\right)^2(x)\,e^{-2R\sigma(T',x)}dx+\|u_{1,t}-u_{2,t}\|^2_{L^2(\omega_T^{t_0})}].$$

Since $z(T',x) = w_t(T',x) = (x^{\alpha}w_x)_x(T',x) + r(T',x)\left(f_1-f_2\right)(x)$, we also have

$$\int_0^1 r^2(T',x)\left(f_1-f_2\right)^2(x)\,e^{-2R\sigma(T',x)}dx \leq 2\int_0^1 z^2(T',x)e^{-2R\sigma(T',x)}dx+2\int_0^1 (x^{\alpha}w_x)_x^2(T',x)e^{-2R\sigma(T',x)}dx,$$

or, equivalently,

$$\begin{aligned}\int_0^1 r^2(T',x)\left(f_1-f_2\right)^2(x)\,e^{-2R\sigma(T',x)}dx \quad &\leq \quad C[\frac{1}{\sqrt{R}}\int_0^1 r^2(T',x)\left(f_1-f_2\right)^2(x)\,e^{-2R\sigma(T',x)}dx\\ &+\|u_{1,t}-u_{2,t}\|^2_{L^2(\omega_T^{t_0})}+\left\|(x^{\alpha}w_x)_x(T',.)\right\|^2_{L^2(0,1)}],\end{aligned}$$

for some constant $C = C(T,t_0,\alpha) > 0$. Choosing $R = R(T,t_0,\alpha)$ large enough such that $\frac{C}{\sqrt{R}} = \frac{1}{2}$, we obtain

$$\int_0^1 r^2(T',x)\left(f_1-f_2\right)^2(x)\,e^{-2R\sigma(T',x)}dx \leq C(\|u_{1,t}-u_{2,t}\|^2_{L^2(\omega_T^{t_0})}+\left\|(x^{\alpha}w_x)_x(T',.)\right\|^2_{L^2(0,1)}).$$

Since R has now been fixed , the dependance on R of all constants appearing hereafter just amounts to the standard one on T, t_0 and α. Let $\gamma = \gamma(T,t_0,\alpha) > 0$ be such that $\forall x \in [0,1]$, $e^{-2R\sigma(T',x)} \geq \gamma$. then

$$\int_0^1 \left(f_1-f_2\right)^2(x)\,e^{-2R\sigma(T',x)}dx \leq C_3\left(\|u_{1,t}-u_{2,t}\|^2_{L^2(\omega_T^{t_0})}+\left\|(x^{\alpha}w_x)_x(T',.)\right\|^2_{L^2(0,1)}\right),$$

where $C_3 = \dfrac{C}{\gamma d^2}$ for some constant $C = C(T, t_0, \alpha, \omega) > 0$. On the other hand, $u_1, u_2 \in C([\varepsilon, T]; D(A)) \cap C^1([\varepsilon, T]; L^2(0,1))$, then $(x^\alpha w_x)_x = (x^\alpha u_{1,x})_x - (x^\alpha u_{2,x})_x$ is valid on $\omega_T^{t_0}$. This completes the proof of Theorem 3.2.1. ■

Chapter 4
Identification of intial conditions

Abstract

THIS chapter concerns the determination of the initial condition for our problem. We follow the method developed in the work [YZ:01].

4.1 Statement of the inverse problem

In this section, we state our inverse problem which consists to determine the initial condition φ. Recall that $w = u_1 - u_2$, $z = \partial_t w$, $F = f_1 - f_2$ and we state our inverse problem for the solutions u_1, u_2 of (3.1) associated with (u_1, φ_1) and (u_1, φ_2) where $u_1(0,x) = \varphi_1(x)$ and $u_2(0,x) = \varphi_2(x)$. Using the measurement of the temperature at a fixed time $T' = \frac{t_0+T}{2}$, the inverse problem corresponding to recover the initial condition φ is given as follows:

$$\begin{cases} z_t - (x^\alpha z_x)_x = \dfrac{\partial r(t,x)}{\partial t} F(x), & (t,x) \in Q_T, \\ z(t,1) = 0, & t \in (0,T), \\ \text{and } \begin{matrix} z(t,0) = 0, \\ (x^\alpha z_x)(t,0) = 0, \end{matrix} & \begin{matrix} \text{for } 0 \le \alpha < 1 \\ \text{for } 1 \le \alpha < 2 \end{matrix}, t \in (0,T), \\ z(T',x) = (x^\alpha w_x)_x (T',x) + r(T',x) F(x) := b(x), & x \in (0,1) \end{cases} \tag{4.1}$$

where $r \in C^1([0,T] \times [0,1])$ and satisfies (3.2).

Now we decompose (4.1) as follows:

$$\begin{cases} h_t - (x^\alpha h_x)_x = \dfrac{\partial r(t,x)}{\partial t} F(x), & (t,x) \in Q_T, \\ h(t,1) = 0, & t \in (0,T), \\ \text{and } \begin{matrix} h(t,0) = 0, \\ (x^\alpha h_x)(t,0) = 0, \end{matrix} & \begin{matrix} \text{for } 0 \le \alpha < 1 \\ \text{for } 1 \le \alpha < 2 \end{matrix}, t \in (0,T), \\ h(0,x) = 0, & x \in (0,1) \end{cases} \tag{4.2}$$

and

$$\begin{cases} k_t - (x^\alpha k_x)_x = 0, & (t,x) \in Q_T, \\ k(t,1) = 0, & t \in (0,T), \\ \text{and } \begin{matrix} k(t,0) = 0, \\ (x^\alpha k_x)(t,0) = 0, \end{matrix} & \begin{matrix} \text{for } 0 \le \alpha < 1 \\ \text{for } 1 \le \alpha < 2 \end{matrix}, t \in (0,T), \\ k(T',x) = b(x) - h(T',x), & x \in (0,1). \end{cases} \tag{4.3}$$

It can be easily seen that

$$z = h + k, \quad z(0,x) = k(0,x). \tag{4.4}$$

4.2 Logarithmic stability result

In this section, we use the same method as in [YZ:01] to state estimate for the initial condition φ. The idea is based on the logarithmic convexity inequality. In order to obtain our stability result, we need more regularity for our solutions. Throughout this section, we consider the following admissible set:

$$\mathscr{U} = \left\{ (f,\varphi), \quad (f,\varphi) \in C^1([0,1]) \times C^2([0,1]); \quad \|f\|_{C^1([0,1])} + \|(x^\alpha \varphi_x)_x\|_{C^1([0,1])} \le M \right\},$$

where M is a given positive constant.

In the first step, we give an L^2 estimate for (h,k).

Lemma 4.2.1. *Let* $(f_1, \varphi_1), (f_2, \varphi_2) \in \mathcal{U}$. *Then, there exits a constant* $C = C\left(T'\right) > 0$, *such that*

$$\|h(t,.)\|_{L^2(0,1)} \leq C\|F\|_{L^2(0,1)}, \qquad 0 \leq t \leq T'.$$

Proof. We muliply the first equation of (4.2) by h and we integrate by parts with respect to the space variable. Hence, we obtain

$$\frac{1}{2}\partial_t \int_0^1 h^2 dx + \int_0^1 x^\alpha h_x^2 dx = \int_0^1 h\frac{\partial r}{\partial t} F dx.$$

Using Cauchy-Schwarz and Young inequalities with integrating over $(0,t)$ for $0 \leq t \leq T'$, we get

$$\|h(t,.)\|^2_{L^2(0,1)} \leq C_1 \|F\|^2_{L^2(0,1)} + \int_0^t \|h(s,.)\|^2_{L^2(0,1)} ds, \qquad 0 \leq t \leq T',$$

where $C_1 = q^2 T'$ with $q = \sup_{(t,x)\in[0,T']\times[0,1]} \left|\frac{\partial r}{\partial t}(t,x)\right|$.

Through Gronwall inequality, we obtain

$$\|h(t,.)\|_{L^2(0,1)} \leq C\|F\|_{L^2(0,1)}, \qquad 0 \leq t \leq T',$$

where $C = \sqrt{C_1 e^{T'}}$. ■

Lemma 4.2.2. *Let* $(f_1, \varphi_1), (f_2, \varphi_2) \in \mathcal{U}$. *Then, there exits a constant* $C = C\left(T', M\right) > 0$, *such that*

$$\|k(t,.)\|_{L^2(0,1)} \leq C\left(\left\|(x^\alpha w_x)_x(T',.)\right\|_{L^2(0,1)} + \|F\|^{\frac{t}{T'}}_{L^2(0,1)}\right), \qquad 0 \leq t \leq T',$$

Proof. Setting $\Phi = \varphi_1 - \varphi_2$, then we have

$$\|z(0,.)\|_{L^\infty(0,1)} = \|w_t(0,.)\|_{L^\infty(0,1)} = \left\|\left(x^\alpha \Phi_x\right)_x + r(0,.) f(.)\right\|_{L^\infty(0,1)} \leq 2CM,$$

where $C = \max\left\{1, \|r(0,.)\|_{L^\infty(0,1)}\right\}$.

It follows that $\|k(0,.)\|_{L^2(0,1)} \leq 2CM$. On the other hand, by using the logarithmic convexity theory for parabolic equations and Lemma 4.2.1, we get, for all $0 \leq t \leq T'$,

$$\begin{aligned}
\|k(t,.)\|_{L^2(0,1)} &\leq \|k(0,.)\|^{1-\frac{t}{T'}}_{L^2(0,1)} \left\|k\left(T',.\right)\right\|^{\frac{t}{T'}}_{L^2(0,1)} \\
&\leq (2CM)^{1-\frac{t}{T'}} \left\|k\left(T',.\right)\right\|^{\frac{t}{T'}}_{L^2(0,1)} \\
&\leq C\left(\left\|(x^\alpha w_x)_x(T',.)\right\|_{L^2(0,1)} + \|F\|_{L^2(0,1)}\right)^{\frac{t}{T'}}_{L^2(0,1)}.
\end{aligned}$$

which completes the proof of this lemma. ■

Now we are ready to state and prove our stability result.

Theorem 4.2.1. *Let* $(f_1,\varphi_1),(f_2,\varphi_2)\in\mathcal{U}$. *Then, there exits a constant* $C=C\left(T',M\right)>0$, *such that*

$$\left\|\varphi_1-\varphi_2\right\|_{L^2(0,1)}\leq\frac{1}{|\ln(E)|}+\left\|(u_1-u_2)(T',.)\right\|_{L^2(0,1)},$$

where

$$E=\left\|\left(x^\alpha u_{1,x}\right)_x(T',.)-\left(x^\alpha u_{2,x}\right)_x(T',.)\right\|_{L^2(0,1)}+\left\|u_{1,t}-u_{2,t}\right\|_{L^2\left(\omega_T^{t_0}\right)}.$$

Proof. Using the above lemmas and (4.4), we have, all $0\leq t\leq T'$,

$$\begin{aligned}
&\|z(t,.)\|_{L^2(0,1)}\\
\leq\ &\|h(t,.)\|_{L^2(0,1)}+\|k(t,.)\|_{L^2(0,1)}\\
\leq\ &C\|F\|_{L^2(0,1)}+C'\left(\left\|(x^\alpha w_x)_x(T',.)\right\|_{L^2(0,1)}+\|F\|_{L^2(0,1)}\right)^{\frac{t}{T'}}_{L^2(0,1)}\\
\leq\ &C''\left[(E)^{\frac{t}{T'}}_{L^2(0,1)}+E\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
\left\|\varphi_1-\varphi_2\right\|_{L^2(0,1)}&=\|w(0,.)\|_{L^2(0,1)}\\
&=\left\|-\int_0^{T'}w_t(s,.)\,ds+w\left(T',.\right)\right\|_{L^2(0,1)}\\
&\leq C''\int_0^{T'}\left[(E)^{\frac{s}{T'}}_{L^2(0,1)}+E\right]ds+\left\|w\left(T',.\right)\right\|_{L^2(0,1)}\\
&\leq C''\frac{E-1}{\ln(E)}+C_2E+\left\|w\left(T',.\right)\right\|_{L^2(0,1)}\\
&\leq\frac{C}{|\ln(E)|}+\left\|w\left(T',.\right)\right\|_{L^2(0,1)}.
\end{aligned}$$

This concludes the proof. ■

Conclusion and prospects

In this memoir, we have proved a Lipschitz stability in inverse source problem for a one-dimensional degenerate parabolic equation with a locally distributed observation using Carleman estimate. We have proved also a logarithmic stability in the identification of initial condition through the logarithmic convexity theory. In the future, we can study the following situations:

1. the identification of diffusion coefficients with an other term source.
2. adding a singular term in the parabolic equation.
3. to consider a system of coupled degenerate parabolic equations.

APPENDIX

A On Hardy-type inequality

In this appendix, we give the proof of some Hardy-type inequalities.

A.1 Hardy-type (I and II) inequalities

Lemma A.1.1 (Hardy inequality (type I)). *Let* $1 < \alpha^* < 2$*. Then, for all* $f \in H^1_{\alpha^*,0}(0,1)$,

$$\int_0^1 \frac{x^{\alpha^*}}{x^2} f^2(x)dx \le \frac{4}{(1-\alpha^*)^2}\int_0^1 x^{\alpha^*} f_x^2(x)dx.$$

Proof. See for instance [CMV:08]. ■

Lemma A.1.2 (Hardy inequality (type II)). *Let* $1 < \alpha^* < 2$*. Then there exists* $C = C(\alpha^*) > 0$ *such that for all* $f \in H^1_{\alpha^*,0}(0,1)$,

$$\int_0^1 \frac{x^{\alpha^*}}{(1-x)^2} f^2(x)dx \le C(\alpha^*)\int_0^1 x^{\alpha^*} f_x^2(x)dx.$$

Proof. As $f \in H^1_{\alpha^*}(0,1)$,

$$\begin{aligned}
\int_0^1 \frac{x^{\alpha^*}}{(1-x)^2} f^2(x)dx &= \int_0^1 \frac{x^{\alpha^*}}{(1-x)^2}\Big(\int_x^1 f_x(s)ds\Big)^2 dx \\
&= \int_0^1 \frac{x^{\alpha^*}}{(1-x)^2}\Big(\int_x^1 f_x(s)(1-s)^{\frac14}\frac{1}{(1-s)^{\frac14}}ds\Big)^2 dx \\
&\le \int_0^1 \frac{x^{\alpha^*}}{(1-x)^2}\Big(\int_x^1 f_x^2(s)\sqrt{(1-s}ds\Big)\Big(\int_x^1 \frac{1}{\sqrt{1-s}}ds\Big)dx \\
&= 2\int_0^1 \frac{x^{\alpha^*}}{(1-x)^2}(1-x)^{\frac12}\Big(\int_x^1 f_x^2(s)\sqrt{1-s}ds\Big)dx \\
&= 2\int_0^1\int_x^1 \frac{x^{\alpha^*}}{(1-x)^{\frac32}} f_x^2(s)\sqrt{1-s}dsdx.
\end{aligned}$$

Then, using Fubini's theorem,

$$\begin{aligned}
\int_0^1 \frac{x^{\alpha^*}}{(1-x)^2} f^2(x)dx &\leq 2\int_0^1\int_0^s \frac{x^{\alpha^*}}{(1-x)^{\frac{3}{2}}} f_x^2(s)\sqrt{1-s}dxds \\
&= 2\int_0^1 f_x^2(s)\sqrt{1-s}(\int_0^s \frac{x^{\alpha^*}}{(1-x)^{\frac{3}{2}}}dx)ds \\
&\leq 2\int_0^1 f_x^2(s)s^{\alpha^*}\sqrt{1-s}(\int_0^s (1-x)^{-\frac{3}{2}}dx)ds \\
&\leq 2\int_0^1 s^{\alpha^*} f_x^2\sqrt{1-s}[2(\frac{1}{\sqrt{1-s}}-1)]ds \\
&\leq 4\int_0^1 x^{\alpha^*} f_x^2(x)dx,
\end{aligned}$$

because $2(1-\sqrt{1-s}) \leq 2$. ■

We can now state and prove Hardy type inequality for all $f \in H^1_{\alpha^*,0}(0,1)$ (see also [CTY:10])

A.2 Hardy-type (III) inequality

Theorem A.2.1 (Hardy inequality (type III))**.** *Let* $1 < \alpha^* < 2$. *Then there exists* $C > 0$ *such that for all* $f \in H^1_{\alpha^*,0}(0,1)$,

$$\int_0^1 \frac{x^{\alpha^*}}{x^2(1-x)^2} f^2(x)dx \leq C\int_0^1 x^{\alpha^*} f_x^2(x)dx.$$

Proof. We have

$$\begin{aligned}
\int_0^1 \frac{x^{\alpha^*}}{x^2(1-x)^2} f^2(x)dx &= \int_0^{\frac{1}{2}} \frac{x^{\alpha^*}}{x^2(1-x)^2} f^2(x)dx + \int_{\frac{1}{2}}^1 \frac{x^{\alpha^*}}{x^2(1-x)^2} f^2(x)dx \\
&\leq 4\int_0^{\frac{1}{2}} \frac{x^{\alpha^*}}{x^2} f^2(x)dx + 4\int_{\frac{1}{2}}^1 \frac{x^{\alpha^*}}{(1-x)^2} f^2(x)dx.
\end{aligned}$$

Consequently, the proof of the theorem follows using the two above lemmas. ■

Bibliography

[CMV:05] P. Cannarsa, P. Martinez P and J. Vancostenoble, Null controllability of degenerate heat equations, *Adv. Differ. Equ*, 10 153–90.

[CMV:08] P. Cannarsa, P. Martinez P and J. Vancostenoble, Carleman estimates for a class of degenerate parabolic operators, *SIAM J. Control Optim*, 47 1–19.

[CTY:10] P. Cannarsa, J. Tort and M. Yamamoto, Determination of source terms in a degenerate parabolic equation, *Inverse Problems*, 26(10), 2010.

[MV:06] P. Martinez P and J. Vancostenoble, Carleman estimates for one-dimensional degenerate heat equations, *J. Evol. Eq*, 6 325–62.

[T:79] H. Tanabe, *Equations of Evolution*, Second edition, (Translated from the Japanese by N Mugibayashi and H Haneda, Monographs and Studies in Mathematics, 6) (Boston, MA, London: Pitman), 1979.

[YZ:01] M. Yamamoto and J. Zou , Simultaneous reconstruction of the initial temperature and heat radiative coefficient, *Inverse Problems*, 17 1181–202.