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Department of Mathematics and Computer Science

# DOCTORAL THESIS

**Field**: Mathematics and Computer Science

**Option**: Mathematics

**Specialty**: Applied mathematics

*Th*eme

# Dynamics of the Lengyel–Epstein Reaction–Diffusion System and its Generalizations

**Presented by**:

***Djamel Mansouri***

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# Dedicace

I dedicate my dissertation work

To my dear patents, may God have mercy on them, source of life, love and affectation

To my wonderful wife

To my childrens Maria, Amir and Amani

To my family for their support and encouragement throughout my study

To all my friends and colleagues

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**Abstract**

The aim of this thesis is to study some generalizations of Lengyel-Epstein Reaction-Diffusion System . Where we proposed and studied the dynamics of a fractional system consistent with the Lengyel - Epstein model, we established sufficient conditions for the stability of the local convergence of the unique equilibrium of the system by the linearization method, we used Lyapunov's direct method to establish the global asymptotic stability of the steady state solution. . Moreover, we studied the stability and instability of the generalized Lengyel - Epstein system as well as examining the Hopf-bifurcation of the system in diffusion-free and diffusive states. The numerical results obtained using the finite difference method were presented to confirm and verify theoretical results.

**Keywords**

**Reaction-diffusion, Lengyel Epstein model (ODE/PDE), CIMA reaction, existence of solutions, asymptotic stability, Turing instability, Hopf-bifurcation .**

**Résumé**

Le but de cette thèse est d'étudier certaines généralisations du système de réaction-diffusion de Lengyel-Epstein, où nous avons proposé et étudié la dynamique d'un système fractionnaire cohérent avec le modèle de Lengyel-Epstein, nous avons établi des conditions suffisantes pour la stabilité de la convergence locale de l'équilibre unique du système par la méthode de linéarisation, nous avons utilisé la méthode directe de Lyapunov pour établir la stabilité asymptotique globale de la solution d'état stationnaire. De plus, nous avons étudié la stabilité et l'instabilité du système Lengyel - Epstein généralisé ainsi que la bifurcation de Hopf du système dans des états sans diffusion et diffusifs. Les résultats numériques obtenus en utilisant la méthode des différences finies ont été présentés pour confirmer et vérifier les résultats théoriques.

**Les Mot-clés**

**Réaction-diffusion, modèle de Lengyel Epstein (ODE / PDE), réaction CIMA, existence de solutions, stabilité asymptotique, instabilité de Turing, bifurcation de hopf.**

# ملخص

الهدف من هذه الأطروحة هو دراسة بعض التعميمات المتعلقة بنظام لنجل ابستين، حيث قمنا باقتراح ودراسة ديناميكيات نظام كسري يتوافق مع نموذج لنجل ابستين، في هذه الحالة قمنا بتهيئة ظروف كافية لاستقرار التقارب المحلي للتوازن الوحيد للنظام بالطريقة الخطية، استخدمنا طريقة ليابونوف المباشرة لإنشاء الاستقرار المقارب الكلي للحل عند نقطة التوازن، علاوة على ذلك، قمنا بدراسة الاستقرار وعدم الاستقرار لنظام لنجل ابستين المعمم، بالإضافة إلى فحص تشعب هوبف للنظام في حالات الانتشار وعدم الانتشار، تم عرض النتائج العددية التي تم الحصول عليها باستخدامطريقة الفروق المنتهية لتأكيد النتائج النظرية و التحقق منها.

### الكلمات المفتاحية

التفاعل والانتشار، نموذج لنجل ابستين (المعادلات التفاضلية الجزئية المعادلة التفاضلية العادية)، تفاعل CIMA، وجود الحلول، الاستقرار المقارب، عدم استقرار تورينغ، تشعب هوبف.

# Contents

# Introduction

For centuries, scientists have struggled to understand the origins of the patterns and forms found in nature from the leopards spots to the graceful spirals of a mollusk shell to the complex designs on a butterflies wing. In the early 1950s, British mathematician Alan Turing proposed a revolutionary model that explains and accounts for pattern formation in morphogenesis, which is the biological process by which organisms develop their physical shapes and colors [40] . In his work, Turing showed mathematically that diffusion driven instability may give rise to spatial concentration patterns. In other words, if a system that is stable in the ODE sense under certain parameters and initial conditions but become unstable once diffusion is accounted for, then special types of patterns arise in the spatial dimensions of the solution. It should be noted that in his work, Turing only explained the linear regime. The physical interpretation and the possibility to have spatial oscillations in far from equilibrium systems were proved in [36] . In this study, the authors showed that the transition to a spatial order is not in contradiction with the second principle of thermodynamics extended for out of equilibrium systems. For more insights into dissipative structures from the theoretical as well as experimental perspectives, the reader may refer to [41, 42]. This pattern formation theory remained untested for decades. One of the earliest physical realizations of this theory was achieved in [11] , where the authors considered a chlorite-iodide malonic-acid (CIMA) chemical reaction across a diffusive membrane. A mathematical model was . first coined in [23, 24] and numerous studies followed that examine the existence and nature of solutions, asymptotic stability conditions, bifurcation analysis, and pattern formation in the so called Lengyel-Epstein reaction-diffusion model. Among the most prominent of these studies are [25, 34, 49, 50]. Turing patterns formed in the chlorite-iodide-malonic acid (CIMA) reaction, a far from equilibrium chemical reaction-diffusion system, are morphologically compared to patterns obtained in the Lengyel-Epstein model, which is based on a nonlinear partial differential equation and believed to model the CIMA reaction.

Numerous modifications have been made to the original model. These new models were either based on modifications made to the reaction itself, such as the intensity of light applied to the reactants, or based on theoretical conceptualization [9, 18, 33, 38, 39, 51, 52]. A number of generalizations have been proposed for the Lengyel–Epstein system with the aim of relaxing existing asymptotic stability and Turing instability conditions as well as extending the results to other similar models [1, 3, 4, 2, 14]. Lengyel and Epstein analyzed the reaction mechanism and the related CDMIA reaction and were able to reduce it to two main variables, iodide and chlorite, assuming that the concentrations of chlorine dioxide, iodine and malonic acid can be taken as being effectively constant compared to the large variants. Changes in the concentrations of the other two species. In 1991, Ouyang and Swinney measured the hexagonal and nearly constant two-dimensional Turing patterns in the CIMA reaction in a two-sided open gel disk reactor, see figure 1. This model was capable of theoretical analysis and explanation of the appearance of Turing patterns in these experiments. Diffusion constants for small ions in solution are usually similar in size, and the low diffusion constants of activated iodide in relation to chlorite arose as a result of this species' binding to starch, a natural polymer that was not able to diffuse in the gel, thus ensuring the conditions necessary for Turing instability.

The CIMA reaction can be described by three chemical reaction schemes as follows

$$\begin{cases} MA + I_2 \rightarrow IMA + I^- + H^+ \\ CIO_2 + I^- \rightarrow \frac{1}{2}I_2 + CIO_2^- \\ CIO_2^- + 4I^- + 4H^+ \rightarrow 2I_2 + CI^- + 2H_2O \end{cases}$$

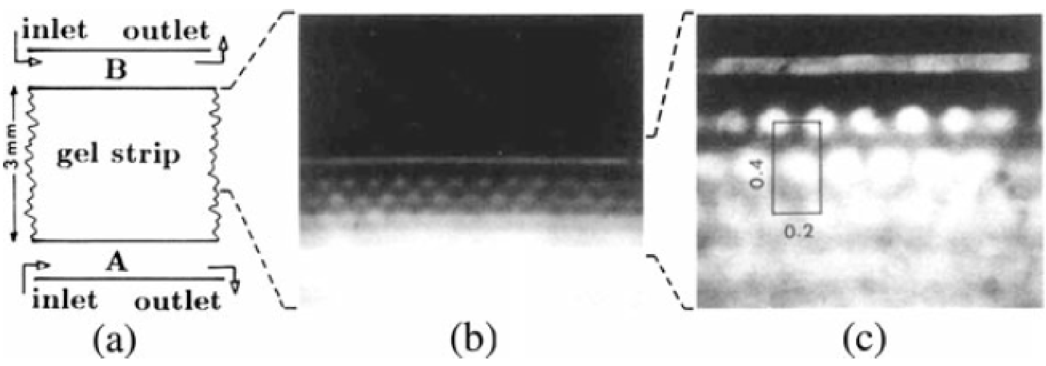


Figure 1: [43] **(a)** Sketch of the two-sided CFUR; dimensions of the gel slab: length $L = 20mm$, width $w = 3mm$, thickness $e = 1mm$. **(b)** Dark regions of the gel correspond to reduced state, colored blue, and clear regions to the oxidized state. **(c)** An enlarged region of the pattern; dimensions are in$mm$.

This thesis is systematized into four chapters.

**Chapter 01**: In the first chapter, we recall some of the necessary nomenclature, and some preliminary definitions concerning stability, bifurecation, fractional calculus and formula that will be useful in the following chapters.

**Chapter 02**: this chapter is divided into four sections : in the section one we give a general introduction to reaction-diffusion systems, with some examples of that. In the section two we study the 2D reaction-diffusion systems by studying the stability local, stability global, Turing instabilities and Hopf-Bifurcation in free diffution .In the section three we applying all previous studies(ODE and PDE) in Lengyel–Epstein reaction–diffusion model .In the last section we give a description of the numerical finite difference method.

**Chapter 03**: We are interested in a fractional version of the Lengyel-Epstein reaction-diffusion system, We have established sufficient conditions for the local asymptotic stability of the system's unique equilibrium in the ODE and PDE senses through the linearization method. In addition, we have employed the direct Lyapunov method to establish the global asymptotic stability of the steady state solution.

**Chapter 04**: In this chapter we will expand the study of the Lengyel - Epstein system previously studied to the generalized system which is proposed by [1, 3] and expand the current study to include sufficient conditions for the existence of Turing patterns in addition to examining the Hopf-bifurcation of the system in the diffusion-free and diffusive cases .

## Nomenclature

The following are some notations the are used in the theses.

- $\mathbb{N}$: Denotes the set of natural numbers.
- $\mathbb{R}$: Denotes the set of real numbers.
- $\mathbb{C}$: Denotes the set of complex numbers.
- $\Omega$: Open bounded subset of $\mathbb{R}^N$.
- $\mathbb{R}^N$: Set of all N-tuples $x = (x_1, x_2, ..., x_N)$.
- $D(A)$: Domain of $A$.
- $A^{-1}$: Denotes the inverse of matrex $A$.
- $\det(A)$: Denotes the determinant of a matrix $A$.
- $A^T$: The transpose of matrix $A$.
- $Re(\lambda)$: Denotes the real part of a complex number $\lambda$.
- $\arg(\lambda)$ : Denotes the argument of a complex number$\lambda$.
- $\frac{\partial f}{\partial x_i} = \partial_{x_i} f = f_{x_i}$: denotes the partial derivative of $f(x_1, ..., x_n)$ with respect to $x_i$.
- $\Delta f = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f$: Denotes the Laplace operator.
- $\nabla f(x) = \left( \frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), ..., \frac{\partial f}{\partial x_n}(x) \right)$: Gradient operator.
- $\langle u, v \rangle$: The scalar product of $u$ and $v$.
- $\langle f, g \rangle_{\mathbb{L}^2(\Omega)} = \int_\Omega fg dx$ : Inner product in $\mathbb{L}^2(\Omega)$.
- $\langle f, g \rangle = \int_\Omega fg dx + \int_\Omega \left( \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} \right) dx$ : Inner product of $\mathbb{H}^1(\Omega)$.
- $\mathbb{H}^m(\Omega) = \left\{ f : [0, T] \to V \text{ Meserable}, \int_\Omega \|f(x)\|_V^p \ dt \ < \infty \right\}$: Sobolev space, functions with weak derivatives in $\mathbb{L}^2$ to order $m$.
- $H^2(\Omega)$: Hilbert space withe norme $\|f(x)\|_2^2 = \int_\Omega |f(x)| dx$
- $\mathbb{C}(\Omega)$: Space of continous function on $\Omega$.
- $\mathbb{C}^k(\Omega)$, $k \in \mathbb{N}$: Space of function continously differentiable to order $k$ on $\Omega$.
- $\mathbb{L}^p(\Omega)$: Space of $p$-integrables function on $\Omega$ .
- $\|f(x)\|_p^p = \int_\Omega |f(x)| dx$: norme of $\mathbb{L}^p(\Omega)$ , $1 \le p \le \infty$.

- $\mathbb{H}_0^1(\Omega) = \left\{ f \in \mathbb{H}^1(\Omega) \setminus f = 0 \text{ in } \partial\Omega \right\}$.
- $\Gamma$ : Gamma fonction.
- ${}_{t_0}^{C}D_t^{\delta}(t)$: Denotes the Caputo fractional derivative .
- $\Re$: Is invariant rectangle.

# Chapter 1

# Preliminary

In this chapter, we present the basic notions used in this thesis, important formulas, stability analysis of ODE systems, and some of the necessary notation concerned of fractional calculus.

## 1.1 Important formulas

### Jordon normal forme

**Definition 1.1.1** *[5] If $A$ is nonsingular matrix, there exist two nonsingular matrixs $J$ and $B$ sush that $A = B^{-1}JB$, or equivalenty $BA = JB$, $J$ is called the Jordon normal forme (or simply jordan matrix) of $A$ . The Jordan matrix $J$ is triangular (but not necessarily diagonal). Let us show what happens if $n = 2$, which is the case we will deal with in the sequel. Let $A$ be a $2 \times 2$ matrix with eigenvalues $\lambda_1, \lambda_2$. Then the Jordan matrix is as follows.*

1. *If $\lambda_1, \lambda_2$ are real and distinct, then their algebric and geometric multiplicity is 1 and hence*

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

2. *If $\lambda_1 = \lambda_2$ is real, then its algebric multiplicity is 2. Either its geometric multiplicity is also 2, a case where*

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{1.1}$$

*or its geometric multiplicity is 1, a case where*

$$J = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}. \tag{1.2}$$

*Furthermore, if the eigenvalues are complex conjugate, $\lambda_{1,2} = \alpha + i\beta$, then one can show that*

$$J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}. \tag{1.3}$$

**Taylor's formula**

If a function $f(x)$ has continuous derivatives up to the $(n+1)$ order, to approximate of $f(u)$ better near $u^*$. The $n$-degree Taylor polymnmial of $f(u)$ for $u$ near the point $u^*$is:

$$f(u) = f(u^*) + \frac{df}{du}(u-u^*) + \frac{f_{uu}}{2!}(u-u^*)^2 + .... + \frac{f^{(n)}}{n!}(u-u^*)^n + o(u-u^*). \quad (1.4)$$

If a function of two variables $f(u,v)$ its partial derivatives through order $(n+1)$ are continuous throughout an open rectangular region $\Re$ centered at the point $(u^*,v^*)$, the $2^{nd}$-degree Taylor polymnmial of $f(u,v)$ for $(u,v)$ near the point $(u^*,v^*)$is:

$$\begin{aligned} f(u,v) &= f(u^*,v^*) + \frac{df}{du}(u-u^*) + \frac{df}{dv}(u-v^*) + f_{uv}(u-u^*)(v-v^*) \quad (1.5) \\ &+ \frac{f_{uu}}{2}(u-u^*)^2 + \frac{f_{vv}}{2}(u-u^*)^2 + o\left[(u-u^*)^2,(v-v^*)^2\right]. \end{aligned}$$

## 1.2 Stability analysis of ODE systems

In this section we present some definitions, theorems, relating to dynamic systems. We consider the system in a way that it depends on a parameter $\sigma$,

$$\begin{cases} \frac{dU}{dt} = (f(U), \sigma) \\ U(0) = p \end{cases}, \quad (1.6)$$

where $U = (u_1, u_2, .., u_n) \in \mathbb{R}^n$, $f = (f_1, f_2, .., f_n) \in \mathbb{R}^n$, $p = (p_1, p_1, ..., p_1) \in \mathbb{R}^n$, and $\sigma \in \mathbb{R}^n$.

**Definition 1.2.1** *[5] Given a system $U'(t) = f(U(t))$ with equilibrium $U^* = 0$, its linearization at $U^* = 0$ is the linear system $U'(t) = AU(t)$, where $A = \nabla f(0)$ . Developing $f$ in Taylor's expansion we find $f(U) = AU + O(|U|)$*

Then the linearized system is $U'(t) = AU$. We have seen that a sufficient condition for the asymptotic stability of $U = 0$ for $U'(t) = AU$ is that all the real parts of the eigenvalues of $A$ be negative, whilst if at least one eigenvalue is positive, or has positive real part, then $U = 0$ is unstable. This result is extended to the nonlinear case in the next theorem.

**Theorem 1.2.1** *[5] Suppose that all the eigenvalues of $\nabla f(0)$ have negative real parts.Then the equilibrium $U^* = 0$ is asymptotically stable with respect to the system $U'(t) = \nabla f(0)U + O(|U|)$.*

*If at least one eigenvalue of $\nabla f(0)$ has positive real part, then the equilibrium $U^* = 0$ is unstable.*

**Theorem 1.2.2** *[5] Suppose that $A$ is a constant $n \times n$ nonsingular matrix.*

*(i) If all the eigenvalues of $A$ have negative real part, then $U^* = 0$ is asymptotically stable. More precisely, for all $p \in \mathbb{R}^n$ one has that the solution $U(t,p) \to 0$ as $t \to \infty$.*

*(ii) If one eigenvalue of $A$ has positive real part, then $U^* = 0$ is unstable.*

### 1.2.1 Stability theory

The stability of fixed points of a system of constant coefficient linear differential equations of the first order can be analyzed using the eigenvalues of the corresponding matrix. In this work we will study the stability of the $(2 \times 2)$ case of system (1.6) , and we are writing the system in the form

$$\begin{cases} \frac{du(t)}{\partial t} = a_{11}u + a_{12}u := F(u,v) \\ \frac{dv(t)}{\partial t} = a_{21}u + a_{22}v := G(u,v) \end{cases}, \tag{1.7}$$

where the coefficients $a_{ij}$ are real numbers. Letting

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

and

$$\begin{cases} a_{11}u + a_{12}u = f(u,v) \\ a_{21}u + a_{22}v = g(u,v), \end{cases}. \tag{1.8}$$

System (1.7) has a unique equilibrium $(u^*, v^*) = (0,0)$ such as:

$$\begin{cases} F(u^*, v^*) = 0 \\ G(u^*, v^*) = 0, \end{cases}.$$

The Jacobian matrix of the system (1.7) is

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

the characteristic polynomial is defined by:

$$\begin{aligned} \rho(\lambda) &= \det(J - \lambda I) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{21}a_{12} \\ &= \lambda^2 - tr(J)\lambda + \det(J), \end{aligned} \tag{1.9}$$

for solve the characteristic polynomial we calculed $\Delta$ when

$$\Delta = tr(J)^2 - 4\det(J), \tag{1.10}$$

where $tr(J)$ and $\det(J)$ designate respectively the trace and the determinant of the Jacobian matrix. So we have,

Case (I): $\Delta \geq 0$ then

$$\lambda_{1,2} = \frac{tr(J) \pm \sqrt{\Delta}}{2}, \ (\lambda_{1,2} \text{ are reals }),$$

and the criterion of stability and the nature of the fixed points are obtained by:

- If the eigenvalues are real and negative, the equilibrium $(u^*, v^*)$is asymptotically stable, whilst if one of the eigenvalues is positive, $(u^*, v^*)$ is unstable.

- If $\lambda_1 < \lambda_2 < 0$, or $\lambda_2 < \lambda_1 < 0$, in any case the origin is asymptotically stable and is called a stable node.

- If $\lambda_1$,$\lambda_2 > 0$, we have an unstable node.

- If $\lambda_1.\lambda_2 < 0$, The origin is unstable and is called a saddle.

Case (II): $\Delta < 0$ then

$$\lambda_{1,2} = \frac{tr\,(J) \pm i\sqrt{-\Delta}}{2}, \ (\lambda_{1,2} \text{ are complex conjugates})$$

- If $\lambda_{1,2} = \alpha \pm \beta i$ and , $\alpha < 0$, then the origin is asymptotically stable, whilst if $\alpha > 0$, the origin is unstable.

- If $\lambda_{1,2} = \alpha \pm \beta i$ and , $\alpha = 0$, the origin is stable, but not asymptotically stable.The equilibrium is called a center.

The following table summarizes the nature of the equilibrium $(u^*, v^*)$ when $A$ is nonsingular.

| *Eigenvalues* | *Equilibrium* |
|---|---|
| $\lambda_{1,2} \in \mathbb{R}, \lambda_1, \lambda_2 < 0$ | *Asymptotically stable node* |
| $\lambda_{1,2} \in \mathbb{R}, \lambda_1, \lambda_2 > 0$ | *Unstable node* |
| $\lambda_{1,2} \in \mathbb{R}, \lambda_1.\lambda_2 < 0$ | *Unstable saddle* |
| $\lambda_{1,2} = \alpha \pm \beta i,\ \alpha < 0$ | *Asymptotically stable focus* |
| $\lambda_{1,2} = \alpha \pm \beta i,\ \alpha > 0$ | *Unstable focus* |
| $\lambda_{1,2} = \pm\beta i$ | *Stable center* |

The typology of the solutions of the planar linear systems which we established with to leave the nature of the eigenvalues of the matrix of the system (1.7) can be also summarized in a plan, (Tr, det).(see Figure 1.1). Eigenvalues of $J$ are solutions of the characteristic equation (1.9).

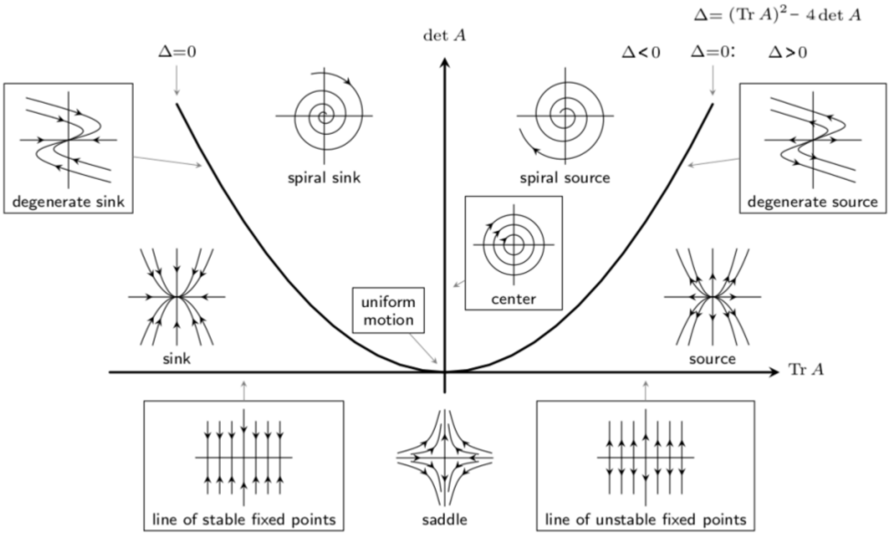


Figure 1.1: Stability diagram

### 1.2.2 Hopf-bifurcation

The term Hopf bifurcation (also sometimes called Poincaré-Andronov-Hopf bifurcation) refers to the local birth or death of a periodic solution (self-excited oscillation) from an equilibrium as a parameter crosses a critical value. It is the simplest bifurcation not just involving equilibria and therefore belongs to what is sometimes called dynamic (as opposed to static) bifurcation theory. In a differential equation a Hopf bifurcation typically occurs when a complex conjugate pair of eigenvalues of the linearised flow at a fixed point becomes purely imaginary. This implies that a Hopf bifurcation can only occur in systems of dimension two or higher.

**Definition 1.2.2** *[13] A closed trajectory in the phase plane such that other trajectories spiral toward it (either from the inside or outside) as* $t \to \infty$ *is called a limit cycle.*

**Theorem 1.2.3** *[13] The Hopf bifurcation occurs when a fixed point exchanges its stability by the generation of a periodic limit cycle . The necessary criteria are*

*i)* $f(U^*, \sigma_0) = 0$

*ii) The Jacobian has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real part*

*iii)* $\frac{dRe\lambda_i}{d\sigma}|_{\sigma_0} \neq 0$

*where* $\sigma_0$ *is Hopf Bifurcation point.*

**Theorem 1.2.4** *[15] (Hopf Bifurcation theorem in* $\mathbb{R}^2$*) Suppose the pummetrezed system* $\dot{U} = f(U, \sigma)$, $U \in \mathbb{R}^2$ , $\sigma \in \mathbb{R}$ *has a fixed point at the origin for all values of the mal prameter* $\sigma$*. Further, suppose the eigenvelues* $\lambda_1(\sigma)$ *and* $\lambda_2(\sigma)$ *of the (*$\sigma$ *dependent) Jacobian of* $f$*, at zero, are purely imaginary for* $\sigma = \sigma_0$ *. If the reat part of the eigenvalues,* $Re\lambda_1(\sigma) = Re\lambda_2(\sigma)$ *since* $\lambda_1 = \overline{\lambda_2}$*, satisfies*

$$\frac{d}{d\sigma}\left(Re\lambda_1(\sigma)\right)|_{\sigma=\sigma_0} > 0,$$

*and the origin is asymptoticallys stable when $\sigma = \sigma_0$ , then:*
*i) $\sigma = \sigma_0$ is a bifurcation point.*
*ii) for $\sigma \in (\sigma_1, \sigma_0)$ some $\sigma_1 < \sigma_0$ the origin is a stable focuse.*
*iii) for $\sigma \in (\sigma_0, \sigma_2)$ some $\sigma_2 > \sigma_0$ the origin is an instable focuse.*

**Generic Hopf-bifurcation**

let us consider the simple two–component system (1.7), which has at $\sigma = \sigma_0$ the equilibrium $(0,0)$ with eigenvalues $\lambda_{1,2} = \pm i\omega_0, \omega_0 > 0$. Thus, the system can be written as

$$\dot{x} = A(\sigma)x + F(x,\sigma). \tag{1.11}$$

The Jacobian matrix $A(\sigma)$ can be written as

$$A(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}. \tag{1.12}$$

Its eigenvalues are the roots of the characteristic equation

$$\lambda^2 - \mu\lambda + \Delta = 0,$$

where

$$\mu(\sigma) = trA(\sigma) = a(\sigma) + d(\sigma),$$

and

$$\Delta(\sigma) = \det A(\sigma) = a(\sigma)d(\sigma) - b(\sigma)c(\sigma).$$

So

$$\lambda_{1,2} = \frac{1}{2}\left(\mu(\sigma) \pm \sqrt{\mu(\sigma)^2 - 4\Delta(\sigma)}\right). \tag{1.13}$$

The Hopf bifurcation condition implies

$$\mu(\sigma) = 0, \Delta(0) = \omega_0^2 > 0.$$

For small $|\sigma|$ we can introduce

$$\beta(\sigma) = \frac{1}{2}\mu(\sigma), \omega(\sigma) = \frac{1}{2}\sqrt{4\Delta(\sigma) - \mu^2(\sigma)}, \tag{1.14}$$

and therefore obtain the following representation for the eigenvalues:

$$\lambda_1(\sigma) = \lambda(\sigma), \lambda_2(\sigma) = \overline{\lambda(\sigma)},$$

where

$$\lambda(\sigma) = \mu(\sigma) + i\omega(\sigma), \mu(0) = 0, \omega(0) = \omega_0 > 0.$$

The matrix $A(\sigma)$ has its canonical real (Jordan) form:

$$J(\sigma) = T(\sigma)A(\sigma)T^{-1}(\sigma) = \begin{pmatrix} \beta(\sigma) & -\omega(\sigma) \\ \omega(\sigma) & \beta(\sigma) \end{pmatrix}. \tag{1.15}$$

Suppose that at $\sigma=0$, the function $F(x,\sigma)$ from (1.11) is represented as

$$F(x,0)=\frac{1}{2}B(x,x)+\frac{1}{6}C(x,x,x)+O\left(\|x\|^4\right), \tag{1.16}$$

where $B(x,y)$ and $C(x,y,u)$ are symmetric multilinear vector functions of $x,y,u\in\mathbb{R}^2$ . In coordinates, we have

$$B_i(x,y)=\sum_{j,k=1}^{2}\left.\frac{\partial^2 F_i(\zeta,0)}{\partial\zeta_j\partial\zeta_k}\right|_{\zeta=0} x_j y_k, i=1,2. \tag{1.17}$$

$$C_i(x,y,u)=\sum_{j,k,l=1}^{2}\left.\frac{\partial^3 F_i(\zeta,0)}{\partial\zeta_j\partial\zeta_k\partial\zeta_l}\right|_{\zeta=0} x_j y_k u_l, i=1,2. \tag{1.18}$$

We can, now, write

$$\begin{pmatrix}x_t\\ y_t\end{pmatrix}=J(b)\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}F(u,v,\sigma)\\ G(u,v,\sigma)\end{pmatrix}, \tag{1.19}$$

with

$$\begin{pmatrix}F(u,v,\sigma)\\ G(u,v,\sigma)\end{pmatrix}=T^{-1}\begin{pmatrix}f(u,v,\sigma)\\ g(u,v,\sigma)\end{pmatrix}. \tag{1.20}$$

Let us, now, use polar coordinates to obtain

$$\begin{cases}\dot r=\beta(\sigma)r+a(\sigma)r^3+...,\\ \dot\theta=\omega(\sigma)+c(\sigma)r^2+....\end{cases}$$

then the Taylor expansion of (1.19) at $\sigma=\sigma_0$ yields

$$\begin{cases}\dot r=\beta'(\sigma_0)(\sigma-\sigma_0)r+a(\sigma_0)r^3+O(...),\\ \dot\theta=\omega(\sigma_0)+\omega'(\sigma_0)(\alpha-\sigma_0)+c(\sigma_0)r^2++O(...).\end{cases}$$

It turns out that the stability of the periodic solution is dependent on the sign of the coefficient $a(\sigma_0)$, which is given by

$$\begin{aligned}a(\sigma_0)=&\frac{1}{16}\left[f^*_{xxx}+f^*_{xyy}+g^*_{xxy}+g^*_{yyy}\right]\\ &+\frac{1}{16\omega(\sigma_0)}\left[f^*_{xy}\left(f^*_{xx}+f^*_{yy}\right)-g^*_{xy}\left(g^*_{xx}+g^*_{yy}\right)-f^*_{xx}g^*_{xx}+f^*_{yy}g^*_{yy}\right].\end{aligned} \tag{1.21}$$

**Theorem 1.2.5** *[13]For the systeme (1.7)*
*i) if $a(\sigma_0)<0$ the periodic solutions bifurcating from $(u^*,v^*)$ at $\sigma=\sigma_0$ are instable, and the direction of the Hopf bifurcation is subcritical.*
*ii) if $a(\sigma_0)>0$ the periodic solutions bifurcating from $(u^*,v^*)$ at $\sigma=\sigma_0$ are stable, and the direction of the Hopf bifurcation is supercritical.*

## 1.3 Fractional calculus

Fractional calculus has been around for centuries. In fact, its inception followed directly after that of conventional calculus. However, it wasn't until the last century that interest grew in the subject due to the many applications it has found in the fields of science and engineering. In this section, we start with some of the necessary notation and stability theory related to the subject. The aim is to clarify the analysis and proofs that will come at later stages in the thesis.

**Definition 1.3.1** *[35] The Riemann–Liouville fractional derivative of order $\delta$ of an integrable function $f(t)$ is defined as*

$$ {}_{t_0}D_t^{-\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\delta}} d\tau. \tag{1.22}$$

*where $0 < \delta \in \mathbb{R}^+$ and $\Gamma(\delta) = \int_0^\infty e^{-t} t^{\delta-1} dt$ is the Gamma function.*

**Definition 1.3.2** *[21] The Caputo fractional derivative of order $\delta > 0$ of a function $f$ of class $C^n$ for $t > t_0$ is defined as*

$$ {}_{t_0}^{C}D_t^{\delta} f(t) = \frac{1}{\Gamma(n-\delta)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\delta-n-1}} d\tau, \tag{1.23}$$

*with $n = \min\{k \in \mathbb{N} \mid k > \delta\}$ and $\Gamma$ representing the gamma function.*

Note that the constant $(u^*, v^*)$ is an equilibrium for the Caputo fractional non–autonomous dynamic system

$$\begin{cases} {}_{t_0}^{C}D_t^{\delta} u = F(u, v), & \text{in } \mathbb{R}^+, \\ {}_{t_0}^{C}D_t^{\delta} v = G(u, v), & \text{in } \mathbb{R}^+, \end{cases} \tag{1.24}$$

if and only if

$$F(u^*, v^*) = G(u^*, v^*) = 0.$$

The following lemmas hold.

**Lemme 1.3.1** *[7]Let $u(t)$ be a continuous and differentiable real function. For any time instant $t \geq t_0$,*

$$ {}_{t_0}^{C}D_t^{\delta} u^2(t) \leq 2u(t) {}_{t_0}^{C}D_t^{\delta} u(t), \tag{1.25}$$

*with $\delta \in (0, 1]$.*

**Lemme 1.3.2** *[31] An equilibrium point $(u^*, v^*)$ of (1.24) is locally asymptotically stable if*

$$|\arg(\lambda_i)| > \frac{\delta\pi}{2}, \quad i = 1, 2, \tag{1.26}$$

*where $\lambda_i$ are the eigenvalues of the Jacobian matrix $J(u^*, v^*)$ and $\arg(\cdot)$ denotes the argument of a complex number.*

# Chapter 2

# Reaction diffusion systems and Lengyel–Epstein model

IN this chapter , we are concerned with the reaction diffusion systems and stability analyses of PDE systems. We make a general introduction to reaction diffusion systems , with some examples of that and study the 2D system by studying the stability local, stability global, Turing instabilities and Hopf-Bifurcation in diffusive case. Then we applying previous studies of 2-D Lengyel–Epstein reaction–diffusion system with homogeneous Neumann boundary condition.

## 2.1 Introduction to Reaction-Diffusion systems

In recent years, reactions-diffsion systems have received a great deal of attention motivated by their widespread incident in models of biological and chemical phenomena, and by the richness of the structure of their sets of solutions. Considering the numerous and varied applications of these systems; The Approaches to modeling certain chemical problems such as reactions oscillating chemicals (Brussellateur). Individuals diverge from one problem to another:

In chemistry, for example, they are chemical substances. In biochemistry, they May represent molecules. In metallurgy, atoms. In dynamics of Populations, they are humans. In population genetics, they represent characters. In biophysics, electrical charges or potential differences. In the environment, they can represent the animals or plants of a forest, a sea or an ocean ....For most of these problems, we show that results in reaction-diffusion systems. The conditions at the edges will be chosen according to the origin and the nature of the problem studied: if there is no immigration of individuals across the boundary of the domain on which the problem is posed, the conditions at the homogeneous edges of Neumann. If there are no individuals on the border, we take the conditions at the edges Homogeneous of Dirichlet. The unknown (the solution one seeks) is a vector of which the components are generally positive functions : in chemistry, for example, it is a vector of chemical concentrations. In biochemistry or metallurgy, a vector of concentrations in numbers of molecules or atoms respectively. In population dynamics and in the environment, it is a vector of densities of Human, animal or plant populations ...

Initial conditions are generally positive; Since they are concentrations, densities, electrical charges, etc. All these problems are in the form of:

$$\frac{\partial U}{\partial t} = D\Delta U + f(U), \tag{2.1}$$

where $U(x,t) = (U_1(x,t), ..., U_m(x,t))$ is a vector of dependent variables and $f(x,t,U(x,t)) = (f_1(x,t,U(x,t)), ..., f_m(x,t,U(x,t)))$ is the reaction (usually nonlinear) and $D$ is a square matrix $m \times m$ positive and diagonalizable called dissemination matrix. The terms of reaction are the result of any interaction between the components of $U$.

For example, in chemistry $u$ is a vector of chemical concentrations and $f$ represents the chemical reactions on these concentrations. The term $D\Delta U$ represents the Molecular diagnoses across the reaction boundary.

## Examples for reaction-diffusion systems

### Brusselator model

THE was proposed by **Ilya Prigogine** (Nobel Laureate in Chemistry 1977) and co-workers in Brussels in 1971. The idea was to describe the autocatalytic oscillations ubiquitous in Nature, especially in the living bodies. The reaction equations for the model are:

$$\begin{aligned} &A \overset{k_1}{\rightarrow} X \\ &2X + Y \overset{k_2}{\rightarrow} 3X \\ &B + X \overset{k_3}{\rightarrow} D + Y \\ &X \overset{k_4}{\rightarrow} E \end{aligned}$$

where $A$ and $B$ are the reactants and $X$ and $Y$ are intermediate products while $D$ and $E$ are the nal products $k_1, k_2, k_3 \text{and} k_4$ being the reaction rate constants. The evolution of the compounds $X$ and $Y$ is given by the equations:

$$\begin{cases} \frac{dX}{dt} = k_1 A + k_2 X^2 Y - k_3 BX + k_4 X, \\ \quad \frac{dY}{dt} = -k_2 X^2 Y + k_3 BX. \end{cases}$$

After the following transformations:$a = \frac{Ak_1}{k_3}, b = \frac{Bk_2}{k_3}, k_4 = k_3, X \to u, Y \to v, \frac{D_X}{k_3} \to D_u$ and $\frac{D_Y}{k_3} \to D_v$, the reaction-diffusion system has Brusselator kinematics in this way

$$\begin{cases} \frac{du}{dt} = D_u \nabla^2 u + a - (b+1)\, u + u^2 v, \\ \quad \frac{dv}{dt} = D_v \nabla^2 v + bu - u^2 v. \end{cases}$$

### Degn and Harrison model

THE chemical model introduced by Degn and Harrison in [10]. It is used to explain the observed oscillatory behavior of respiration rate in the continuous cultures of the bacteria Klebsiella aerogenes, and follows the form of the three-step reaction scheme:

$$\begin{aligned} A &\longrightarrow Y \\ B &\rightleftharpoons X \\ X + Y &\longrightarrow P \end{aligned}$$

where $X$ and $Y$ are the intermediate reactants, and represent oxygen and nutrient respectively, $A$ and $B$ explain "sources" or external parameters whose concentrations are to be kept at a constant level all over the reactor vessel, $P$ is the final product whose concentration is also assumed to be constant. In the reaction process, the last step is considered to be inhibited by excess of oxygen in the reactor. The first and last steps are assumed to be irreversible, however the second step is reversible. Degn and Harrison first proposed that the last step followed a nonlinear rate equation of the type $\frac{XY}{1+qX^2}$ , where $q$ measures the strength of the inhibitory law. With the homogeneous Neumann boundary conditions, the above Degn–Harrison reaction scheme is given by

$$\begin{cases} X_t - D_X \Delta X = k_2 B - k_3 X - \frac{k_4 XY}{1+qX^2}, \quad x \in \Omega,\ t > 0, \\ Y_t - D_Y \Delta Y = k_1 A - \frac{k_4 XY}{1+qX^2}, \quad x \in \Omega,\ t > 0, \\ \frac{\partial X}{\partial \eta} = \frac{\partial X}{\partial \eta} = 0, \quad x \in \partial\Omega,\ t > 0, \end{cases}$$

where $A, B, X$ and $Y$ denote dimensionless concentrations of the reactants, the constants $k_I$ $(i = 1, 2, 3, 4)$ are reaction rates, $D_X$ and $D_Y$ , respectively, denote the Fickian molecular diffusion coefficients of $X$and $Y$ , and they are assumed to be positive constants all over the reactor vessel. The rate and diffusion constants are parameters characteristic for a given system, and the concentrations $A$ and $B$ are variable parameters which can be controlled in the reaction process.

## 2.2 Stability analysis of PDE systems

### 2.2.1 Local stability

**Definition 2.2.1** *Let $\Omega$ be a domain which is enclosed by a simple curve $\partial\Omega$ (in the phase plane), $\Omega$ is called invariant set for (2.1) any solution with initial conditions in $\Omega$ stays inside $\Omega$ for all $t > 0$.*

**Definition 2.2.2** *A rectangle $\Re$ is said to be an invariant rectangle if the vector field $(F, G)$ on the boundary $\partial\Re$ points inside, i.e.*

$$\begin{cases} F(0, v) \geq 0 \text{ and } F(r_1, v) \leq 0 \text{ for } 0 < v < r_2, \\ G(u, 0) \geq 0 \text{ and } G(u, r_2) \leq 0 \text{ for } 0 < u < r_1. \end{cases} \tag{2.2}$$

We consider a general R–D system subject to Neumann boundary condition on spatial domain $\Omega = (0, l\pi)$ with $l \in \mathbb{R}$

$$\begin{cases} \frac{\partial u}{\partial t} - D_u \Delta u = F(u, v, \sigma), \qquad x \in (0, l\pi), t > 0, \\ \frac{\partial v}{\partial t} - D_v \Delta v = G(u, v, \sigma), \qquad x \in (0, l\pi), t > 0, \\ u_x(0, t) = u_x(l\pi, t) = v_x(0, t) = u_x(l\pi, t), t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), \quad x \in (0, l\pi)), \end{cases} \tag{2.3}$$

where $D_u, D_v, \sigma \in \mathbb{R}_+$, and $F, G : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ are $\mathbb{C}^K$ $(k > 5)$ with $f(0, 0, \sigma) = g(0, 0, \sigma) = 0$. Define the real-valued Sobolev space

$$X = \left\{(u, v) \in H^2(0, l\pi) \times H^2(0, l\pi) : u_x = (0, t) = u_x = (l\pi, t) = v_x = (0, t) = v_x = (l\pi, t) = 0\right\} \tag{2.4}$$

Properties of the Eigenvalues of the Laplace Operator In order to study the local asymptotic stability in the $PDE$ sense, one of the most commonly used methods is that of eigenfunction expansion [8]. It is important to recall some of the theory related to the eigenvalues of the Laplace operator. Let us denote these eigenvalues by $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \leq ...$and the corresponding normalized eigenfunctions in $\mathbb{R}$ by $\phi_0$,$\phi_1, ..., \phi_k$....We assume Neumann boundary conditions. These eigenvalues and eigenfunctions satisfy the eigenvalue problem

$$-\Delta\phi_k = \lambda_k\phi_k$$

in $\Omega$ with $\frac{\partial\phi_k}{\partial\nu} = 0$ on$\Omega$ , and

$$\int\limits_{\mathbb{R}} \phi_k^2(x)\, dx = 1$$

The linearization reaction–diffusion system(2.3)at point equilibrium has the form:

$$LU = \frac{\partial U}{\partial t} = D\Delta U + J_0 U, \tag{2.5}$$

where $J_0$ is the Jacobian matrix of the corresponding $ODE$ system evaluated at the equilibrium point , and $D$ is the matrix of diffusion coefficients given by :

$$J_0 = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}, \text{and } D = \begin{pmatrix} D_u & 0 \\ 0 & D_v \end{pmatrix}.$$

where $D_u$ and $D_v$denote the diffusivity constants for substances $u$ and $v$, respectively. The eigenvalues of the Laplace operator $\Delta$ over the interval $[0, l\pi]$ are the roots of the characteristic polynomial

$$\left|J_0 - Dk^2 - \lambda I\right| = 0,$$

where $k = \frac{n^2}{l^2}$. We consider the following characteristic equation of the operator $L$:

$$\begin{cases} L(\sigma)\binom{\phi}{\psi} = \xi\binom{\phi}{\psi}, & x \in \Omega, \\ \partial_\nu\phi = \partial_\nu\psi = 0, & x \in \partial\Omega. \end{cases} \tag{2.6}$$

Let $(\phi, \psi)^t$ be an eigenfunction of $L(\sigma)$ corresponding to the eigenvalue $\xi$, and let

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_0^\infty \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos\left(\frac{n}{l}x\right), \text{where } a_n \text{ and } b_n \text{are coefficients } \backslash n = (0, 1, 2......)$$

where

$$L_n(\sigma) = \begin{pmatrix} f_u - D_u\left(\frac{n}{l}\right)^2 & f_v \\ g_u & g_v - D_v\left(\frac{n}{l}\right)^2 \end{pmatrix}, n = (0, 1, 2......) \tag{2.7}$$

It follows that $L(\sigma)$ are given by the eigenvalues of $L_n(\sigma)$for $(n = 0, 1, 2, ...)$. The characteristic polynomial can be rewritten in the form

$$\lambda^2 - \lambda T_n(\sigma) + D_n(\sigma) = 0 \tag{2.8}$$

where

$$T_n(\sigma) = \left( tr(J_0) - \left(\frac{n}{l}\right)^2 (D_u + D_v) \right), \tag{2.9}$$

and

$$D_k(\sigma) = \det(J_0) - \left(\frac{n}{l}\right)^2 (f_u D_v + g_v D_u) + \left(\frac{n}{l}\right)^4 D_u D_v. \tag{2.10}$$

The eigenvalues $\xi_n$ are given by

$$\xi_n = \frac{T_n(\sigma) \pm \sqrt{T_n^2(\sigma) - 4D_n(\sigma)}}{2}, n = 0, 1, ..... \tag{2.11}$$

From the standard linear operator theory [5], it is known that if all the eigenvalues of the operator$L$ have negative real parts, then positive equilibrium $(u^*, v^*)$ is asymptotically stable, and if some eigenvalues have positive real parts, the positive equilibrium $(x^*, y^*)$ is unstable.

**Theorem 2.2.1** *[49] i) The equilibrium of (2.1) is globally asymptotically stable if for each nonnegative integer $n$ the eigenvalues of $A - \lambda_n D$ have negative real parts. Further there exist positive constants $K$ and $\omega$ such that for any $t > 0$,*

$$\|U(t, x)\| \leq K e^{-\omega t} \|U(t, x)\|.$$

*ii) The equilibrium is stable if for each nonnegative integer $n$ the eigenvalues of $A - \lambda_n D$ have nonpositive real parts and those with zero real parts have simple elementary divisors.*

*iii) The equilibrium is unstable if for some $n$ there exists an eigenvalue of $A - \lambda_n D$ with either a positive real part or a zero real part with a nonsimple elementary divisor.*

### 2.2.2 Diffusion-driven instability

In this subsection we study the onset of Turing instabilities in more detail for two variable systems

**Definition 2.2.3** *A diffusion-driven instability, or Turing instability, occurs when a steady state, stable in the absence of diffusion, becomes unstable when diffusion is present.*

Diffusion-driven instability refers to the situation where the solution is stable in the absence of diffusion, but unstable in the presence of diffusion

if $D_u = D_v$, the initial conditions for diffusion-driven instability become clear

$$\begin{cases} T_n(\sigma) = tr(J_0) > 0, \\ D_k(\sigma) = \det(J_0) < 0, \end{cases}$$

if $D_u \neq D_v$ Equation (2.9) will always be negative. So, for instability to occur, we must look at Equation (2.10).

$$D_k(\sigma) = \det(J_0) - \left(\frac{n}{l}\right)^2 (f_u D_v + g_v D_u) + \left(\frac{n}{l}\right)^4 D_u D_v < 0,$$

To find the minimum value of the $D_n(\sigma)$, we set the derivative with respect to $k^2$ equal to zero then :

$$\left(\frac{n}{l}\right)^2 = \frac{(f_u D_v + g_v D_u)}{2D_u D_v}.$$

The minimum value, substitute this expression for $\left(\frac{n}{l}\right)^2$ into the $D_n(\sigma)$. This is the third condition required for diffusion-driven instability to occur.

$$\begin{aligned}\det(J_0) - \left(\frac{(f_u D_v + g_v D_u)}{2D_u D_v}\right)(f_u D_v + g_v D_u) + \left(\frac{(f_u D_v + g_v D_u)}{2D_u D_v}\right)^2 D_u D_v &< 0,\\ \det(J_0) - \left(\frac{(f_u D_v + g_v D_u)}{2D_u D_v}\right)^2 D_u D_v + \left(\frac{(f_u D_v + g_v D_u)}{2D_u D_v}\right)^2 D_u D_v &< 0,\end{aligned}$$

$$\det(J_0) - \left(\frac{(f_u D_v + g_v D_u)}{2D_u D_v}\right)^2 D_u D_v < 0,$$

$$(f_u D_v + g_v D_u) > 2\sqrt{D_u D_v \det(J_0)}.$$

As $\det(J_0) > 0$the third condition becomes

$$(f_u D_v + g_v D_u) > 2\sqrt{D_u D_v \det(J_0)} > 0.$$

Consequently, the sufficient conditions for the existence of Turing instability in a linear 2–component reaction–diffusion system are

$$\begin{cases} tr(A) = f_u + g_v < 0, \\ \det(A) = f_u g_v - f_v g_u > 0, \\ (f_u D_v + g_v D_u) > 2\sqrt{D_u D_v (f_u g_v - f_v g_u)} > 0. \end{cases}$$

### 2.2.3 Global stability

**Lyapunov stability theory** At the beginning of the 1900's, the Russian mathematician Aleksandr Liapunov developed what is called the Liapunov Direct Method for determining the stability of an equilibrium point. We will describe this method and illustrate its applications. Let $f : X \longrightarrow X$ be a continuous mapping where $X$ is a metric space.

**Definition 2.2.4** *[13] Let $U^* \in \mathbb{R}^n$ is an equilibrium point of (2.1), and $\Omega \subseteq \mathbb{R}^n$ is an open set containing $U^*$, then the real valued function $V \in C^1(\Omega, \mathbb{R})$ is called a Lyapunov function for (2.1) if $U \in \Omega, U \neq U^*$*

$$V(U) > V(U^*),$$

*and*

$$\frac{dV(U(t))}{dt} :=< \nabla V(U), f(U) >\leq 0,$$

*for all $U \in \Omega$ . It follows that if (2.1) has a Lyapunov function, then $U^*$ is stable. Furthermore, if for all $U \neq 0$, $\frac{dV(U(t))}{dt} < 0$ then $U^*$ is asymptotically stable.*

In Definition 2.2.4 above, $< .,. >$ denotes the usual Euclidean scalar product. Note that the asymptotic stability condition

$$\frac{dV\left(U\left(t\right)\right)}{dt} < 0,$$

for all $U = 0$ simply means that the Lyapunov function is nonincreasing when we travel along the trajectory $U(t)$.

### 2.2.4 Hopf-Bifurcation

In this part we are interested, of derive an explicit algorithm for determining the direction of Hopf bifurcation and stability of the bifurcating periodic solutions for a reaction–diffusion system consisting of two equations with Neumann boundary condition, by using the center manifold theory and normal form method, introduced by Hassard [17]. Where he summed up their results [49, 48].

We consider Hopf bifurcations, we assume that for some $\sigma_0 \in \mathbb{R}$, the following condition holds:

$(H1)$ There exists a neighborhood $O$ of $\sigma_0$ such that for $\sigma \in O$, $L(\sigma)$ has a pair of complex, simple, conjugate eigenvalues $\alpha(\sigma) \pm i\omega(\sigma)$, continuously differentiable in $\sigma$, with $\alpha(\sigma_0) = 0$, $\omega\left(\sigma_0\right) = \omega_0 > 0$, and $\alpha^{'}(\sigma_0) \neq 0$; all other eigenvalues of $L(\sigma)$ have non-zero real parts for $\sigma \in O$. We rewrite system (2.3) in the abstract form

$$\frac{\partial U}{\partial t} = L(\sigma)U + F\left(\sigma, U\right), \text{ with } U = (u, v)^t \tag{2.12}$$

At $\sigma = \sigma_0$ the system (2.12) reduces to

$$\frac{\partial U}{\partial t} = L(\sigma_0)U + F\left(\sigma_0, U\right), \tag{2.13}$$

Let

$$\langle U_1, U_2 \rangle = \int_0^{l\pi} \left[(\overline{u_1}u_2) + (\overline{v_1}v_2)\right] dx,$$

with $U_i = (u_i, v_i)$ , $i = (1, 2)$ be the complex-valued $L^2$ inner product on Hilbert space $X_{\mathbb{C}}$. Let $L^*\left(\sigma\right)$ be the conjugate operator of $L\left(\sigma\right)$ defined in (2.5)

$$L^*\left(\sigma\right) = \begin{pmatrix} D_u\frac{\partial^2}{\partial x^2} + f_u & f_v \\ g_u & D_v\frac{\partial^2}{\partial x^2} + g_v \end{pmatrix}.$$

Such that $\langle U, L\left(\sigma\right)V\rangle = \langle L^*\left(\sigma\right)U, V\rangle$ .for any $U \in D_{L^*(\sigma)}, V \in D_{L(\sigma)}$, and $L^*q^* = -i\omega_0 q^*, Lq = -i\omega_0 q, \langle q^*, q\rangle = 1$, $\langle q^*, \overline{q}\rangle = 0$.We can choose $q^* = \cos\left(\frac{n}{l}\right)x\left(a_n, d_n\right)^t \in X_{\mathbb{C}}$. We decompose $X = X^c \oplus X^s$,with

$$\begin{cases} X^c = \left\{zq + \overline{zq} \backslash z \in \mathbb{C}\right\}, \\ X^s = \left\{u \in X \backslash \left\langle q^*, u\right\rangle = 0\right\}. \end{cases}$$

For any $(u, v) \in X$, there exits $z \in \mathbb{C}$ and $w = (w_1, w_2) \in X^s$ such that

$$(u, v)^t = zq + \overline{zq} + (w_1, w_2)^t, z = \left\langle q^*, (u, v)^t\right\rangle$$

The system(2.13) in $(z,w)$coordinates becomes

$$\begin{cases} \frac{\partial z}{\partial t} = i\omega_0 z + \langle q^*, F_0\rangle, \\ \frac{\partial w}{\partial t} = L(\sigma_0)w + H(z,\overline{z},w), \end{cases} \tag{2.14}$$

where

$$H(z,\overline{z},w) = F_0 - \langle q^*, F_0\rangle q - \langle \overline{q}^*, F_0\rangle \overline{q}, \quad \text{and} \quad F_0 = F_0(zq+\overline{zq}+w). \tag{2.15}$$

We write $F_0$ in the form:

$$F_0(U) = \frac{1}{2}B(U,U) + \frac{1}{6}C(U,U,U) + O\left(\|U\|^4\right), \text{ where } U=(u,v) \tag{2.16}$$

Let

$$H(z,\overline{z},w) = \frac{H_{20}}{2}z^2 + H_{11}z\overline{z} + \frac{H_{02}}{2}\overline{z}^2 + O\left(\|z\|^3\right),$$

then by (2.14) and (2.15), we have

$$\begin{aligned} H_{20} &= B_{qq} + \langle q^*, B_{qq}\rangle q + \langle \overline{q}^*, B_{qq}\rangle \overline{q}, \text{ where } B_{qq} = B(q,q), \\ H_{11} &= B_{q\overline{q}} + \langle q^*, B_{q\overline{q}}\rangle q + \langle \overline{q}^*, B_{q\overline{q}}\rangle \overline{q}, \text{ where } B_{q\overline{q}} = B(q,\overline{q}), \end{aligned}$$

We can write $w$ in the form: $w = \frac{w_{20}}{2}z^2 + w_{11}z\overline{z} + \frac{w_{02}}{2}\overline{z}^2 + O\left(\|z\|^3\right)$, the laste equation in $z,\overline{z}$ coordinates then

$$L(\sigma_0)w + H(z,\overline{z},w) = \frac{\partial w}{\partial z}\frac{dz}{dt} + \frac{\partial w}{\partial \overline{z}}\frac{d\overline{z}}{dt},$$

we have $w_{20} = (2i\omega_0 I - L(\sigma_0))^{-1} H_{20}$ and $w_{11} = -(L(\sigma_0))^{-1} H_{11}$. The reaction–diffusion system restricted to the center manifold is given by

$$\frac{\partial z}{\partial t} = i\omega_0 z + \langle q^*, F_0\rangle = i\omega_0 z + \sum_{2\le i+j\le 3} \frac{g_{ij}}{i!j!} z^i \overline{z}^j + O\left(\|U\|^4\right). \tag{2.17}$$

where

$$\begin{aligned} g_{20} &= \langle q^*, B_{qq}\rangle, \\ g_{11} &= \langle q^*, B_{q\overline{q}}\rangle, \\ g_{02} &= \langle q^*, B_{\overline{qq}}\rangle, \\ g_{21} &= 2\langle q^*, B_{w_{11}q}\rangle + \langle q^*, B_{w_{20}\overline{q}}\rangle + \langle q^*, C_{qq\overline{q}}\rangle. \end{aligned}$$

The dynamics of (2.14) can be determined by the dynamics of (2.17). We write the Poincaré normal form of (2.17) (for $\sigma$ in a neighborhood of $\sigma_0$) in the form:

$$\dot{z} = (\alpha(\sigma) + iw(\sigma))z + z\sum_{j=1}^{M} a_j(\sigma)(z\overline{z})^j,$$

where $z$ is a complex variable, $M \ge 1$ and $a_j(\sigma)$ are complex-valued coefficients.We have

$$\begin{aligned} a_1(\sigma) &= \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2} \\ &= \frac{i}{2\omega_0}\langle q^*, B_{qq}\rangle . \langle q^*, B_{q\overline{q}}\rangle + \langle q^*, B_{w_{11}q}\rangle + \frac{1}{2}\langle q^*, B_{w_{20}\overline{q}}\rangle + \frac{1}{2}\langle q^*, C_{qq\overline{q}}\rangle. \end{aligned}$$

**Theorem 2.2.2** *Suppose* $(H1)$ *is satisfied. Then (2.3) possesses a family of real-valued* $T\ (s)$ *periodic solutions* $(\sigma(s), u(s)(x,t), v(s)(x,t))$, *for s sufficiently small, bifurcating from* $(\sigma_0, 0, 0)$ *at* $\sigma = \sigma_0$ *in the space* $\mathbb{R} \times X$, *and there exists a unique* $n \in \mathbb{N}$, *such that* $(u(s)(x,t), v(s)(x,t))$ *can be parameterized in the form*

$$\begin{cases} u(s)(x,t) = s\left(a_n e^{\frac{2\pi it}{T(s)}} + \overline{a}_n e^{-\frac{2\pi it}{T(s)}}\right)\cos\left(\frac{n}{l}\right)x + O\left(s^2\right), \\ v(s)(x,t) = s\left(b_n e^{\frac{2\pi it}{T(s)}} + \overline{b}_n e^{-\frac{2\pi it}{T(s)}}\right)\cos\left(\frac{n}{l}\right)x + O\left(s^2\right). \end{cases}$$

*Furthermore:*

*1) The bifurcation is supercritical (resp. subcritical) if*

$$\frac{1}{\alpha'(\sigma_0)} Re\left(a\left(\sigma_0\right)\right) < 0 \quad (resp. > 0,)$$

*2) If in addition all other eigenvalues of* $L(\sigma_0)$ *have negative real parts, then the bifurcating periodic solutions are stable (resp. unstable) if* $Re\left(a\left(\sigma_0\right)\right) < 0 \quad (resp. > 0)$.

## 2.3 The CIMA Reaction-Diffusion model

The first experimental realization of a Turing pattern forming system is CIMA reaction that involves chlorite-iodide and malonic acid proposed from Lengyel- Epstein Where identified the key reactants in the system and the crucial role of the starch indicator as a complexing agent i.e. it weakly binds iodide and iodine to slow down the effective diffusion rates. In the CIMA reaction chlordioxide and iodine are produced as intermediates with near constant concentration . The reaction is described by three stoichiometric equations of the five independent chemical ingredients $MA, I_2, CIO_2, CIO_2^-$ and $I^-$ as shown in [24]. The first is the reaction of malonic acid $(MA)$ and iodine $(I_2)$:

$$MA + I_2 \rightarrow I\ M\ A + I^- + H^+ \tag{2.18}$$

along with the associate rate law

$$r_1 = \frac{\partial\left[I_2\right]}{dt} = k_{1a}\frac{\left[MA\right]\left[I_2\right]}{k_{1b} + \left[I_2\right]}. \tag{2.19}$$

The second is a reaction between chlorine dioxide $(CIO_2)$ and iodide $(I^-)$:

$$CIO_2 + I^- \rightarrow \frac{1}{2}I_2 + CIO_2^- \tag{2.20}$$

with rate

$$r_2 = 2\frac{\partial\left[I_2\right]}{dt} = k_2\left[CIO_2\right]\left[I_2\right]. \tag{2.21}$$

And the third component reaction between chlorite

$$CIO_2^- + 4I^- + 4H^+ \rightarrow 2I_2 + CI^- + 2H_2O, \tag{2.22}$$

with rate

$$r_3 = k_{3a}\left[CIO_2^-\right]\left[I^-\right]\left[H^+\right] + \frac{k_{3a}\left[CIO_2^-\right]\left[I_2\right]\left[I^-\right]}{\alpha + \left[I^-\right]^2}. \tag{2.23}$$

where $\alpha = 1\times 10^{-13}M^2$ is added to alleviate the situation when $I^-$ goes to zero. The denominator in (2.23) was introduced in [24]. The model consisting of the reactions (2.18)-(2.23) along with their empirical rate equations describes both the batch oscillation in the $CIO2 - I_2 - MA$system and the oscillatory behavior of the $CIO2 - I^-$ reaction in a flow reactor. Treating the concentration of $CIO_2$ , $I_2$ and $MA$ as constants, we obtain a two-variable $\left[CIO_2^-\right]$ and $\left[I^-\right]$ model, in agreement with the observed dynamics. The two-variable model shows the change of concentrations of the species $\left[CIO_2^-\right]$and $\left[I^-\right]$ with respect to time. Generally, this change can be obtained using the formula

$$\frac{\partial\left[A\right]}{\partial t} = \sum_{A\rightarrow} m \times \textit{rate of reaction} - \sum_{A\rightarrow} n \times \ \textit{rate of reaction}, \tag{2.24}$$

where $A$ is any component in the reaction and $m$ and $n$ are the stoichiometric factors of component $A$. The notation $(\rightarrow A)$ refers to component $A$ being a product and $(A \rightarrow)$indicates that it is a reactant. Therefore, the change in concentration for $\left[CIO_2^-\right]$ and $\left[I^-\right]$ may be described by

$$\frac{\partial\left[I^-\right]}{\partial t} = r_1 - r_2 - 4r_3, \text{ and } \frac{\partial\left[CIO_2^-\right]}{\partial t} = r_2 - r_3, \tag{2.25}$$

respectively. Considering that the concentrations of $CIO_2$, $I_2$ and $MA$ are treated as constants, the first two rates of reaction can be described as: $r_1 = k_1'$and $r_2 = k_2'\left[I^-\right]$ where

$$k_1' = k_{1a}\frac{\left[MA\right]\left[I_2\right]}{k_{1b} + \left[I_2\right]}, \text{and } k_2' = k_2\left[CIO_2\right]. \tag{2.26}$$

As for $r_3$, the first term has a negligible value compared to the second term under the applied conditions of the experiment, and thus the rate of reaction can be given by

$$r_3 = k_3'\frac{\left[CIO_2^-\right]\left[I^-\right]}{\alpha + \left[I^-\right]^2} \text{ withe } k_3' = k_{3a}\left[I_2\right]. \tag{2.27}$$

Assigning the notations $u$ and $v$ for $\left[I^-\right]$ and $\left[CIO_2^-\right]$, respectively, Eq. (2.25) yield the ODE system of the form

$$\begin{cases} \frac{\partial u}{\partial t} = k_1' - k_2'u - 4k_3'\frac{uv}{\alpha+u^2}, \\ \frac{\partial v}{\partial t} = k_2'u + k_3'\frac{uv}{\alpha+u^2}. \end{cases} \tag{2.28}$$

Simple normalization of (2.28) and considering the spatial diffusion of substances yields the reaction–diffusion system

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - a - \frac{4uv}{1+u^2}, \\ \frac{\partial v}{\partial t} = (\sigma b)\Delta v + (\sigma c)\left(u - \frac{uv}{1+u^2}\right). \end{cases} \tag{2.29}$$

Now we can create the Lengyel–Epstein model in this form

$$\begin{cases} u_t - \Delta u = a - u - \frac{4uv}{1+u^2}, & in\ \mathbb{R}^+ \times \Omega, \\ v_t - \sigma c \Delta v = \sigma b\left(u - \frac{4uv}{1+u^2}\right), & in\ \mathbb{R}^+ \times \Omega\ , \\ \partial_\nu u = \partial_\nu v = 0, & on\ \mathbb{R}^+ \times \partial\Omega\ , \\ 0 < u(x,0) = u_0(x), 0 < v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{2.30}$$

Where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$is the Laplacian operator on $\Omega$. The constants $a, b, c$ , and $\sigma$ are assumed to be positive.

### 2.3.1 Local stability in the ODE sense

Several authors who have studied the model of Lengyle-Epstien were concerned with local stability [25, 49, 50] Omit the diffusion terms in system (2.30), we have the following local system

$$\begin{cases} u_t = a - u - \frac{4uv}{1+u^2} = f(u,v,\sigma), \\ v_t = \sigma b\left(u - \frac{uv}{1+u^2}\right) = g(u,v,\sigma). \end{cases} \tag{2.31}$$

It is easy to see that (2.31) has a unique constant positive solution $(u^*, v^*) = \left(\alpha, 1+\alpha^2\right)$, where $\alpha = \frac{a}{5}$. Linearizing system (2.31) at the positive equilibrium $(u^*, v^*) = (0,0)$, we have

$$J_0 = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} \frac{3\alpha^2-5}{1+\alpha^2} & -\frac{4\alpha}{1+\alpha^2} \\ \frac{2\sigma\alpha^2 b}{1+\alpha^2} & -\frac{\sigma\alpha b}{1+\alpha^2} \end{pmatrix}, \tag{2.32}$$

and the corresponding equation:

$$\zeta^2 - \{tr\,(J)_0\,(u^*, v^*)\,\zeta + \det J_0\,(u^*, v^*) = 0,$$

where

$$\begin{cases} \{tr\,(J_0)\,(u^*, v^*) = \frac{3\alpha^2-5-\sigma\alpha b}{1+\alpha^2}, \\ \det J_0\,(u^*, v^*) = \frac{5\sigma\alpha b}{1+\alpha^2}. \end{cases} \tag{2.33}$$

For stability, we require that the trace be negative and the determinant be positive. The constants $a, b$, and $\sigma$ are positive, $\det J_0\,(u^*, v^*) > 0$ is evident. As for the trace, we can see that it is negative subject to the condition

$$\frac{3\alpha^2 - 5}{\alpha} < \sigma b. \tag{2.34}$$

Holds, then the equilibrium $(u^*, v^*)$ of system (2.31) is locally asymptotically stable.

### 2.3.2 Local stability in the PDE sense

The linearization Lengyel–Epstein reaction–diffusion system(2.30) at point equilibrium $(u^*, v^*) = (0,0)$ has the form:

$$LU = \frac{\partial U}{\partial t} = D\Delta U + J_0 U, \tag{2.35}$$

where $J_0$ is the Jacobian matrix of the corresponding $ODE$ system evaluated at the equilibrium point , and $D$ is the matrix of diffusion coefficients given by

$$\begin{pmatrix} 1 & 0 \\ 0 & \sigma c \end{pmatrix}. \tag{2.36}$$

From the standard linear operator theory [8], it is known that if all the eigenvalues of the operator $L$ have negative real parts, then $(u^*, v^*)$ is asymptotically stable, and if some eigenvalues have positive real parts, the $(u^*, v^*)$ is unstable. Suppose that $(\phi(x), \psi(x))$ is an eigenfunction of $L$ corresponding to the eigenvalue $\lambda$. Then

$$\begin{pmatrix} \Delta + \frac{3\alpha^2-5}{1+\alpha^2} - \lambda(\zeta_i) & -\frac{4\alpha}{1+\alpha^2} \\ (\sigma b)\frac{2\sigma\alpha^2 b}{1+\alpha^2} & \sigma c\Delta + (\sigma b)\frac{-\sigma\alpha b}{1+\alpha^2} - \lambda(\zeta_i) \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.37}$$

with

$$\phi = \sum_{0\le i\le\infty, 1\le j\le m_i} a_{ij}\Phi_{ij} \text{ and } \psi = \sum_{0\le i\le\infty, 1\le j\le m_i} b_{ij}\Phi_{ij},$$

We assume

$$\Gamma_k = J_0 - Dk^2, (k = 1, 2, 3, ...).$$

The eigenvalues of $L$ are given by the eigenvalues of $\Gamma_k$ for $k = 0, 1, 2, ...$The characteristic equation of $\Gamma_k$ given by $\lambda^2 - P_k\lambda - Q_k = 0$,
where

$$\begin{aligned} P_k &= \{tr(J)\Gamma_k = \{tr(J)_0 - (1+\sigma c)k^2 \\ &= \frac{3\alpha^2 - 5 - \sigma\alpha b}{1+\alpha^2} - (1+\sigma c)k^2, \end{aligned} \tag{2.38}$$

and

$$\begin{aligned} Q_k &= \det\Gamma_k = \det J_0 - k^2\left(\frac{(\sigma b)3\alpha^2 - 5}{1+\alpha^2} + \frac{-\sigma\alpha b}{1+\alpha^2}\right) + (\sigma b)k^4 \\ &= (\sigma b)k^4 - k^2\sigma\left(\frac{b(3\alpha^2-5)}{1+\alpha^2} - \frac{\alpha b}{1+\alpha^2}\right) + \frac{5\sigma\alpha b}{1+\alpha^2}. \end{aligned} \tag{2.39}$$

Consider the following system with the no-flux boundary condition in a one-dimensional over the interval $(0, \pi)$

$$\begin{cases} u_t = \frac{\partial^2 u}{\partial^2 x}a - u - \frac{4uv}{1+u^2}, \\ v_t = \sigma\left[c\frac{\partial^2 u}{\partial^2 x} + b\left(u - \frac{uv}{1+u^2}\right)\right], \end{cases} \tag{2.40}$$

with the above no-flux boundary condition

$$u_x(0,t) = u_x(\pi,t) = v_x(0,t) = u_x(\pi,t).$$

**Theorem 2.3.1** *Suppose that* $b > b_0 = \frac{(3\alpha^2-5)}{\sigma\alpha}$, *so that* $(u^*, v^*)$ *is a locally asymptotically stable equilibrium for(2.31). Then* $(u^*, v^*)$ *is an unstable equilibrium solution of (2.40) if*

$$\alpha^2 > 3 \text{ and } c > \frac{3\alpha b}{\alpha^2 - 3},$$

*and* $(u^*, v^*)$ *is a locally asymptotically stable equilibrium solution of (2.40) if*

$$\begin{cases} \frac{5}{3} < \alpha^2 \leq 3, \\ \quad or \\ \alpha^2 < 3 \text{ and } 0 < c \leq \frac{3\alpha b}{\alpha^2 - 3}. \end{cases}$$

### 2.3.3 Diffusion-driven instability

Based on the sufficient conditions for the existence of turing in above , we can go back to our Lengyel-Epstein model (2.30) and derive sufficient conditions for the existence or absence of turing instabilities. The main related results that can be found in the literature of [49] and [45].

**Theorem 2.3.2** *[49] Assuming*
$b > \frac{\left(3\alpha^2-5\right)}{\sigma\alpha}$.
*So that* $(u^*, v^*)$ *is a locally asymptotically stable equilibrium for ODE system(2.31).Then* $(u^*, v^*)$ *is an unstable equilibrium solution of PDE (2.30) if*

$$\alpha^2 > 3 \text{ and } c > \frac{3\alpha b}{\alpha^2 - 3}.$$

**Theorem 2.3.3** *[45] Suppose that* $b > \frac{\left(3\alpha^2-5\right)}{\sigma\alpha}$*.is satisfied, the equilibrium* $(u^*, v^*)$ *is Turing-unstable*
*if*
$\alpha^2 > \frac{3}{5}$,
$c > max \left\{ \frac{\alpha b}{13\alpha^2+5-4\alpha\sqrt{10(\alpha^2+1)}}, \frac{3\alpha^2-5}{\sigma\left(13\alpha^2+5-4\alpha\sqrt{10(\alpha^2+1)}\right)} \right\}$,
*and* $\exists\ n, m, l \in \mathbb{N}$ *and* $(n, m, l) \neq (0, 0, 0)$ *satisfying*

$$H_1 < \frac{n^2 + m^2}{l^2} < H_2,$$

*where*

$$H_1 = \frac{-\left(\frac{\alpha\sigma b}{1+\alpha^2} - \sigma c \frac{3\alpha^2-5}{1+\alpha^2}\right) - \sqrt{\Delta_H}}{2\sigma c},$$

*and*

$$H_2 = \frac{-\left(\frac{\alpha\sigma b}{1+\alpha^2} - \sigma c \frac{3\alpha^2-5}{1+\alpha^2}\right) + \sqrt{\Delta_H}}{2\sigma c},$$

*with*

$$\Delta_H(b) = \frac{\alpha^2\sigma^2 b^2 - 26\alpha^3\sigma^2 bc - 10\alpha\sigma^2 bc +^2 \sigma^2 c^2 \left(3\alpha^2 - 5\right)^2}{\left(1 + \alpha^2\right)^2}.$$

### 2.3.4 Global asymptotic stability

The authors focus on the global stability of $(u^*, v^*)$ assuming that the feeding rate of iodide $a$ is less than $\sqrt{27}$, without any restriction on the other parameters, by we establish sufficient conditions for the global asymptotic stability of its unique constant steady state[25, 34, 50].

**Theorem 2.3.4** *[25]*
*The equilibrium point $(u^*, v^*)$ is globally asymptotically stable for the ODE system (2.31 ) if*

$$\sigma b > \left(\frac{32}{25} + \alpha^2\right) - 4.$$

**Theorem 2.3.5** *Suppose* $0 < a^2 < 27$ *Then for any nonnegative* $u_0(x)$, $v_0(x) \in C^2(\Omega) \cap C(\overline{\Omega})$, *the solution.*$(u(x), v(x))$*of the system (2.30) converges uniformly in* $x$ *to* $(u^*, v^*)$.

The detailed proof of Theorem (2.3.5 ) above can be found in[50]. It suffices here to note that the proof is based on the direct Lyapunov method. The authors establish that the function:

$$E(t) = \int_\Omega \left[\int \left(u^2 - (u^*)^2\right) du + \int (4(v - v^*)) dv\right] dx. \tag{2.41}$$

**Corollary 2.3.1** *If* $0 < a^2 < 27$. *Then* $(u^*, v^*)$ *is the only equilibrium solution of (2.30), and it is globally asymptotically stable.*

**Theorem 2.3.6** *Assume* $a^2 \leq \frac{125}{4}$ . *Then, for any solution* $(u, v)$ *to (2.30), we obtain*

$$\lim_{t\to\infty} \|u(.,t) - u^*\|_{L^2(\Omega)} = \lim_{t\to\infty} \|v(.,t) - v^*\|_{L^2(\Omega)} .$$

*To prove this theorm, we use new lyapunov function*

$$E(t) = \int_\Omega \left[(\sigma b)(u - u^*) \frac{(u + 2u^*)}{3} + 2(v - v^*)^2\right] dx.$$

### 2.3.5 Hopf-Bifurcation

The Hopf–bifurcation of the Lengyel–Epstein system in the absence of diffusion was studied in [49] We analyze the Hopf bifurcation occurring at $(u^*, v^*)$ by choosing $b$ as the bifurcation parameter. Denote

$$b_0 = \frac{3\alpha^2 - 5}{\sigma\alpha}.$$

By using the translations $\widetilde{u} \to u - u^\star$ and $\widetilde{v} \to v - v^\star$, we obtain the modified system

$$\begin{cases} \frac{\partial \widetilde{u}}{\partial t} = 4\alpha - \widetilde{u}\left[\frac{4(\widetilde{u}+\alpha)\left(\widetilde{v}+\left(1+\alpha^2\right)\right)}{1+(\widetilde{u}+\alpha)^2}\right], \\ \frac{\partial \widetilde{v}}{\partial t} = \sigma b\left[(\widetilde{u} + \alpha) - \frac{4(\widetilde{u}+\alpha)\left(\widetilde{v}+\left(1+\alpha^2\right)\right)}{1+(\widetilde{u}+\alpha)^2}\right]. \end{cases} \tag{2.42}$$

From Poincaré–Andronov–Hopf Bifurcation Theorem in [46], $\beta'(b_0) = -\frac{\sigma\alpha}{2(1+\alpha^2)} < 0$ and calculate the sign of

$$a_0(\sigma) = \frac{2\alpha^4 - 27\alpha^2 - 5}{2\alpha^2(1 + \alpha^2)}.$$

we get new theorem

**Theorem 2.3.7** *[49] Suppose* $\sigma, \alpha > 0, b_0 = \frac{3\alpha^2-5}{\sigma\alpha}$ *and* $3\alpha^2 - 5 > 0$.

*1) The equilibrium* $(u^*, v^*)$ *of system (2.31 ) is locally asymptotically stable when* $b > b_0$*, and unstable when* $b < b_0$.

*2) The system (2.31) undergoes a Hopf bifurcation at* $(u^*, v^*)$ *when* $b = b_0$ *The direction of the Hopf bifurcation is subcritical and the bifurcating periodic solutions are orbitally asymptotically stable if*

$$\frac{5}{3} < \alpha^2 < \frac{27 + \sqrt{769}}{4}.$$

*and the direction of the Hopf bifurcation is supercritical and the bifurcating periodic solutions are unstable if*

$$\alpha^2 > \frac{27 + \sqrt{769}}{4}.$$

**Theorem 2.3.8** *Suppose* $\sigma, \alpha > 0, b_0 = \frac{3\alpha^2-5}{\sigma\alpha}$ *and* $3\alpha^2 - 5 > 0$*. Then system* $(ODE)$ *has at least one stable periodic solution satisfying* $0 < u(t) < a$*, and* $0 < v(t) < 1 + \varepsilon$ *for some* $\varepsilon > \frac{\alpha^2}{25}$.

**Theorem 2.3.9** *Subject to* $3\alpha^2 - 5 > 0$*, the system (2.40) undergoes a Hopf–bifurcation at* $(u^*, v^*)$*when* $b_0 = \frac{3\alpha^2-5}{\sigma\alpha}$ *. Then*

*1) If* $\frac{5}{3} < \alpha^2 < \frac{27+\sqrt{769}}{4}$*, then the direction of the Hopf bifurcation is supercritical and the bifurcating periodic solutions are unstable.*

*2) If* $\alpha^2 > \frac{27+\sqrt{769}}{4}$*, then the direction of the Hopf bifurcation is subcritical and the bifurcating periodic solutions are unstable.*

*3) If* $\frac{5}{3} < \alpha^2 \leq 3$ *or* $\alpha^2 < 3$ *and* $0 < c \leq \frac{3\alpha b}{\alpha^2-3}$*, then the direction of the Hopf bifurcation is subcritical and the bifurcating periodic solutions are orbitally asymptotically stable.*

## 2.4 Finite difference method

The finite-difference method among the first approaches applied to the numerical solution of differential equations . In this part we are interested by the parabolic partial differential equation we consider is the heat or diffusion equation[37]. We consider the parabolic partial differential equation

$$\frac{\partial u}{\partial t} u(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq l, \quad t > 0, \tag{2.43}$$

subject to the conditions:

$$u(0,t) = u(l,t) = 0, \quad t > 0, \text{ and } u(x,0) = f(x), \ 0 \leq x \leq l.$$

Let us assume the solution exists and focus on the finite difference method , which involves the following steps.

**Step 1: Generate a grid**. First select an integer $m > 0$ and define the $x$-axis step size $h = \frac{l}{m}$. Then select a timestep size $k$. The grid points for this situation are $(x_i, t_j)$, where $x_i = ih$, $for$ $i = 0, 1, ..., m$, and $t_j = jk$, for $j = 0, 1, ....$

**Step 2: Substitute derivatives with finite difference formulas at each grid point**

We obtain the difference method using the Taylor series in $t$ to form the difference quotient

$$\frac{\partial u}{\partial t} u\left(x_i, t_j\right) = \frac{u\left(x_i, t_{j+k}\right) - +u\left(x_i, t_j\right)}{k} - \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \mu_j), \quad \text{for } \mu_j \in \left(t_j, t_{j+k}\right), \tag{2.44}$$

and the Taylor series in $x$ to form the difference quotient

$$\frac{\partial^2 u}{\partial x^2}\left(x_i, t_j\right) = \frac{u\left(x_{i+h}, y_j\right) - 2u\left(x_i, y_j\right) + u\left(x_{i-h}, y_j\right)}{h^2} + \frac{h^2}{12}\frac{\partial^4 u}{\partial^2 x^4}(\zeta_i, y_j), \tag{2.45}$$

where $\zeta_i \in (x_{i-1}, x_{i+1})$. The parabolic partial differential equation ( 2.43) implies that at interior gridpoints $(x_i, t_j)$, for each $i = 1, 2, ..., m-1$ and $j = 1, 2, ...,$ we have

$$\frac{\partial u}{\partial t} u\left(x, t\right) - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

so the difference method using the difference quotients (2.44) and (2.45) is

$$\frac{w_{i,j+k} - w_{i,j}}{k} - \alpha^2 \frac{w_{i+h,j} - 2w_{i,j} + w_{i-h,j}}{h^2} = 0, \tag{2.46}$$

where $w_{i,j}$ approximates $u\left(x_i, t_j\right)$. The local truncation error for this difference equation is

$$\tau_{i,j} = \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \mu_j) - \frac{h^2}{12}\frac{\partial^4 u}{\partial^2 x^4}(\zeta_i, y_j), \tag{2.47}$$

Solving (2.46) for $w_{i,j+1}$ gives

$$w_{i,j+k} = \left(1 - \frac{2\alpha^2 k}{h^2}\right) w_{i,j} + \frac{\alpha^2 k}{h^2}\left(w_{i+h,j} + w_{i-h,j}\right),$$

for each $i = 1, 2, ..., m-1$ and $j = 1, 2, ...,$ So we have

$$w_{0,0} = f\left(x_0\right),\ w_{1,0} = f\left(x_1\right), ......w_{m,0} = f\left(x_m\right).$$

Then we generate the next $t$-row by

$$w_{0,1} = u\left(0, t_1\right) = 0,$$

$$w_{1,1} = \left(1 - \frac{2\alpha^2 k}{h^2}\right) w_{1,0} + \frac{\alpha^2 k}{h^2}\left(w_{2,0} + w_{0,0}\right),$$

$$w_{2,1} = \left(1 - \frac{2\alpha^2 k}{h^2}\right) w_{2,0} + \frac{\alpha^2 k}{h^2}\left(w_{3,0} + w_{1,0}\right),$$

.

$$w_{m-1,1} = \left(1 - \frac{2\alpha^2 k}{h^2}\right) w_{m-1,0} + \frac{\alpha^2 k}{h^2}\left(w_{m,0} + w_{m-2,0}\right),$$

$$w_{m,1} = u\left(m, t_1\right) = 0.$$

**Step3: Solve the system of algebraic equations.**

Now we can use the $w_{i,1}$ values to generate all the $w_{i,2}$ values and so on. The explicit nature of the difference method implies that the $(m-1)\times(m-1)$ matrix associated with this system can be written in the tridiagonal form

$$A = \begin{bmatrix} (1-2\lambda) & \lambda & 0 & . & . & . & 0 \\ \lambda & (1-2\lambda) & \lambda & . & & & . \\ 0 & \lambda & . & . & . & & . \\ 0 & . & . & . & . & . & . \\ . & & . & . & . & \lambda & 0 \\ . & & & . & \lambda & . & \lambda \\ 0 & . & . & 0 & 0 & \lambda & (1-2\lambda) \end{bmatrix},$$

where $\lambda = \frac{2\alpha^2 k}{h^2}$. If we let

$$w^{(0)} = \left(f\left(x_1\right), f\left(x_2\right), ..., f\left(x_{m-1}\right)\right)^t,$$

and

$$w^{(j)} = \left(w_{1,j}, w_{2,j}, ..., w_{m-1,j}\right)^t,$$

for eatch $(j = 1, 2...)$.

Then the approximate solution is given by

$$w^{(j)} = Aw^{(j-1)} \text{ , for eatch } (j = 1, 2...).$$

Chapter 3

# On the asymptotic stability of the time-fractional Lengyel-Epstein system

In this chapter [1] interesting us a time fractional version of the conventional Lengyel-Epstein CIMA reaction model. We establish sufficient conditions for the unique equilibrium's local and global asymptotic stability . Numerical results are presented to illustrate the effect of the fractional order on system dynamics.

[1] D. Mansouri, S. Abdelmalek, S. Bendoukha, On the asymptotic stability of the time-fractional Lengyel Epstein system, Computers and Mathematics with Applications, Vol 78,(2019),pp.415-1430

## 3.1 System model

In this chapter, we consider the time fractional Lengyel–Epstein system

$$\begin{cases} {}_0^C D_t^{\delta} u - d_1 \Delta u = a - u - \frac{4uv}{1+u^2} =: F(u,v), & \text{in } \mathbb{R}^+ \times \Omega, \\ {}_0^C D_t^{\delta} v - d_2 \Delta v = \sigma b \left( u - \frac{uv}{1+u^2} \right) =: G(u,v), & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{3.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n = 2,3$ in practice) with smooth boundary $\partial\Omega$, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $0 < \delta \leq 1$ is the fractional order, ${}_0^C D_t^{\delta}$ denotes the Caputo fractional derivative over $(0, \infty)$ as defined in (1.23), and $d_1, d_2, a, b$ and $\sigma$ are strictly positive constants. We assume the nonnegative initial conditions

$$0 \leq u(0,x) = u_0(x), \quad 0 \leq v(0,x) = v_0(x), \quad \text{in } \Omega, \tag{3.2}$$

with $u_0, v_0 \in C^2(\Omega) \cap C(\overline{\Omega})$, and impose homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on} \quad \mathbb{R}^+ \times \partial\Omega, \tag{3.3}$$

where $\nu$ is the unit outer normal to $\partial\Omega$. Before we study the local and global asymptotic stability of the solutions of the proposed system, let us define its invariant region. We start with a definition of the term invariant region following the lines of [32, 50]. Note that when $F(u,v) = 0$, the curves in the $u$–$v$ plane are called $u$–isoclines. Similarly, they are called $v$–isoclines when $G(u,v) = 0$. in addition, if the vector field $(F, G)$ does not point outwards at the boundary of a certain rectangle $\partial\Re$, then $\Re$ is said to be an invariant rectangle (see definition 2.2.2).

**Proposition 3.1.1** *System (3.1) admits the region of attraction*

$$\Re_a = (0, a) \times \left(0, 1 + a^2\right). \tag{3.4}$$

**Lemme 3.1.1** *If an equilibrium point $(u^*, v^*)$ of (1.24) is locally asymptotically stable for the standard system*

$$\begin{cases} u_t = F(u,v), & \textit{in } \mathbb{R}^+, \\ v_t = G(u,v), & \textit{in } \mathbb{R}^+, \end{cases} \tag{3.5}$$

*then, it is also locally asymptotically stable for (1.24).*

**Preuve.** Assuming that $(u^*, v^*)$ is a locally asymptotically stable equilibrium for (3.5), then all the eigenvalues of the Jacobian matrix have negative real parts, i.e.

$$|\arg(\lambda_i)| > \frac{\pi}{2}, \quad i = 1, 2.$$

Since $\delta < 1$, it is trivial to see that (1.26) holds, which leads to the local asymptotic stability of $(u^*, v^*)$ as an equilibrium of (1.24). □

**Corollary 3.1.1** *In the diffusion case, if an equilibrium point* $(u^*, v^*)$ *of (1.24) is locally asymptotically stable for the integer system*

$$\begin{cases} u_t - d_1 \Delta u = F(u,v), & in\ \mathbb{R}^+ \times \Omega, \\ v_t - d_2 \Delta v = G(u,v), & in\ \mathbb{R}^+ \times \Omega, \end{cases}$$

*then it is also locally asymptotically stable for*

$$\begin{cases} {}_0^C D_t^\delta u - d_1 \Delta u = F(u,v), & in\ \mathbb{R}^+ \times \Omega, \\ {}_0^C D_t^\delta v - d_2 \Delta v = G(u,v), & in\ \mathbb{R}^+ \times \Omega. \end{cases}$$

## 3.2 Asymptotic stability conditions

### 3.2.1 Local stability

In this section, we derive sufficient conditions for the local asymptotic stability of the equilibrium point of (3.1). The free diffusions system corresponding to (3.1) is

$$\begin{cases} {}_0^C D_t^\delta u = a - u - \frac{4uv}{1+u^2}, \\ {}_0^C D_t^\delta v = \sigma b \left( u - \frac{uv}{1+u^2} \right). \end{cases} \tag{3.6}$$

Let us assume that $(u^*, v^*) = \left(\alpha, 1+\alpha^2\right)$ where $(\alpha = \frac{a}{5})$ be the equilibrium point of the system (3.6). In order to study the asymptotic stability of the equilibrium point $(u^*, v^*)$, consider the translation $(u,v) = (u - u^\star, v - v^\star)$, and since by design we obtain ${}_0^C D_t^\delta u^* ={}_0^C D_t^\delta v^* = 0$ , the new system dynamics

$$\begin{cases} {}_0^C D_t^\delta (U + u^*) ={}_0^C D_t^\delta U = a - (U+u^*) - \frac{4(U+u^*)(V+v^*)}{1+(U+u^*)^2}, \\ {}_0^C D_t^\delta (V + v^*) ={}_0^C D_t^\delta V = \sigma b \left( (U+u^*) - \frac{(U+u^*)(V+v^*)}{1+(U+u^*)^2} \right). \end{cases}$$

Using the Taylor series expansion of $F((U+u^*),(V+v^*))$ and $G((U+u^*),(V+v^*))$, we get

$$\begin{cases} F((U+u^*),(V+v^*)) = F(u^*,v^*) + U \left.\frac{\partial F}{\partial U}\right|_{(0,0)} U + \left.\frac{\partial F}{\partial V}\right|_{(0,0)} V + \textit{higher order terms}, \\ G((U+u^*),(V+v^*)) = G(u^*,v^*) + \left.\frac{\partial G}{\partial U}\right|_{(0,0)} U + \left.\frac{\partial G}{\partial V}\right|_{(0,0)} V + \textit{higher order terms}. \end{cases}$$

Given that $F(u^*,v^*) = G(u^*,v^*) = 0$, and whenever $(U,V)$ is small enough, we have

$$\begin{cases} F((U+u^*),(V+v^*)) = U \left.\frac{\partial F}{\partial U}\right|_{(0,0)} U + \left.\frac{\partial F}{\partial V}\right|_{(0,0)} V, \\ G((U+u^*),(V+v^*)) = \left.\frac{\partial G}{\partial U}\right|_{(0,0)} U + \left.\frac{\partial G}{\partial V}\right|_{(0,0)} V. \end{cases} \tag{3.7}$$

Substituting this into the system dynamics, we obtain the linearized fractional order form

$${}_0^C D_t^\delta \binom{U}{V} = \left.\begin{pmatrix} \frac{\partial F}{\partial U} & \frac{\partial F}{\partial V} \\ \frac{\partial G}{\partial U} & \frac{\partial G}{\partial V} \end{pmatrix}\right|_{(0,0)} \binom{U}{V},$$

where the Jacobian matrix is obtained as

$$\left(\begin{array}{cc} \frac{\partial F}{\partial U} & \frac{\partial F}{\partial V} \\ \frac{\partial G}{\partial U} & \frac{\partial G}{\partial V} \end{array}\right)\Bigg|_{(0,0)} = \left(\begin{array}{cc} \frac{\partial F}{\partial U} & \frac{\partial F}{\partial V} \\ \frac{\partial G}{\partial U} & \frac{\partial G}{\partial V} \end{array}\right)\Bigg|_{(u^*,v^*)}.$$

The local stability of equilibrium points of system (3.6) reduces to stability of zero solution of linear system (3.7). More on the properties of the linearized system in comparison to the original nonlinear one can be found in[22].

**Proposition 3.2.1** *System (3.6) has the unique equilibrium*

$$(u^*, v^*) = \left(\alpha, 1+\alpha^2\right), \tag{3.8}$$

*with*

$$\alpha = \frac{a}{5}. \tag{3.9}$$

*Subject to*

$$\Upsilon = \left(\frac{3\alpha^2 - 5 - \sigma b\alpha}{1+\alpha^2}\right)^2 - 20\frac{\sigma b\alpha}{\alpha^2+1} \geq 0,$$

$(u^*, v^*)$ *is asymptotically stable if*

$$tr\,(J) < 0,$$

*and unstable if*

$$tr\,(J) > 0,$$

*where*

$$J = \left(\begin{array}{cc} \frac{3\alpha^2-5}{1+\alpha^2} & -\frac{4\alpha}{1+\alpha^2} \\ \sigma b\frac{2\alpha^2}{1+\alpha^2} & -\sigma b\frac{\alpha}{1+\alpha^2} \end{array}\right).$$

*Alternatively, if* $\Upsilon < 0$*, then* $(u^*, v^*)$ *is asymptotically stable whenever* $trJ \leq 0$ *or*

$$|\arg(\lambda_1)| > \delta\frac{\pi}{2} \text{ and } |\arg(\lambda_2)| > \delta\frac{\pi}{2}, \tag{3.10}$$

*where*

$$\lambda_{1,2} = \frac{1}{2}\left[\frac{3\alpha^2 - 5 - \sigma b\alpha}{1+\alpha^2} \pm i\sqrt{-\Upsilon}\right]. \tag{3.11}$$

**Preuve.** The Jacobian matrix in $(u^*, v^*)$ is given by

$$J\,(u^*, v^*) = \left(\begin{array}{cc} \frac{3\alpha^2-5}{1+\alpha^2} & -\frac{4\alpha}{1+\alpha^2} \\ \sigma b\frac{2\alpha^2}{1+\alpha^2} & -\sigma b\frac{\alpha}{1+\alpha^2} \end{array}\right).$$

Its determinant and trace are given by

$$\det J\,(u^*, v^*) = 5\sigma b\frac{\alpha}{\alpha^2+1},$$

and

$$tr\,(J)\,(u^*, v^*) = \frac{3\alpha^2 - 5 - \sigma b\alpha}{1+\alpha^2},$$

respectively. The characteristic equation of the Jacobian matrix is

$$\lambda^2 - (tr\,(J)\,\lambda + \det J = 0,$$

and its discriminant is

$$\Upsilon = tr(J)^2 - 4\det J.$$

We study the different cases separately [30]. First, if $\Upsilon > 0$, then the eigenvalues $\lambda_{1,2}$ are real and can be rewritten as

$$\lambda_{1,2} = \frac{1}{2}\left[tr(J) \pm \sqrt{\Upsilon}\right].$$

Note that $\det J > 0$. Hence, the negativity of the eigenvalues rests on the sign of the trace tr$J$:

- If tr$J < 0$, then

  $$\lambda_1 = \frac{1}{2}\left[tr(J) - \sqrt{\Upsilon}\right] < 0,$$

  and, therefore, $\arg(\lambda_1) = \pi$. Since both eigenvalues are real, the trace is negative, and the determinant is positive, it is evident that $|\arg(\lambda_2)| = |\arg(\lambda_1)| = \pi > \frac{\delta\pi}{2}$ as $\delta \in (0,1]$. It follows that the equilibrium $(u^*, v^*)$ is asymptotically stable.

- If tr$J > 0$, we have

  $$tr\,(J) - \sqrt{\Upsilon} > 0,$$

  leading to

  $$\lambda_1 = \frac{1}{2}\left[tr\,(J) - \sqrt{\Upsilon}\right] > 0,$$

  and thus

  $$|\arg(\lambda_1)| = 0.$$

  So, $(u^*, v^*)$ is asymptotically unstable.

- If tr$J = 0$, then

  $$\Upsilon > 0 \Rightarrow -4\det J > 0,$$

  which is a contradiction. Hence, this case does not show up.

Next, we consider the case of the discriminant $\Upsilon$ being equal to zero. Since $\det J > 0$, then it is impossible that tr$J = 0$. The eigenvalues reduce to

$$\lambda_{1,2} = \frac{1}{2} tr\,(J).$$

The sign of the eigenvalues is identical to that of the trace. Consequently, $(u^*, v^*)$ is asymptotically stable for all $\delta \in (0,1]$ if $tr\,(J) < 0$ and unstable if $tr\,(J) > 0$.
Finally, if the discriminant $\Upsilon < 0$, then

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{2}\left[tr\,(J) \pm \sqrt{\Upsilon}\right] \\ &= \frac{1}{2}\left[tr\,(J) \pm i\sqrt{-\Upsilon}\right].\end{aligned}$$

We, now, have three cases:

- If $tr\,(J) < 0$, then by means of Lemma 3.1.1, $(u^*, v^*)$ is asymptotically stable.
- If $tr\,(J) = 0$, then
$$\left|\arg\left(\lambda_{1,2} = \pm\frac{1}{2}i\sqrt{-\Upsilon}\right)\right| = \frac{\pi}{2}.$$
Hence, for $\delta < 1$, $(u^*, v^*)$ is asymptotically stable.
- If $tr\,(J) > 0$, then $(u^*, v^*)$ is asymptotically stable subject to (3.10).

The proof is complete. □

Now, let us move on to the complete system (3.1). For this, we are going to use the eigenfunction expansion method [8]. We denote the eigenvalues of the spectral problem with Neumann boundary conditions by $0 = \eta_0 < \eta_1 \leq \eta_2 \leq ..... \leq \eta_n..$ and the corresponding normalized eigenfunctions by $\phi_0, \cdots, \phi_k, \cdots$. Let us set

$$J_i = \begin{pmatrix} F_0 - d_1\eta_i & F_1 \\ \sigma G_0 & \sigma G_1 - d_2\eta_i \end{pmatrix}, \tag{3.12}$$

and

$$L = \begin{pmatrix} d_1\Delta + F_0 & F_1 \\ \sigma G_0 & d_2\Delta + \sigma G_1 \end{pmatrix}, \tag{3.13}$$

where

$$F_0 = \frac{3\alpha^2 - 5}{1+\alpha^2},\ F_1 = -\frac{4\alpha}{1+\alpha^2},\ G_0 = b\frac{2\alpha^2}{1+\alpha^2},\ \text{and } G_1 = -b\frac{\alpha}{1+\alpha^2}. \tag{3.14}$$

In addittion, if $d_1 > d_2$, we define $\eta_{01} < \eta_{02}$ as the roots of

$$\Upsilon_i = (d_1 - d_2)^2\eta_i^2 + 2(d_1 - d_2)(-F_0 + \sigma G_1)\eta_i + \Upsilon. \tag{3.15}$$

The following proposition describes the conditions for the asymptotic stability of the steady state assuming $F_0 > 0$.

**Proposition 3.2.2** *If $d_1 = d_2$, then the asymptotic stability conditions are identical to the free diffusions case as stated in Proposition 3.2.1. Alternatively, if $d_1 \neq d_2$, $tr\,(J) < 0$ and $\Upsilon > 0$, then $(u^*, v^*)$ is an asymptotically stable constant steady state if $d_1 < d_2$ and*

$$\begin{cases} \eta_1 d_1 \geq F_0, & or \\ \eta_1 d_1 < F_0 & and \quad 0 < d_2 < \tilde{d}, \end{cases} \tag{3.16}$$

*where*

$$d_i = \sigma b\frac{\alpha}{1+\alpha^2}\frac{(\eta_i d_1 + 5)}{(F_0 - \eta_i d_1)\eta_i}, \tag{3.17}$$

*and*

$$\tilde{d} = \min_{i\geq 0} d_i. \tag{3.18}$$

*If $d_1 > d_2$, the euilibrium $(u^*, v^*)$ is asymptotically stable if $\eta_1 d_1 \geq F_0$ and the eigenvalues*

$$\xi_{1,2}(\eta_i) = \frac{1}{2}\left[tr\,(J)_i \pm i\sqrt{4\det J_i - (tr\,(J)_i)^2}\right] \tag{3.19}$$

*satisfy*

$$\left|\arg\left(\xi_1\left(\eta_i\right)\right)\right| > \delta\frac{\pi}{2} and \left|\arg\left(\xi_2\left(\eta_i\right)\right)\right| > \delta\frac{\pi}{2} \tag{3.20}$$

*for all* $\eta_i \in (\eta_{01}, \eta_{02})$.

**Preuve.** In order to study the local asymptotic stability in the PDE sense, we will linearize the system. Following the standard linear operator theory (see [8]), and keeping in mind the fractional nature of the system, we can state that $(u^*, v^*)$ is asymptotically stable if the eigenvalues of the linearized system satisfy the conditions of Lemma 1.3.2.

Suppose that $(\phi(x), \psi(x))^T$ is an eigenfunction of $L$ corresponding to the eigenvalue $\xi$. Then,

$$\begin{pmatrix} d_1\Delta + F_0 - \xi(\eta_i) & F_1 \\ \sigma G_0 & d_2\Delta + \sigma G_1 - \xi(\eta_i) \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

With

$$\phi = \sum_{0\le i\le\infty, 1\le j\le m_i} a_{ij}\Phi_{ij} \text{ and } \psi = \sum_{0\le i\le\infty, 1\le j\le m_i} b_{ij}\Phi_{ij},$$

we obtain

$$\sum_{0\le i\le\infty, 1\le j\le m_i} \begin{pmatrix} F_0 - d_1\eta_i - \xi(\eta_i) & F_1 \\ \sigma G_0 & \sigma G_1 - d_2\eta_i - \xi(\eta_i) \end{pmatrix} \begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix} \Phi_{ij} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It holds that

$$\begin{pmatrix} F_0 - d_1\eta_i - \xi(\eta_i) & F_1 \\ \sigma G_0 & \sigma G_1 - d_2\eta_i - \xi(\eta_i) \end{pmatrix} = J_i - \xi(\eta_i) I,$$

with $J_i$ as defined in (3.12). The characteristic equation of matrix $J_i$ is

$$\xi^2(\eta_i) - tr(J)_i\ \xi(\eta_i) + \det J_i = 0, \tag{3.21}$$

where

$$tr(J)_i = -(d_1 + d_2)\eta_i + tr(J),$$

and

$$\det J_i = (\eta_i d_1 - F_0)\eta_i d_2 + \frac{\sigma b\alpha}{1+\alpha^2}(\eta_i d_1 + 5).$$

In order to investigate the stability of $(u^*, v^*)$, we examine the nature of the eigenvalues by taking the discriminant of (3.21), which is given by

$$\begin{aligned} \Upsilon_i &= (tr(J)_i)^2 - 4\det J_i \\ &= (d_1 - d_2)^2 \eta_i^2 + 2(d_1 - d_2)(-F_0 + \sigma G_1)\eta_i + \left((-F_0 + \sigma G_1)^2 + 4\sigma F_1 G_0\right) \\ &= (d_1 - d_2)^2 \eta_i^2 + 2(d_1 - d_2)(-F_0 + \sigma G_1)\eta_i + \Upsilon. \end{aligned}$$

The sign of $\Upsilon_i$ is important for the stability of $(u^*, v^*)$. The discriminant of $\Upsilon_i$ with respect to $\eta_i$ is

$$\Delta_\eta = 32(d_1 - d_2)^2 \sigma b \frac{\alpha^3}{(1+\alpha^2)^2}.$$

We have a number of cases for $\Delta_\eta$:

- If $d_1 = d_2$, we notice that

$$\Upsilon_i = \Upsilon_0 = \Upsilon.$$

Hence, the exact same conditions for OFDE stability as described in Proposition 3.2.1 apply here.

- If $d_1 \neq d_2$, then $\Delta_\eta > 0$. Hence, $\Upsilon_i$ has two real roots and we have two cases:

➢ If $d_1 < d_2$, then using $tr\,(J)_i > 0$, we have

$$2\,(d_1 - d_2)\,(-F_0 + \sigma G_1) > 0.$$

Thus, since $\Upsilon > 0$, the solutions $\eta_{01}$ and $\eta_{02}$ of the equation $\Upsilon_i = 0$ are both negative regardless of $i$. Hence, $\Upsilon_i > 0$ for all $i$ and the roots of (3.21)

$$\xi_1\,(\eta_i) = \frac{tr\,(J_I) - \sqrt{tr\,(J_i)^2 - 4detJ_i}}{2},$$

and

$$\xi_1\,(\eta_i) = \frac{tr\,(J_I) + \sqrt{tr\,(J_i)^2 - 4detJ_i}}{2}.$$

are real. Note that

$$tr\,(J) < 0 \Rightarrow tr\,(J)_i < 0,$$

which leads to $\xi_1\,(\eta_i) < 0$. Also, if $\eta_1 d_1 \geq F_0$, then $\xi_2\,(\eta_i) < 0$. This leads to

$$|\arg\left(\xi_1\,(\eta_i)\right)| = |\arg\left(\xi_2\,(\eta_i)\right)| = \pi,$$

which guarantees the asymptotic stability of $(u^*, v^*)$.

Alternatively, if $\eta_1 d_1 < F_0$ and $0 < d_2 < \tilde{d}$, then

$$\eta_i d_1 < F_0 \text{ and } d_2 < d_i \text{ for } i \in [1, i_\alpha]\,.$$

It follows that $\det J_i > 0$ for all $i \in [1, i_\alpha]$. Furthermore, if $i > i_\alpha$ then $\eta_i d_1 \geq F_0$ and $\det J_i > 0$. The argument leads to the asymptotic stability of $(u^*, v^*)$ again.

➢ If $d_1 > d_2$, we have

$$2\,(d_1 - d_2)\,(-F_0 + \sigma G_1) > 0,$$

and since $\Upsilon > 0$, we have $0 < \eta_{01} \leq \eta_{02}$. Hence,

$$\left\{\begin{array}{l} \eta_i \geq \eta_{02} \\ \quad\text{or} \\ \eta_i \leq \eta_{01} \end{array}\right. \Rightarrow \Upsilon_i \geq 0,$$

which takes us back to the previous case. Again, for $\eta_1 d_1 \geq F_0$, we have $\det J_i > 0$ and thus $\xi_1$ and $\xi_2$ are negative. Next, if $\eta_{01} < \eta_i < \eta_{02}$, we have $\Upsilon_i < 0$ and $\det J_i > 0$. The eigenvalues are, thus, complex, see (3.19). Hence, $(u^*, v^*)$ is an asymptotically stable equilibrium subject to (3.20) for all $\eta_i$ in the interval $(\eta_{01}, \eta_{02})$.

□

### 3.2.2 Global stability

In this section, we derive conditions for the global asymptotic stability. First of all, let us define the function

$$f_a(u) = \frac{a - \mu u}{\varphi(u)}, \tag{3.22}$$

where

$$\varphi(u) = \frac{u}{1+u^2}. \tag{3.23}$$

Obviously, we have

$$f_a(u^*) = \lambda \frac{\alpha}{\varphi(\alpha)}. \tag{3.24}$$

Also, setting

$$U = u - u^* \text{ and } V = v - v^*, \tag{3.25}$$

we obtain the modified system

$$\begin{cases} {}_{t_0}^{C}D_t^{\delta} U - d_1 \Delta U = \varphi(U+u^*)\left[\left(f_a(U+u^*) - f_a(u^*)\right) - 4V\right], \\ {}_{t_0}^{C}D_t^{\delta} V - d_2 \Delta V = \sigma b \varphi(U+u^*)\left[U(U+2u^*) - V\right]. \end{cases} \tag{3.26}$$

**Theorem 3.2.1** *Subject to*

$$0 < a^2 \leq 27, \tag{3.27}$$

*equilibrium* $(u^*, v^*)$ *is globally asymptotically stable.*

**Preuve.** In order to establish the global asymptotic stability, we use the Lyapunov method. Let

$$L(t) = \int_{\Omega} \left[\frac{\sigma b}{3} U^3 + \sigma b u^* U^2 + 2V^2\right] dx. \tag{3.28}$$

Taking the fractional Caputo derivative of (3.28) and using (1.25), we obtain

$$\begin{aligned} {}_{t_0}^{C}D_t^{\delta} L(t) &= \int_{\Omega} \left[\left(\frac{\sigma b}{3}\right) {}_{t_0}^{C}D_t^{\alpha} U^3 + (\sigma b u^*)\ {}_{t_0}^{C}D_t^{\alpha} U^2 + 2\,{}_{t_0}^{C}D_t^{\alpha} V^2\right] dx \\ &\leq \int_{\Omega} \left[\sigma b U^2\, {}_{t_0}^{C}D_t^{\alpha} U + 2(\sigma b u^*)\, U\ {}_{t_0}^{C}D_t^{\alpha} U + 4V\ {}_{t_0}^{C}D_t^{\alpha} V\right] dx, \end{aligned}$$

see [6]. Further simplification yields

$$\begin{aligned} {}_{t_0}^{C}D_t^{\delta} L(t) &\leq \int_{\Omega} \left[\sigma b U(U+2u^*)\ {}_{t_0}^{C}D_t^{\alpha} U + 4V\ {}_{t_0}^{C}D_t^{\alpha} V\right] dx \\ &\leq \int_{\Omega} \varphi(U+u^*)\left\{\sigma b U(U+2u^*)\left[\left(f_a(U+u^*) - f_a(u^*)\right) - 4V\right]\right. \\ &\quad \left. + 4V\sigma b\left[U(U+2u^*) - V\right]\right\} dx + \int_{\Omega} \sigma b U(U+2u^*)\, d_1 \Delta U dx \\ &\quad + \int_{\Omega} 4V d_2 \Delta V dx \\ &\leq \int_{\Omega} \sigma b \varphi(U+u^*)\left\{U(U+2u^*)\left(f_a(U+u^*) - f_a(u^*)\right)\right. \\ &\quad \left. - 4U(U+2u^*)V + 4VU(U+2u^*) - 4V^2\right\} dx \\ &\quad + \sigma b \int_{\Omega} U(U+2u^*)\, d_1 \Delta U dx + 4d_2 \int_{\Omega} V \Delta V dx, \end{aligned}$$

leading to

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\delta}L(U,V) \leq & \underbrace{\sigma b\int_{\Omega}\varphi(U+u^*)\left\{U(U+2u^*)\left(f_a(U+u^*)-f_a(u^*)\right)-4V^2\right\}dx+}_{I_1(t)} \\
& \underbrace{+\sigma bd_1\int_{\Omega}U(U+2u^*)\Delta U dx+4d_2\int_{\Omega}V\Delta V dx.}_{I_2(t)}
\end{aligned} \tag{3.29}
$$

We note that the function $f_a$ is strictly decreasing over the interval $(0,a)$ when $0 < a^2 \leq 27$. Hence, by the mean value theorem, there exists some $c$ between $u$ and $u^*$ such that

$$
f_a(U+u^*) - f_a(u^*) = U f_a'(c).
$$

Substituting in $I_1(t)$ yields

$$
I_1(t) = \sigma b\int_{\Omega}\varphi(U+u^*)\left\{U^2(U+2u^*)f_a'(c)-4V^2\right\}dx < 0.
$$

For $I_2(t)$, we have

$$
\begin{aligned}
I_2(t) &= \sigma bd_1\int_{\Omega}U(U+2u^*)\Delta U dx + 4d_2\int_{\Omega}V\Delta V dx \\
&= -\sigma bd_1\int_{\Omega}\nabla\left(U^2+2u^*U\right)\nabla U dx - 4d_2\int_{\Omega}|\nabla V|^2 dx \\
&= -\sigma bd_1\int_{\Omega}2(U+u^*)|\nabla U|^2 dx - 4d_2\int_{\Omega}|\nabla V|^2 dx < 0.
\end{aligned}
$$

Hence,

$$
{}_{t_0}^{C}D_t^{\delta}L(U,V) < 0, \tag{3.30}
$$

and ${}_{t_0}^{C}D_t^{\delta}L(t) = 0$ if and only if $(U,V) = (0,0)$. Therefore, by the direct Lyapunov method, the constant steady state $(u^*, v^*)$ is globally asymptotically stable subject to (3.27). ☐

## 3.3 Numerical examples

In this section, we present some numerical examples to show the effect of $\delta$ on the dynamics of the fractional Lengyel–Epstein system (3.1). Consider the parameter set $(a,b,\sigma,d_1,d_2) = (15,1,7,1,10)$ and initial conditions

$$
\begin{cases}
u(x,0) = 1 + 0.3\sin\left(\frac{x}{2}\right), \\
v(x,0) = 2 + 0.6\sin\left(\frac{x}{2}\right).
\end{cases} \tag{3.31}
$$

The solutions of system (3.1) with zero Neumann boundary conditions and different values of $\delta$ were obtained numerically for $t \in [0,10]$ and $x \in [0,20]$ with $\Delta t = 0.001$ and $\Delta x = 0.5$. Figures 3.1 and 3.2 show the one–dimensional spatio–temporal states $u(x,t)$ and $v(x,t)$, respectively. We see that for $\delta = 1$, the solution is oscillatory in nature and thus asymptotically unstable. This is confirmed by means of the phase–space plot taken at a single spatial point $x = 10$ as depicted in Figure 3.3. The solution converges to an ellipse signifying a periodic nature. As $\delta$ is made smaller, the solution becomes asymptotically stable and converges to the unique spatially homogeneous constant steady state

$$(u^*, v^*) = \left(\frac{a}{5}, 1 + \left(\frac{a}{5}\right)^2\right) = (3, 10)\,. \tag{3.32}$$

Furthermore, we see that the smaller $\delta$, the faster the solution converges to the steady state. This strong dependence of the asymptotic stability on $\delta$ is very interesting as it gives us a new perspective into the control and dynamics of the CIMA chemical reaction. In addition to these one–dimensional examples, we have also examined the two–dimensional case. We consider the parameter set $(a, b, \sigma, d_1, d_2) = (15, 1.2, 8, 1, 24)$ with initial conditions

$$\begin{cases} u\,(x, y, 0) = 3.5\,(1 + 0.2w_u\,(x, y))\,, \\ v\,(x, y, 0) = 10.5\,(1 + 0.2w_v\,(x, y))\,. \end{cases} \tag{3.33}$$

with $w_u\,(x, y)$ and $w_v\,(x, y)$ being Gaussian distributed random functions with zero mean and unit variance. Figure 3.4 shows snap shots of the concentrations $u\,(x, y, t)$ and $v\,(x, y, t)$ taken at time instances $t = 0$, $t = 5$, and $t = 20$ with $\delta = 1$. We see that the diffusion–driven or Turing instability leads to the formation of patterns in the form of dots and stripes. Reducing the fractional order to $\delta = 0.98$ leads to a different type of patterns as shown in Figure 3.5. This means that the fractional order has an impact on the Turing patterns evolving over time, which is an interesting observation. Reducing the fractional order further to $\delta = 0.95$ also yields slightly different patterns as shown in Figure 3.6.

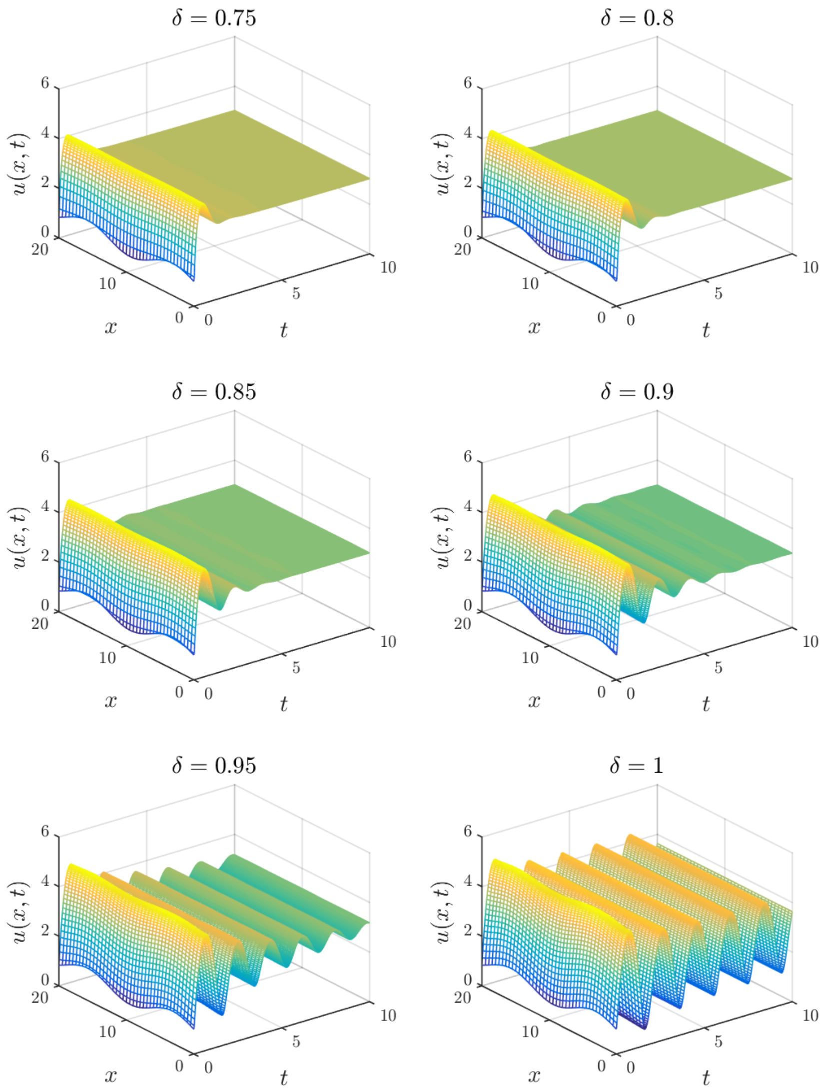


Figure 3.1: One dimensional concentration $u(x,t)$ as a solution of (3.1) with $(a,b,\sigma,d_1,d_2) = (15,1,7,1,10)$, initial conditions (3.31), zero Nuemann boundaries, and different values for $\delta$.

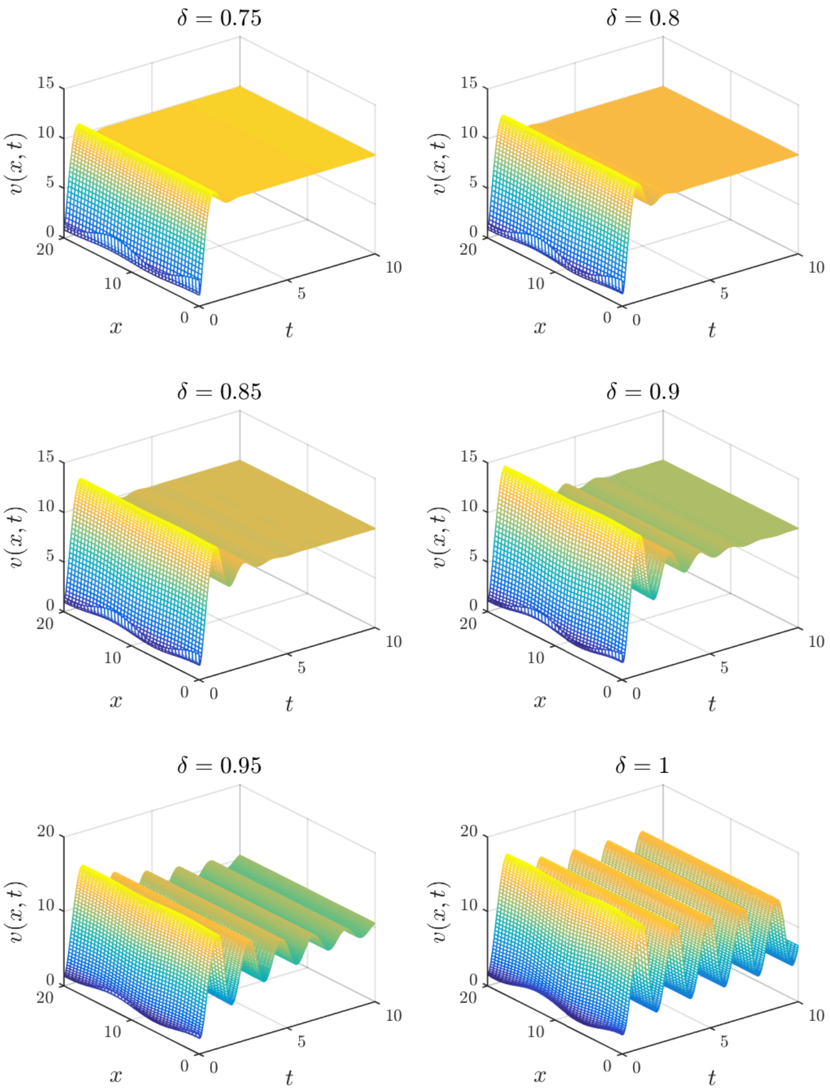


Figure 3.2: One dimensional concentration $v(x,t)$ as a solution of (3.1) with $(a,b,\sigma,d_1,d_2) = (15,1,7,1,10)$, initial conditions (3.31), zero Nuemann boundaries, and different values for $\delta$.

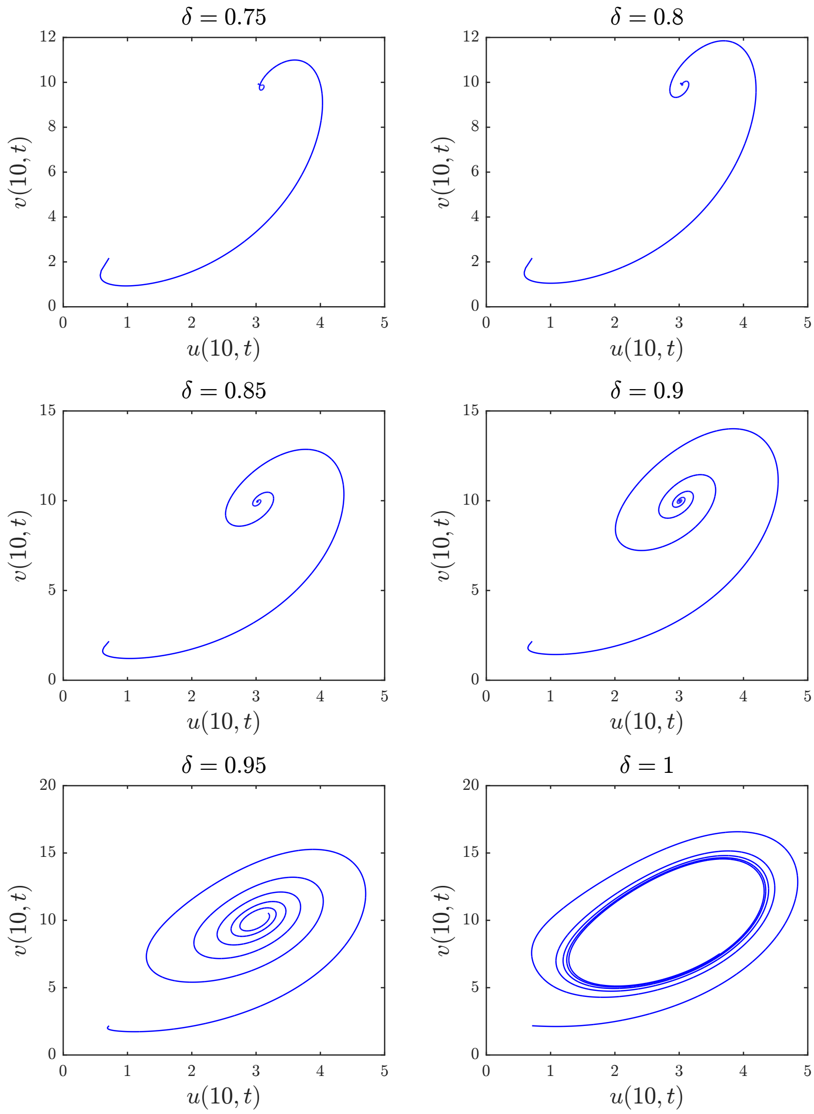


Figure 3.3: Phase plot of system (3.1) taken at $x = 10$ with $(a, b, \sigma, d_1, d_2) = (15, 1, 7, 1, 10)$ , initial conditions (3.31), zero Nuemann boundaries, and different values for $\delta$.

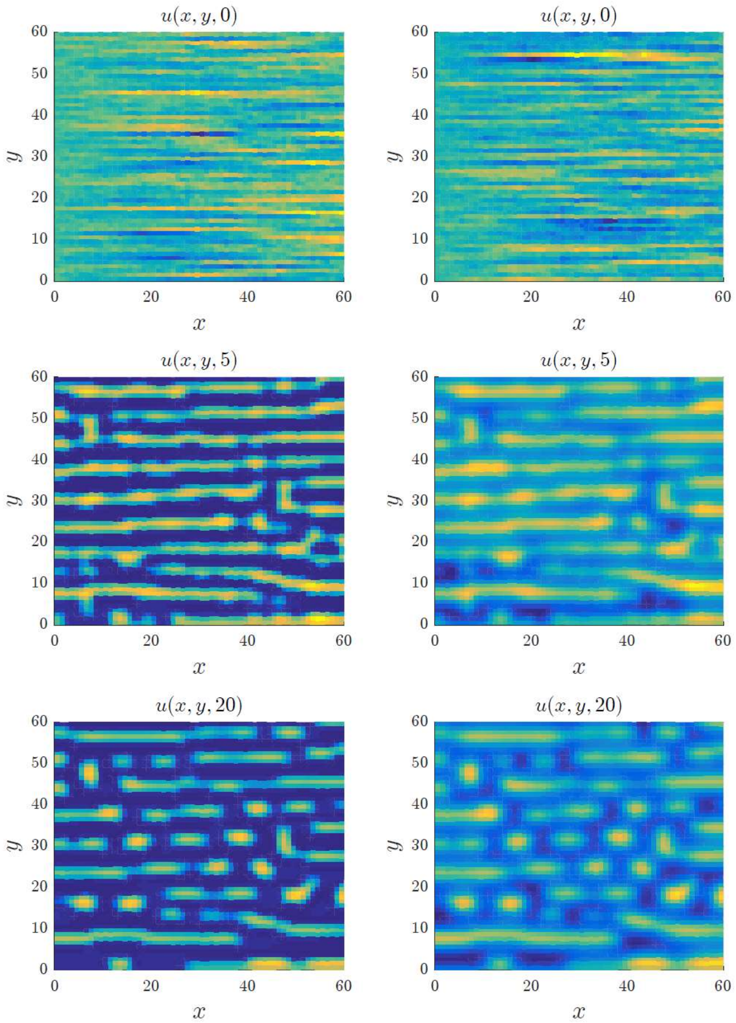


Figure 3.4: Two dimensional concentrations $u(x, y, t)$ and $v(x, y, t)$ for $(a, b, \sigma, d_1, d_2) = (15, 1.2, 8, 1, 24)$, initial conditions (3.33), zero Nuemann boundaries, and $\delta = 1$.

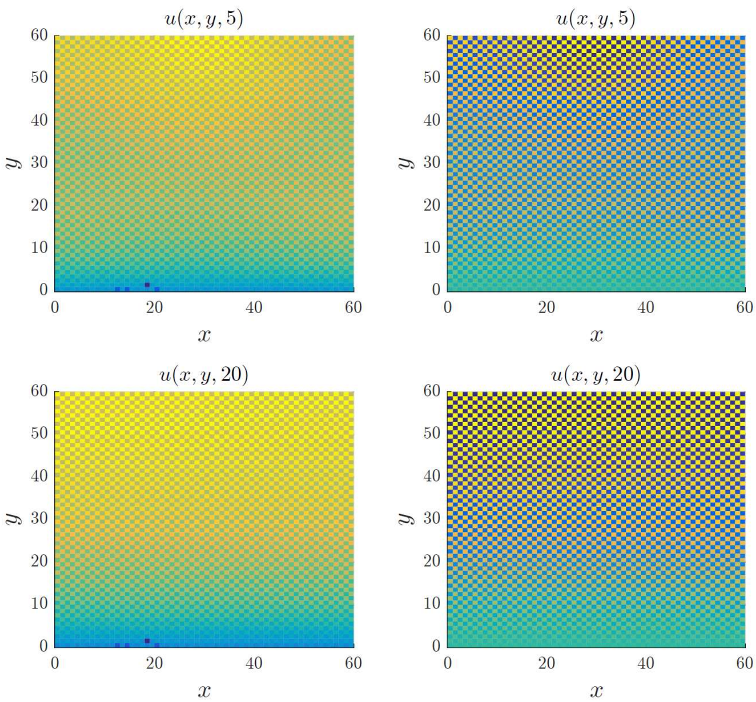


Figure 3.5: Two dimensional concentrations $u(x, y, t)$ and $v(x, y, t)$ for $(a, b, \sigma, d_1, d_2) = (15, 1.2, 8, 1, 24)$, initial conditions (3.33), zero Nuemann boundaries, and $\delta = 0.98$.

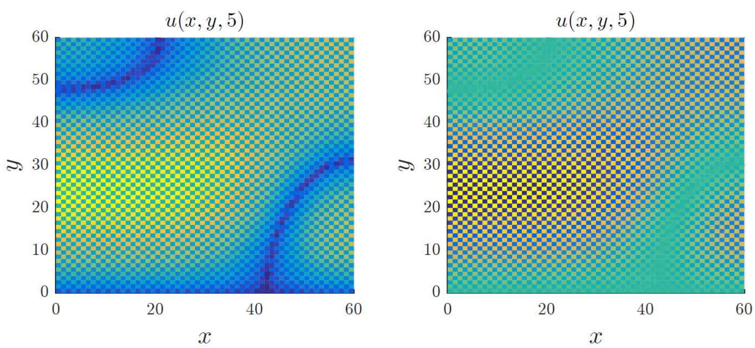


Figure 3.6: Two dimensional concentrations $u(x, y, t)$ and $v(x, y, t)$ for $(a, b, \sigma, d_1, d_2) = (15, 1.2, 8, 1, 24)$, initial conditions (3.33), zero Nuemann boundaries, and $\delta = 0.95$.

Chapter 4

# Bifurcations and pattern formation in a generalized Lengyel-Epstein Reaction-Diffusion model

THIS chapter [1] investigates the formation of spatial patterns in a reaction-diffusion system based on the generalized Lengyel-Epstein CIMA model. By analyzing the properties of the system's unique positive equilibrium in the ODE and PDE cases, we establish the existence of non-constant steady state solutions thereby con rming the existence of Turing instability. Hopf-bifurcation analysis of the system show the existence of periodic solutions in the absence and presence of diffusion. Numerical simulations are presented to validate the theoretical results of the work.

[1]D. Mansouri, S. Abdelmalek, S. Bendoukha, Bifurcations and pattern formation in a generalized Lengyel–Epstein reaction–diffusion model, Chaos, Solitons and Fractals, Vol 132 (2020), 109579.

In this work, we generalized the last step of (2.30) by using $\varphi(u)v$ to replace $\frac{4uv}{1+u^2}$. After resealing, we get the following reaction-difusion system:

$$\begin{cases} u_t - d_1\Delta u = a - \mu u - \lambda\varphi(u)v, & \text{in } \mathbb{R}^+ \times \Omega, \\ v_t - d_2\Delta v = \sigma(u - \varphi(u)v), & \text{in } \mathbb{R}^+ \times \Omega \end{cases} . \tag{4.1}$$

Where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$is the Laplacian operator on $\Omega$. In [1], the initial conditions

$$\begin{cases} u(x,0) = u_0(x) > 0, u \in C^2(\Omega) \cap C(\overline{\Omega}), & x \in \Omega, \\ v(x,0) = v_0(x) > 0, v \in C^2(\Omega) \cap C(\overline{\Omega}), & x \in \Omega, \end{cases}$$

and homogeneous Neumann boundary conditions

$$\partial_\nu u = \partial_\nu v = 0, \text{ on } \mathbb{R}^+ \times \partial\Omega,$$

where imposed. In addition, the following conditions were assumed to hold:

(H1) $d_1$, $d_2$, $a$, $\lambda$, $\mu$, $\sigma > 0$, and $\varphi(u)$is a continuously differentiable function on $\mathbb{R}^+$.

(H2) $\varphi(0) = 0$.

(H3) $\varphi(u) > 0$ for $u \in \left(0, \frac{a}{\mu}\right)$.

(H4) $\varphi(u) > \varphi'(u)u$ for $u \in \left(0, \frac{a}{\mu}\right)$

The constants $d_1$, $d_2$, $a$, $\lambda$, $\mu$, and $\sigma$ are assumed to be positive. The function $\varphi(u)$ is assumed to be nonnegative and continuously differentiable on $\mathbb{R}^+$. For system (4.1), the local system is an ordinary differential equation in the form of

$$\begin{cases} \frac{du}{dt} = a - \mu u - \lambda\varphi(u)v = F(u,v), & t > 0, \\ \frac{dv}{dt} = \sigma(u - \varphi(u)v) = G(u,v), & t > 0 \end{cases} . \tag{4.2}$$

Note that the system (4.2) is an activator–inhibitor system under the condition ($a_{11} > 0$ and $a_{21} > 0$ the Jacobian matrix ),

$$F_0 := -\mu - \lambda\frac{\alpha}{\varphi(\alpha)}\varphi'(\alpha) > 0,$$

since Observe that $F_v(u^*,v^*) < 0$, $G_v(u^*,v^*) < 0$, $G_u(u^*,v^*) > 0$ and $F_u(u^*,v^*) > 0$ (see discussion of activator–inhibitor systems in [1].

## 4.1 Analysis of the ODE model

In this cection we study the characteristics of the solutions to ODE system (4.2 ) by means of stability analysis and Hopf-bifurcation.

### 4.1.1 Stability and bifurcation

**Proposition 4.1.1** *The system (4.2) admits a unique positive constant equilibrium given by*

$$(u^*,v^*) = \left(\alpha, \frac{\alpha}{\varphi(\alpha)}\right), \tag{4.3}$$

*where*

$$\alpha = \frac{a}{\mu+\lambda} \, and \, \varphi(\alpha) = \frac{\alpha}{1+\alpha^2} > 0.$$

**Preuve.** In the ODE case, the constant steady state solutions satisfy

$$\begin{cases} a - \mu u + \lambda \varphi(u) v = 0, \\ \sigma(u - \varphi(u) v) = 0. \end{cases} \tag{4.4}$$

The second equation satisfed $\sigma(u - \varphi(u) v) = 0$ we know $\sigma > 0$ then

$$u = \varphi(u) v \;. \tag{4.5}$$

We compensate(4.5) in the first equation of (4.4), we find

$$a - \mu\varphi(u) v \; - \lambda\varphi(u) v = 0,$$

then

$$v = \frac{a}{(\mu+\lambda)\varphi(u)}, \text{ and } u = \frac{a}{\mu+\lambda},$$

we pose $\alpha = \frac{a}{\mu+\lambda}$.
This leads to the solution

$$(u^*, v^*) = \left(\alpha, \frac{\alpha}{\varphi(\alpha)}\right).$$

□

**Proposition 4.1.2** *The solution $(u^*, v^*)$ s locally asymptotically stable as an equilibrium of(4.2) ubject to*

$$\sigma > \sigma_0 = \frac{-\left[\mu\varphi(\alpha) + \lambda\alpha\varphi'(\alpha)\right]}{\varphi^2(\alpha)}. \tag{4.6}$$

*Alternatively, if $\sigma < \sigma_0$ then $(u^*, v^*)$ is asymptotically unstable.*

**Preuve.** In order to assess the local asymptotic stability of the equilibrium $(u^*, v^*)$, we examine the trace and determinant of the Jacobian matrix associated with (4.2) , whose general form is

$$J(u,v) = \begin{pmatrix} -\mu - \lambda v\varphi'(u) & -\lambda\varphi(u) \\ \sigma(1 - v\varphi'(u)) & -\sigma\varphi(u) \end{pmatrix}.$$

Evaluating the Jacobian at the equilibrium yields

$$J_0(\sigma) = J_0(u^*, v^*) = \begin{pmatrix} -\mu - \lambda\frac{\alpha}{\varphi(\alpha)}\varphi'(\alpha) & -\lambda\varphi(\alpha) \\ \sigma\left(1 - \frac{\alpha}{\varphi(\alpha)}\varphi'(\alpha)\right) & -\sigma\varphi(\alpha) \end{pmatrix}. \tag{4.7}$$

The characteristic equation of $J_0(\sigma)$ is given by

$$\zeta^2 - T\zeta + D = 0, \tag{4.8}$$

where

$$T = tr\,(J)_0\,(\sigma) = -\left(\mu + \lambda\alpha\frac{\varphi'(\alpha)}{\varphi(\alpha)} + \sigma\varphi(\alpha)\right),$$
$$D = \det J_0\,(\sigma) = \sigma\,(\mu+\lambda)\,\varphi(\alpha) > 0.$$

Given $\sigma_0$ as defined in (4.6) , we know that $\sigma = \sigma_0$ is the only root of $tr\,(J)_0\,(\sigma) = 0$ . Hence, it is easy to conclude that $(u^*, v^*)$ is locally asymptotically stable if $\sigma > \sigma_0$ and unstable $\sigma < \sigma_0$.

□

What remains to be examined is the nature of the solutions when $\sigma = \sigma_0$. In the following, we investigate the Hopf-bifurcation occurring at $(u^*, v^*)$ with as the critical parameter. Obviously, when $\sigma = \sigma_0$, the characteristic polynomial (4.8) has a pair of purely imaginary solutions of the form

$$\zeta = \pm i\frac{1}{2}\sqrt{4\,(\mu+\lambda)\,\sigma_0\varphi(\alpha)} = \pm i\sqrt{(\mu+\lambda)\,\sigma_0\varphi(\alpha)}.$$

Let us denote the general solutions of (4.7) by $\zeta^2 - T\zeta + D = 0$, with

$$\beta(\sigma) = \frac{1}{2}tr\,(J)_0 = \frac{-1}{2}\left(\mu + \lambda\alpha\frac{\varphi'(\alpha)}{\varphi(\alpha)} + \sigma\varphi(\alpha)\right), \tag{4.9}$$

and

$$\omega(\sigma) = \frac{1}{2}\sqrt{4\det J_0 - tr\,(J_i)^2} = \sqrt{(\mu+\lambda)\,\sigma\varphi(\alpha)}.$$

The first derivative of $\beta(\sigma)$ evaluated at (4.9) is given by

$$\beta'(\sigma)\big|_{\sigma=\sigma_0} = \frac{-1}{2}\varphi(\alpha) < 0.$$

Hence, the well known Poincaré-Andronov–Hopf Bifurcation Theorem tells us that for $\sigma = \sigma_0$, system (4.2) undergoes a Hopf-bifurcation at $(u^*, v^*)$ when $\sigma = \sigma_0$. For more details on the stated theorem, the reader may refer to Theorem 3.1.3 of [46]. In order to gain a broader understanding of the nature of this Hopf-bifurcation,we must further analyze the normal form of the system. We apply the transformation

$$\begin{cases} \widetilde{u} = u - u^*, \\ \widetilde{v} = v - v^*, \end{cases}$$

we gete

$$\begin{cases} \widetilde{u}_t = \lambda\alpha - \mu\widetilde{u} - \lambda\varphi(\widetilde{u}+\alpha)\left(\widetilde{u} + \frac{\alpha}{\varphi(\alpha)}\right), \\ \widetilde{v}_t = \frac{\sigma}{\lambda}\left[\lambda\alpha + \lambda\widetilde{u} - \lambda\varphi(\widetilde{u}+\alpha)\left(\widetilde{v} + \frac{\alpha}{\varphi(\alpha)}\right)\right], \end{cases}$$

to shift the equilibrium to the origin. In order to simplify the notation, we will drop the tilde symbol and simply denote the new dependent variables by $u$ and $v$. The transformed system becomes

$$\begin{cases} u_t = \lambda\alpha - \mu u - \lambda\varphi(u+\alpha)\left(v + \frac{\alpha}{\varphi(\alpha)}\right) := F_1(u,v,\sigma), \\ v_t = \frac{\sigma}{\lambda}\left[\lambda\alpha + \lambda u - \lambda\varphi(u+\alpha)\left(v+\frac{\alpha}{\varphi(\alpha)}\right)\right] := G_1(u,v,\sigma). \end{cases} \tag{4.10}$$

We consider two functions:

$$f(u,v,\sigma) = \frac{1}{2}B_1((u,v),(u,v)) + \frac{1}{6}C_1((u,v),(u,v),(u,v)) + O\left(\|(u,v)\|^4\right), \tag{4.11}$$

and

$$g(u,v,\sigma) = \frac{1}{2}B_2((u,v),(u,v)) + \frac{1}{6}C_2((u,v),(u,v),(u,v)) + O\left(\|(u,v)\|^4\right), \tag{4.12}$$

where

$$B_1((u,v),(u,v)) = \left.\frac{\partial^2 F_1(u,v,\sigma)}{\partial u^2}\right|_{u=v=0} u^2 + 2\left.\frac{\partial^2 F_1(u,v,\sigma)}{\partial u\partial v}\right|_{u=v=0} uv + \left.\frac{\partial^2 F_1(u,v,\sigma)}{\partial v_2^2}\right|_{u=v=0} v^2,$$

by calculating each terms of second member, we get

$$\begin{aligned} \left.\frac{\partial^2 F_1(u,v,\sigma)}{\partial u^2}\right|_{u=v=0} &= \left.\frac{\partial^2}{\partial u^2}\left(\lambda\alpha - \mu u - \lambda\varphi(u+\alpha)\left(v+\frac{\alpha}{\varphi(\alpha)}\right)\right)\right|_{u=v=0} \\ &= \left.\frac{\partial}{\partial u}\left(-\mu - \lambda\left(v+\frac{\alpha}{\varphi(\alpha)}\right)\varphi'(u+\alpha)\right)\right|_{u=v=0} \\ &= -\lambda\left(v+\frac{\alpha}{\varphi(\alpha)}\right)\left.\frac{d^2}{du^2}\varphi(u+\alpha)\right|_{u=v=0} \\ &= \frac{-\lambda\alpha}{\varphi(\alpha)}\left.\frac{d^2}{du^2}\varphi(u+\alpha)\right|_{u=0}, \end{aligned}$$

$$\begin{aligned} \left.\frac{\partial^2 F_1(u,v,\sigma)}{\partial u\partial v}\right|_{u=v=0} &= \left.\frac{\partial^2}{\partial u\partial v}\left(\lambda\alpha - \mu u - \lambda\varphi(u+\alpha)\left(v+\frac{\alpha}{\varphi(\alpha)}\right)\right)\right|_{u=v=0} \\ &= \left.\frac{\partial}{\partial u}\left(-\lambda\varphi(u+\alpha)\right)\right|_{u=0} \\ &= -\lambda\left.\frac{d}{du}\varphi(u+\alpha)\right|_{u=0}, \end{aligned}$$

and

$$\begin{aligned} \left.\frac{\partial^2 F_1(u,v,\sigma)}{\partial v_2^2}\right|_{u=v=0} &= \left.\frac{\partial^2}{\partial v_2^2}\left(\lambda\alpha - \mu u - \lambda\varphi(u+\alpha)\left(v+\frac{\alpha}{\varphi(\alpha)}\right)\right)\right|_{u=v=0} \\ &= 0. \end{aligned}$$

So

$$B_1((u,v),(u,v)) = \frac{-\lambda\alpha}{\varphi(\alpha)}\left.\frac{d^2}{du^2}\varphi(u+\alpha)\right|_{u=0} u^2 - 2\lambda\left.\frac{d}{du}\varphi(u+\alpha)\right|_{u=0} uv.$$

and

$$C_1\left((u,v),(u,v),(u,v)\right) = \left.\frac{\partial^3 F_1(u,v,\sigma)}{\partial u^3}\right|_{u=v=0} u^3 + 3\left.\frac{\partial^3 F_1(u,v,\sigma)}{\partial u^2 \partial v}\right|_{u=v=0} u^2 v + 3\left.\frac{\partial^3 F_1(u,v,\sigma)}{\partial u \partial v^2}\right|_{u=v=0} uv^2 + \left.\frac{\partial^3 F_1(u,v,\sigma)}{\partial v^3}\right|_{u=v=0} v^3.$$

By calculating each terms of second member, we get

$$\begin{aligned}\left.\frac{\partial^3 F_1(u,v,\sigma)}{\partial u^3}\right|_{u=v=0} &= \left.\frac{\partial^3}{\partial u^3}\left(\lambda\alpha - \mu u - \lambda\varphi(u+\alpha)\left(v+\frac{\alpha}{\varphi(\alpha)}\right)\right)\right|_{u=v=0} \\ &= \left.\frac{\partial^2}{\partial u^2}\left(-\mu - \lambda\frac{\alpha}{\varphi(\alpha)}\frac{d}{du}\varphi(u+\alpha)\right)\right|_{u=0} \\ &= -\lambda\frac{\alpha}{\varphi(\alpha)}\left.\frac{d^3}{du^3}\varphi(u+\alpha)\right|_{u=0},\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial^3 F_1(u,v,\sigma)}{\partial u^2\partial v}\right|_{u=v=0} &= \left.\frac{\partial^3}{\partial u^2\partial v}\left(\lambda\alpha - \mu u - \lambda\varphi(u+\alpha)\left(v+\frac{\alpha}{\varphi(\alpha)}\right)\right)\right|_{u=v=0} \\ &= \left.\frac{\partial^2}{\partial u^2}\left(-\lambda\varphi(u+\alpha)\right)\right|_{u=0} \\ &= -\lambda\left.\frac{d^2}{du^2}\varphi(u+\alpha)\right|_{u=0},\end{aligned}$$

$$\left.\frac{\partial^3 F_1(u,v,\sigma)}{\partial u\partial v^2}\right|_{u=v=0} = 0,$$

and

$$\left.\frac{\partial^3 F_1(u,v,\sigma)}{\partial v^3}\right|_{u=v=0} = 0.$$

So

$$C_1\left((u,v),(u,v),(u,v)\right) = -\lambda\frac{\alpha}{\varphi(\alpha)}\left.\frac{d^3}{du^3}\varphi(u+\alpha)\right|_{u=0} u^3 - 3\lambda\left.\frac{d^2}{du^2}\varphi(u+\alpha)\right|_{u=0} u^2 v,$$

therfour

$$\begin{aligned}f(u,v,\sigma) &= \frac{1}{2}B_1\left((u,v),(u,v)\right) + \frac{1}{6}C_1\left((u,v),(u,v),(u,v)\right) + O\left(\|(u,v)\|^4\right) \\ &= \frac{-\lambda\alpha}{2\varphi(\alpha)}\left.\varphi_{uu}(u+\alpha)\right|_{u=0} u^2 - \lambda\left.\varphi_u(u+\alpha)\right|_{u=0} uv \\ &\quad - \lambda\frac{\alpha}{6\varphi(\alpha)}\left.\varphi_{uuu}(u+\alpha)\right|_{u=0} u^3 - \frac{\lambda}{2}\left.\varphi_{uu}(u+\alpha)\right|_{u=0} u^2 v + O\left(\|(u,v)\|^4\right).\end{aligned}$$

Same method we find the function

$$g(u,v,\sigma) = \frac{\sigma}{\lambda}f(u,v,\sigma),$$

Using these functions, system (4.10) can be rewritten in vector form as

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v,\sigma) \\ f(u,v,\sigma) \end{pmatrix}. \tag{4.13}$$

Let us, now, define the matrix

$$P = \begin{pmatrix} 1 & 0 \\ N & M \end{pmatrix},$$

where

$$N = \frac{\left(\mu + \lambda \frac{\alpha}{\varphi(\alpha)}\varphi'(\alpha) - \sigma\varphi(\alpha)\right)}{2\lambda\varphi(\alpha)}, \text{ and } M = \frac{\sqrt{4\det J - tr(J)^2}}{2\lambda\varphi(\alpha)},$$

whose inverse is simply

$$P^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{-N}{M} & \frac{1}{M} \end{pmatrix},$$

At the critical point $\sigma = \sigma_0$, we have

$$M_0 = M|_{\sigma=\sigma_0} = \frac{\sqrt{(\mu+\lambda)\left(-\mu - \lambda\alpha\frac{\varphi'(\alpha)}{\varphi(\alpha)}\right)}}{\lambda\varphi(\alpha)},\; N_0 = N|_{\sigma=\sigma_0} = \frac{\sigma_0}{\lambda}, \text{and } \omega(\sigma_0) = \sqrt{(\mu+\lambda)\sigma_0\varphi(\alpha)}.$$

Applying the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix},$$

system (4.13) becomes

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \widetilde{J}(\sigma)\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f^*(u,v,\sigma) \\ g^*(u,v,\sigma) \end{pmatrix}, \tag{4.14}$$

where

$$\widetilde{J}(\sigma) = P^{-1}JP = \begin{pmatrix} \beta(\sigma) & -\omega(\sigma) \\ \omega(\sigma) & \beta(\sigma) \end{pmatrix},$$

and

$$P^{-1}\begin{pmatrix} f(u,v,\sigma) \\ g(u,v,\sigma) \end{pmatrix} = \begin{pmatrix} f^*(u,v,\sigma) \\ g^*(u,v,\sigma) \end{pmatrix}.$$

Here

$$f^*(u,v,\sigma) = A_{20}x^2 + A_{11}xy + A_{21}x^2y + A_{30}x^3 + O\left(\|(x,y)\|^4\right), \tag{4.15}$$

and

$$g^*(u,v,\sigma) = \left(-\frac{N}{M} + \frac{\sigma}{\lambda M}\right) f^*(u,v,\sigma), \tag{4.16}$$

where

$$\begin{aligned}
A_{20} &= -\lambda\left\{\frac{\alpha}{2\varphi(\alpha)}\,\varphi_{uu}(u+\alpha)|_{u=0} + \varphi_u(u+\alpha)|_{u=0}\,N\right\},\\
A_{11} &= -\lambda M\,\varphi_u(u+\alpha)|_{u=0},\\
A_{21} &= -\frac{\lambda M}{2}\,\varphi_{uu}(u+\alpha)|_{u=0},\\
A_{30} &= -\frac{\lambda}{2}\left\{\frac{\alpha}{3\varphi(\alpha)}\,\varphi_{uuu}(u+\alpha)|_{u=0} + \varphi_{uu}(u+\alpha)|_{u=0}\,N\right\}.
\end{aligned}$$

Rewrite (4.13) in the following polar coordinates form:

$$\begin{cases} \dot{r} = \beta(\sigma) r + a(\sigma) r^3 + ..., \\ \dot{\theta} = \omega(\sigma) + c(\sigma) r^2 + ..., \end{cases} \tag{4.17}$$

then the Taylor expansion of (4.17) at $\sigma = \sigma_0$ yields

$$\begin{cases} \dot{r} = \beta'(\sigma_0)(\sigma - \sigma_0) r + a(\sigma_0) r^3 + O(...), \\ \dot{\theta} = \omega(\sigma_0) + \omega'(\sigma_0)(\sigma - \sigma_0) + c(\sigma_0) r^2 + +O(...), \end{cases} \tag{4.18}$$

In order to determine the stability of the periodic solution, we need to calculate the sign of the coefficient $a(\sigma_0)$, which is given by

$$\begin{aligned} a(\sigma_0) = &\frac{1}{16}\left[f^*_{xxx} + f^*_{xyy} + g^*_{xxy} + g^*_{yyy}\right] \\ &+ \frac{1}{16\omega(\sigma_0)}\left[f^*_{xy}\left(f^*_{xx} + f^*_{yy}\right) - g^*_{xy}\left(g^*_{xx} + g^*_{yy}\right) - f^*_{xx}g^*_{xx} + f^*_{yy}g^*_{yy}\right], \end{aligned} \tag{4.19}$$

Note here that the derivatives in (4.19) are all evaluated at the point $(x, y, \sigma) = (0, 0, \sigma_0)$., which is the bifurcation point. It is easy to see that

$$f^*_{xyy}(0,0,\sigma_0) = g^*_{yyy}(0,0,\sigma_0) = f^*_{yy}(0,0,\sigma_0) = g^*_{yy}(0,0,\sigma_0) \equiv 0.$$

and

$$g^*_{xxy}(0,0,\sigma_0) = g^*_{xx}(0,0,\sigma_0) = g^*_{xy}(0,0,\sigma_0) = 0.$$

Therefore, (4.19) reduces to

$$a(\sigma_0) = \frac{1}{16} f^*_{xxx}(0,0,\sigma_0) + \frac{1}{16\omega(\sigma_0)} f^*_{xy}(0,0,\sigma_0) f^*_{xx}(0,0,\sigma_0), \tag{4.20}$$

where

$$\begin{aligned} f^*_{xxx}(0,0,\sigma_0) &= -3\lambda\left\{\frac{\alpha}{3\varphi(\alpha)}\, \varphi_{uuu}(u+\alpha)|_{u=0} + \varphi_{uu}(u+\alpha)|_{u=0} N_0\right\}, \\ f^*_{xy}(0,0,\sigma_0) &= -\lambda M_0\, \varphi_u(u+\alpha)|_{u=0}, \\ f^*_{xx}(0,0,\sigma_0) &= -2\lambda\left\{\frac{\alpha}{2\varphi(\alpha)}\, \varphi_{uu}(u+\alpha)|_{u=0} + \varphi_u(u+\alpha)|_{u=0} N_0\right\}. \end{aligned}$$

Finally, the following result follows directly from the above mentioned Poincaré-Andronov-Hopf theorem (1.2.5).

**Theorem 4.1.1** *The system (4.2) undergoes a Hopf bifurcation at $(u^*, v^*)$ The nature of the bifurcation has two distinct cases:*
*case 1: if $a(\sigma_0) < 0$ the periodic solutions bifurcating from $(u^*, v^*)$ at $\sigma = \sigma_0$ are instable, and the direction of the Hopf bifurcation is subcritical.*
*case 2: if $a(\sigma_0) > 0$ the periodic solutions bifurcating from $(u^*, v^*)$ at $\sigma = \sigma_0$ are stable, and the direction of the Hopf bifurcation is supercritical.*

### 4.1.2 Numerical example

We choose the particular special case with

$$\varphi(u) = \frac{u^p}{1+u^q},$$

with $q \in \mathbb{R}^+$and $0 < p \leq 1$. This particular example was considered in [53]. It is also similar to the generalization of the Lengyel–Epstein model considered in [14]. The following three examples examine the types of solutions for different parameter sets.

In this section, we would like to put the theoretical results of the previous two sections to the test.

**Exemple 4.1.1** *In the first example, we let $p = \frac{1}{2}$, $q = 2$, $\mu = \frac{1}{2}$, $\lambda = \frac{3}{2}$, $a = 2$, and $\sigma = \frac{2}{3}$, yielding*

$$\begin{cases} \frac{du}{dt} = 2 - \frac{1}{2}u - \frac{3}{2}\frac{\sqrt{u}}{1+u^2}v, \\ \frac{dv}{dt} = \frac{2}{3}\left(u - \frac{\sqrt{u}}{1+u^2}v\right). \end{cases} \tag{4.21}$$

*We are interested in the nature of the solutions in the absence of diffusion (ODE case). The equilibrium point is given by*

$$(u^*, v^*) = \left(\alpha, \frac{\alpha}{\varphi(\alpha)}\right) = (1, 2).$$

*From (4.6), we have*

$$\sigma_0 = \frac{-\left[\mu\varphi(\alpha) + \lambda\alpha\varphi'(\alpha)\right]}{\varphi^2(\alpha)} = \frac{2}{3}.$$

According to Theorem 4.1.1, the system undergoes a *Hopf–bifurcation at $(u^*, v^*)$ when $\sigma = \sigma_0$.* The nature of the periodic solutions depends on the $a(\sigma_0)$ as defined in (4.19). In this case, we have $a(\sigma_0) = -\frac{217}{1296} < 0$, which implies that the periodic solutions are unstable, and the direction of the Hopf–bifurcation is subcritical. Figure 4.1 shows the numerical solutions obtained by means of Matlab simulations. The solutions seem to converge towards a circle in phase space, which agrees with the theoretical results.

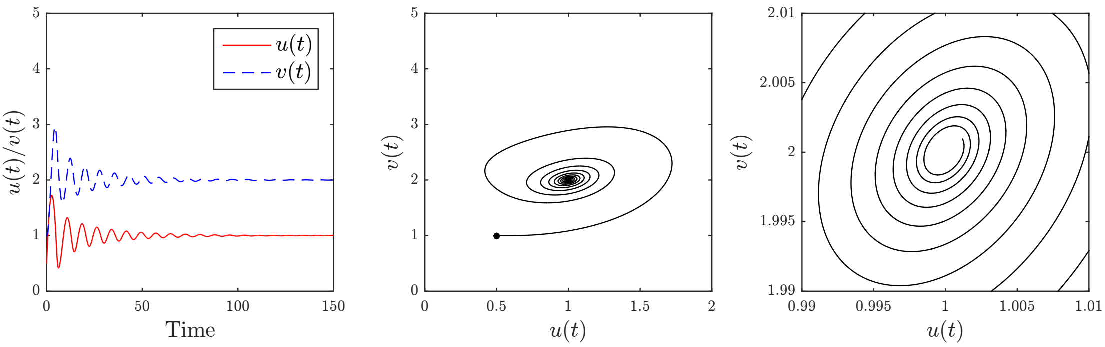


Figure 4.1: Numerical solutions of the local system (4.22) with initial conditions $(u_0, v_0) = (0.5, 1)$ and $\sigma = 1$.

**Exemple 4.1.2** *Let us now consider the case where* $p = \frac{1}{3}$, $q = 2$, $\mu = \frac{1}{2}$ , $\lambda = \frac{3}{2}$, $a = 2$, *and* $\sigma = 1$, *yielding*

$$\begin{cases} \frac{du}{dt} = 2 - \frac{1}{2}u - \frac{3}{2}\frac{u^{\frac{1}{3}}}{1+u^2}v, \\ \frac{dv}{dt} = \left(u - \frac{u^{\frac{1}{3}}}{1+u^2}v\right). \end{cases} \tag{4.22}$$

In this case, $\sigma_0 = 1$ and $a(\sigma_0) = \frac{1}{24} > 0$, which implies that periodic solutions bifurcating from $(u^*, v^*) = (1, 2)$ at $\sigma = 1$ are stable, and the direction of the Hopf bifurcation is supercritical. This, again, agrees with the solutions obtained numerically as depicted in Figure 4.2.

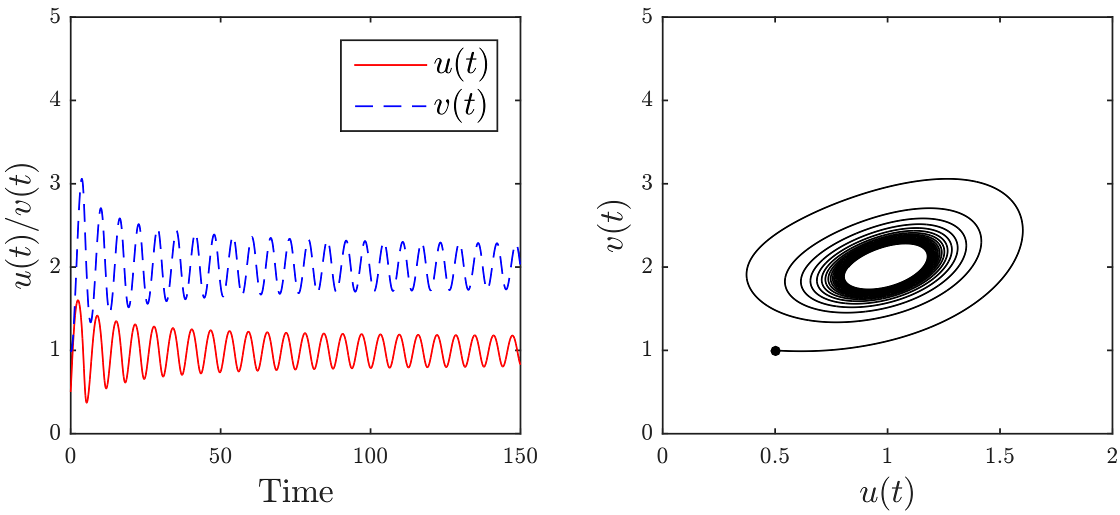


Figure 4.2: Numerical solutions of the local system (4.22) with initial conditions $(u_0, v_0) = (0.5, 1)$ and $\sigma = \frac{11}{5}$.

## 4.2 Analysis of the PDE model

In this section, consider the stability of $(u^*, v^*)$, Turing instability and Hopf bifurcation near $(u^*, v^*)$. Throughout this section, $\mathbb{N}$ is the set of natural numbers .The eigenvalues of operator $-\Delta$ with homogeneous Neumann boundary condition in $\Omega$ are denoted by $0 = \eta_0 < \eta_1 \leq \eta_2 \leq \ldots.. \leq \eta_n..$ , and the eigenfunction corresponding to $\eta_n$ is $\phi(x)$. The stability of $(u^*, v^*)$ as a steady state solution of (4.2) rests on the nature of the eigenvalue problem obtained through the linearization of the steady statem, which can be formulated as

$$\begin{cases} L(\sigma)\binom{\phi}{\psi} = \xi\binom{\phi}{\psi}, & x \in \Omega \\ \partial_\nu \phi = \partial_\nu \psi = 0, & x \in \partial\Omega \end{cases} \tag{4.23}$$

where the linearization operator $L(\sigma)$ is defned by

$$L(\sigma) = \begin{pmatrix} d_1 \Delta u - \mu - \lambda \frac{\alpha}{\varphi(\alpha)} \varphi'(\alpha) & -\lambda \varphi(\alpha) \\ \sigma\left(1 - \frac{\alpha}{\varphi(\alpha)}\varphi'(\alpha)\right) & d_2 \Delta v - \sigma \varphi(\alpha) \end{pmatrix}. \tag{4.24}$$

For each $n \in \mathbb{N}$, we define a $2\times 2$ matrix

$$L_n(\sigma) = \begin{pmatrix} -d_1\eta_n - \mu - \lambda\frac{\alpha}{\varphi(\alpha)}\varphi'(\alpha) & -\lambda\varphi(\alpha) \\ \sigma\left(1-\frac{\alpha}{\varphi(\alpha)}\varphi'(\alpha)\right) & -d_2\eta_n - \sigma\varphi(\alpha) \end{pmatrix}. \tag{4.25}$$

Obviously, for any eigenvalue of (4.23) denoted by $\xi$ , there exists $n \in \mathbb{N}$ such that $\xi$ is also an eigenvalue of $L_n(\sigma)$. In order to establish the local asymptotic stability of $(u^*, v^*)$ as a constant steady state for (4.1), it suffices to ensure that all the eigenvalues of $L_n(\sigma)$ have negative real parts. The characteristic equation of $L_n(\sigma)$is

$$\xi^2 - P_n(\sigma)\xi + Q_n(\sigma) = 0. \tag{4.26}$$

Then

$$P_n(\sigma) = -(d_1+d_2)\eta_n - \varphi(\alpha)(\sigma-\sigma_0), \tag{4.27}$$

$$Q_n(\sigma) = d_1d_2\eta_n^2 + \varphi(\alpha)(d_1\sigma - d_2\sigma_0)\eta_n + \sigma\varphi(\alpha)(\mu+\lambda). \tag{4.28}$$

Then $(u^*, v^*)$ is locally asymptotically stable if $P_n(\sigma) < 0$ and $Q_n(\sigma) > 0$ for all $n \in \mathbb{N}$, and it is unstable if there exists $n \in \mathbb{N}$ such that $P_n(\sigma) > 0$ or $Q_n(\sigma) < 0$. If $\sigma_0 \leq 0$**,** we have $P_n(\sigma) < 0$ and $Q_n(\sigma) > 0$,for all $n \in \mathbb{N}$, then $(u^*, v^*)$ is locally asymptotically stable.
Next we consider the case that $\sigma_0 > 0$. we define the functions

$$\sigma_H(\eta) = \frac{-(d_1+d_2)\eta}{\varphi(\alpha)} + \sigma_0, \text{and } \sigma_S(\eta) = \frac{-d_1d_2\eta^2 + \varphi(\alpha)d_2\sigma_0\eta}{\varphi(\alpha)((\mu+\lambda)+d_1\eta)}. \tag{4.29}$$

**Lemme 4.2.1** *.The functionsand $\sigma_H(\eta)$ ) and $\sigma_S(\eta)$ have the following properties:*

*(i) For $\eta \geq 0$ $\sigma_H(\eta)$ is strictly decreasing wthe such $\sigma_H(\eta_{H_0}) = 0$, where*

$$\eta_{H_0} = \frac{\varphi(\alpha)\sigma_0}{(d_1+d_2)}. \tag{4.30}$$

*(ii)-$\sigma_H(\eta)$ and $\sigma_H(\eta)$ intersect at a single point $(\eta_c, \sigma_S(\eta_c))$, with*

$$\eta_{c\sigma_0} = \frac{-((d_2-d_1)\varphi\sigma_0 + (d_1+d_2)(\mu+\lambda))}{d_1^2} + \frac{\sqrt{((d_2-d_1)\varphi\sigma_0 + (d_1+d_2)(\mu+\lambda))^2 + 4d_1^2\varphi\sigma_0(\mu+\lambda)}}{d_1^2}. \tag{4.31}$$

*(iii) $\sigma_S(\eta)$is strictly increasing for $\eta \in (0, \eta^*)$, and strictly decreasing for $\eta \in (\eta^*, +\infty)$,withe denoting the unique root of $\sigma_S(\eta)$ given by*

$$\eta^* = \frac{\sqrt{(\mu+\lambda)((\mu+\lambda)+\varphi(\alpha)\sigma_0)} - (\mu+\lambda)}{d_1} < .\frac{\varphi(\alpha)\sigma_0}{d_1} = \eta_{S_0}. \tag{4.32}$$

*Then* $\sigma_s(0) = \sigma_s(\eta_{S_0}) = 0$, $\sigma^* = \sup_{\eta\in(0,\infty)} \sigma_S(\eta)$ *,leading to*

$$\sigma^* = \sigma_S(\eta^*) = \frac{d_2}{d_1\varphi(\alpha)(\mu+\lambda)} \left(\sqrt{(\mu+\lambda)^2 + (\mu+\lambda)\varphi(\alpha)\sigma_0} - (\mu+\lambda)\right)^2, \tag{4.33}$$

*Furthermore,* $\sigma^* > (=,<)\,\sigma_0$ *if and only if* $\frac{d_2}{d_1} > (=,<)\,\chi$ *where*

$$\chi = \frac{\varphi(\alpha)(\mu+\lambda)\sigma_0}{\left(\sqrt{(\mu+\lambda)^2 + (\mu+\lambda)\varphi(\alpha)\sigma_0} - (\mu+\lambda)\right)^2}, \tag{4.34}$$

**Preuve.** The first result of Lemma (4.2.1) $(i)$ is evident. With some manipulations we can see that $\sigma_H(\eta) = \sigma_S(\eta)$ if and only if satisfies.

$$d_1^2\eta^2 - ((d_2-d_1)\varphi(\alpha)\sigma_0 + (d_1+d_2)(\mu+\lambda))\eta + \varphi(\alpha)\sigma_0(\mu+\lambda) = 0.$$

This leads to result $(ii)$. Result $(iii)$ is obvious as

$$\sigma_S^{'}(\eta) = -\frac{\left(\eta^2 d_1^2 + 2(\mu+\lambda)\eta d_1 - (\mu+\lambda)\varphi(\alpha)\sigma_0\right)}{((\mu+\lambda)+\eta d_1)^2}.$$

□

Before presenting the asymptotic stability conditions for the PDE case, let us define the constant

$$\overline{\sigma} = \max_{n\in N^*} \sigma_S(\eta_n) \le \sigma^*. \tag{4.35}$$

The following results hold.

**Theorem 4.2.1** *The constant equilibrium* $(u^*, v^*)$ *is locally asymptotically stable as a constant steady state of (4.24) if*

$$\begin{cases} \sigma_0 \le 0 \text{ or,} \\ \sigma_0 > 0 \text{ and } \sigma > \max(\sigma_0, \overline{\sigma}). \end{cases}$$

*Alternatively,if* $\sigma_0 > 0$ *and* $\sigma < \max(\sigma_0, \overline{\sigma})$ *, then* $(u^*, v^*)$ *is asymptotically unstable.*

**Preuve.** First of all, we assume that $\sigma_0 \le 0$,which leads to $Q_n(\sigma) > 0$,and $P_n(\sigma) < 0$,and consequently $(u^*, v^*)$ is locally asymptotically stable constant steady state.Next, we let $\sigma_0 > 0$,which case we can use Lemma (4.2.1). The constant steady state $(u^*, v^*)$ is locally asymptotically stable if

$$\sigma > \max(\sigma_0, \overline{\sigma}). \tag{4.36}$$

as $P_n(\sigma) < 0$ and $Q_n(\sigma) > 0$ for all $n \in \mathbb{N}$,
Alternatively, if

$$\sigma < \max(\sigma_0, \overline{\sigma}), \tag{4.37}$$

then there exists $n \in \mathbb{N}$ sush that $P_n(\sigma) > 0$ or $Q_n(\sigma) < 0$ and $(u^*, v^*)$ is instable. □

### 4.2.1 Turing instability

Now that we have established su cient conditions for the asymptotic stability of the equilibrium in the ODE and PDE senses, we move our attention to the important issue of Turing instability. It is well known that the term Turing instability refers to the equilibrium being stable in the diffusion-free case (4.2) but becomes unstable when diffusion is considered as in (4.1). In light of4.2.1 and Theorem 4.1.1, we conclude that the diffusion-driven instability occurs specifically when is in the interval

$$0 < \sigma_0 < \sigma < \overline{\sigma}. \tag{4.38}$$

Hence, we can state the following results.

**Lemme 4.2.2** *The equation $\sigma_S(\eta) = \sigma_0$ has two different positive roots.*

**Preuve.** With some minor manipulations we nd that the equation $\sigma_S(\eta) = \sigma_0$is satisfied if and only if is a root of the quadratic polynomialt

$$d_1 d_2 \eta^2 + \varphi(\alpha)\, \sigma_0 (d_1 - d_2)\, \eta + \sigma_0 \varphi(\alpha)(\mu + \lambda) = 0,$$

which has the roots

$$\begin{cases} \eta_l = \dfrac{\varphi(\alpha)\sigma_0\left(\frac{d_2}{d_1}-1\right)-\sqrt{\left[\varphi(\alpha)\sigma_0\left(\frac{d_2}{d_1}-1\right)\right]^2-4\frac{d_2}{d_1}\sigma_0\varphi(\alpha)(\mu+\lambda)}}{2d_2}, \\ \eta_r = \dfrac{\varphi(\alpha)\sigma_0\left(\frac{d_2}{d_1}-1\right)-\sqrt{\left[\varphi(\alpha)\sigma_0\left(\frac{d_2}{d_1}-1\right)\right]^2-4\frac{d_2}{d_1}\sigma_0\varphi(\alpha)(\mu+\lambda)}}{2d_2}. \end{cases} \tag{4.39}$$

□

**Theorem 4.2.2** *Assuming $\sigma_0 > 0$ and $d_2/d_1 > \chi$ ,where $\chi$ is defined in (4.34). Then Turing instability happens if there exists $n \in N$ such that $\eta \in (\eta_l, \eta_r)$ holds.*

**Preuve.** Since $\overline{\sigma} < \sigma^*$ ,using Lemma 4.2.2, it is straight forward to see that the diffusion coefficients must satisfy $d_2/d_1 > \chi$ . Hance (4.38) holds if there exists $n \in N$ such that $\eta \in (\eta_l, \eta_r)$ . □.

The preceding lemma and theorem show that the occurrence of a Turing instability in the present model is governed by the nature of the functions $\sigma_H(\eta)$ and $\sigma_S(\eta)$ and the values of $\sigma^*$and $\sigma_0$ . These results are summarized in figure (4.3).

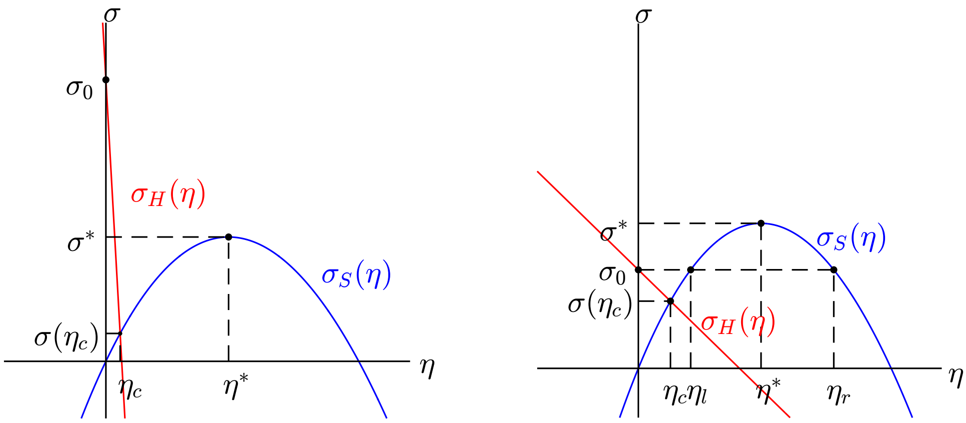


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Figure 4.3: Graphs of the functions $\sigma_H(\eta)$ and $\sigma_S(\eta)$ for the cases $\sigma^* < \sigma_0$ (left) and $\sigma^* > \sigma_0$ (right)

### 4.2.2 Hopf bifurcation

In order to understand more about the nature of the constant steady state $(u^*, v^*)$ we examine its Hopf-bifurcation with $\sigma$ as the critical parameter subject to $\sigma_0 \geq 0$ which makes the system stable. The eigenvalues $\eta_i$ of the Laplacian operator are assumed to be simple. A necessary and sufficient condition for the existence of a Hopf-bifurcation value $\sigma_H$ was stated in[48, 49, 17] : Let us define the simpler notation

$$\sigma_{n,H} = \sigma_H(\eta_n) \text{ and } \sigma_{n,S} = \sigma_S(\eta_n). \tag{4.40}$$

**Theorem 4.2.3** *Subject to $\sigma_0 > 0$ and assuming a smooth domain so that all eigenvalues $\eta_n$, $n \in \mathbb{N}$ are simple, then there exists an index $n_0 \in \mathbb{N}$ such that $\eta_{n_0} < \eta_{n_{\sigma_0}} < \eta_{n_0+1}$ It follows that if $\sigma_H(\eta_n) \neq \sigma_H(\eta_m)$ for any index $n \in \{0, ..., n_0\}$ and $m > n_0$ there exist $n_0 + 1$ possible Hopf bifurcation points satisfuing the relation*

$$\sigma(\eta^*) < \sigma(\eta_{n_0}) < \sigma(\eta_{n_0-1}) < \ldots\ldots < \sigma(\eta_0) = \sigma_0. \tag{4.41}$$

*Consequently, at $\sigma_{n,H}$ as defined in (4.40), system (4.25) undergoes a Hopf bifurcation, and the periodic orbits near $(\sigma, u, v) = (\sigma_{n,H}, u^*, v^*)$ can be parameterized by $(\sigma_n(\tau), u_n(\tau), v_n(\tau))$ so that $\sigma_n(\tau) = \sigma_{n,H} + o(\tau)$ for $\tau \in (0, \rho)$ for som constant $\rho > 0$, and*

$$\begin{aligned} u_n(\tau)(x,t) &= u^* + \tau a_n \cos(\alpha(\sigma_{n,H})t)\phi_n(x) + o(\tau), \\ v_n(\tau)(x,t) &= v^* + \tau b_n \cos(\alpha(\sigma_{n,H})t)\phi_n(x) + o(\tau), \end{aligned}$$

*where $\alpha(\sigma_{n,H}) = \sqrt{Q_n(\sigma_{n,H})}$with $Q_n(\sigma_{n,H})$ given in (4.28 ) is the corresponding time frequency, $\phi_n(x)$ is the corresponding spatial eigenfunction, and $(a_n, b_n)$ is the corresponding eigenvector, i.e.,*

$$(L(\sigma_{n,H}) - i\omega(\sigma_{n,H}) I) \begin{pmatrix} a_n\phi_n(x) \\ b_n\phi_n(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

*where $L(\sigma)$is given in (4.23). Moreover:*

*1) The bifurcation periodic orbit from $\sigma_{n,H} = \sigma_0$ is spatially homogeneous.*

*2) The bifurcation periodic orbits from $\sigma_{n,H}, n \in \{0, ..., n_0\}$ are spatially nonhomogeneous.*

**Preuve.** Using the Hopf bifurcation theorem in [48], in order to prove the results of this theorem, it su ces to prove that (i) There exists and index $n \in \mathbb{N}$ such that

$$P_n(\sigma_H) = 0, \; Q_n(\sigma_H) > 0,$$

and

$$P_m(\sigma_H) \neq 0, \quad Q_m(\sigma_H) \neq 0 \; form \in \mathbb{N} \backslash \{n\}, \tag{4.42}$$

where $P_n(\sigma)$ and $Q_n(\sigma)$ as defined earlier in (4.27) and (4.28) respectively. (ii)For the unique pair of complex eigenvalues $\beta(\sigma) \pm i\omega(\sigma)$ near the imaginary axis.

$$\beta'(\sigma) \neq 0 \text{ and } \omega(\sigma) > 0. \tag{4.43}$$

In order to establish these conditions, we make use of the fact that $\sigma_{n,H}$ is strictly deceasing as proven in Lemma (4.2.1).Hence $\sigma_H(\eta_n) > \sigma_H(\eta_c)$ for all $n \in \{0, ..., n_0\}$and consequently, there existe a unique index $n_0 \in \mathbb{N}$ such that $\eta_{n_0} < \eta_{n_{\sigma_0}} < \eta_{n_0+1}$ and $\sigma(\eta_{n_0}) > \sigma(\eta_c) > \sigma(\eta_{n_0+1})$ . Therefore, we have $n_0 + 1$ Hopf bifurcation points at $\sigma_H(\eta_n)$ if $n \in \{0, ..., n_0\}$. This establishes the correctness of (4.22). On the other hand we have $Q_n(\sigma_{n,H}) > 0$ for all $n = 0, 1, 2, 3, ......n_0$ such that $Q_n(\sigma_{n,H}) = 0$ isequivalent to $\sigma_H(\eta_n) = \sigma_S(\eta_m)$ . If $\eta_n \neq \eta_m$ then $m \neq n$ and $P_m(\sigma_H) \neq 0, \quad Q_m(\sigma_H) \neq 0$.
This yields

$$\beta(\sigma) = \frac{P_n(\sigma_H)}{2} = \frac{1}{2}\left(-(d_1 + d_2)\right)\eta_n - \varphi(\alpha)(\sigma_H - \sigma),$$

which leads to

$$\beta'(\sigma_{n,H}) = \frac{P_n'(\sigma_{n,H})}{2} = \frac{-1}{2}\varphi(\alpha) < 0.$$

Hanse

$$\omega(\sigma_{n,H}) = \sqrt{Q_n(\sigma_{n,H})} > 0. \tag{4.44}$$

Then, conditions in (i) and (ii) are satisfied if $n \in \{0, ..., n_0\}$. Direct application of the Hopf bifurcation theorem in [48] concludes the proof of the theorem. □

### 4.2.3 Numerical example

**Exemple 4.2.1** *In this last example, we let $p = 2$, $q = 5$, $\mu = \frac{1}{2}$, $\lambda = \frac{3}{2}$, and $a = 2$. The ODE system (4.2) becomes:*

$$\begin{cases} \frac{du}{dt} = 2 - \frac{1}{2}u - \frac{3}{2}\frac{u^2}{1+u^5}v, \\ \frac{dv}{dt} = 5\left(u - \frac{u^2}{1+u^5}v\right). \end{cases} \tag{4.45}$$

*The equilibrium solution is given by $(u^*, v^*) = (1, 2)$ and $\sigma_0 = \frac{1}{2}$. It follows from Proposition 4.1.2 that the positive equilibrium $(u^*, v^*) = (1, 2)$ is locally asymptotically stable when $\sigma > 5$*

*and unstable when $\sigma < 5$. Moreover, according to Theorem 4.1.1, when $\sigma$ passes from the right side of $\frac{1}{2}$, $(u^*, v^*)$ will lose its stability and a Hopf–bifurcation occurs.*
*Let us consider two separate examples with different diffusivity constants and analyze the existence of patterns by means of numerical solutions in the ODE and 2D PDE cases. First, consider the diffusivity constants $d_1 = 0.1$ and $d_2 = 1.5$. It follows that*

$$\sigma^* = 0.22078 < \sigma_0,$$

$$\eta^* = 1.2132,$$

$$\eta_c = 0.14079,$$

$$\sigma_H(\eta_c) \approx \sigma_H(\eta_c) \approx 4.9472 \times 10^{-2}$$

*and*

$$\chi = 33.971.$$

*Since $\sigma^* < \sigma_0$, we have the case on the left hand side of Figure 4.3. In this case, no spatial patterns occur. The numerical solutions of the local system (4.45) when $\sigma = 0.02$ and $\sigma = 1$ are depicted in Figures 4.4 and 4.5, respectively. We see that the ODE system is asymptotically unstable for $\sigma = 0.02$ but asymptotically stable for $\sigma = 1$, which agrees completely with the results of Section 4.1. Numerical simulations have shown that no patterns exist in the diffusive case for a wide range of $\sigma$ values. The results have been omitted as they do not present any novelty.*
*Next, let $d_1 = 0.1$ and $d_2 = 10$. Hence,*

$$\sigma^* = 1.4719 > \sigma_0,$$

$$\eta^* = 0.6066,$$

$$\eta_c = 2.2051 \times 10^{-2},$$

$$\sigma_H(\eta_c) \approx \sigma_S(\eta_c) \approx 5.457 \times 10^{-2}$$

*and*

$$\chi = 33.971.$$

*Since $\sigma^* > \sigma_0$, we have the case on the right hand side of Figure 3.1. Given that*

$$\begin{cases} \eta_l = 0.2219, \\ \eta_r = 2.2531, \end{cases}$$

*it is easy to verify that there exist eigenvalues that satisfy $\eta \in (\eta_l, \eta_r)$, which according to Theorem 4.2.2 implies the existence of Turing patterns for $\sigma_0 < \sigma < \overline{\sigma}$. This is meant to lead to the formation of spatial patterns in the solutions as time progresses. We choose the value $\sigma = 0.505$, which clearly satisfies the condition. The numerical solutions of the local system (4.21) given initial conditions $(u_0, v_0) = (0.5, 1)$ are depicted in Figure 4.6 . It is easy to see that the solutions are asymptotically stable and in fact converge towards the equilibrium*

*$(u^*, v^*) = (1, 2)$ in sufficient time. The numerical diffusive system corresponding to (4.21) with zero Neumann boundaries are depicted in Figure 4.7 at $t = 500$ units of time. The same initial states are chosen as in the local ODE case but an additive white Gaussian noise perturbation with zero mean and a variance of 0.2 is included to introduce spatial non-homogeneity. We see that spots appear in both substances confirming the existence of patterns as a result of the Turing instability of the system.*

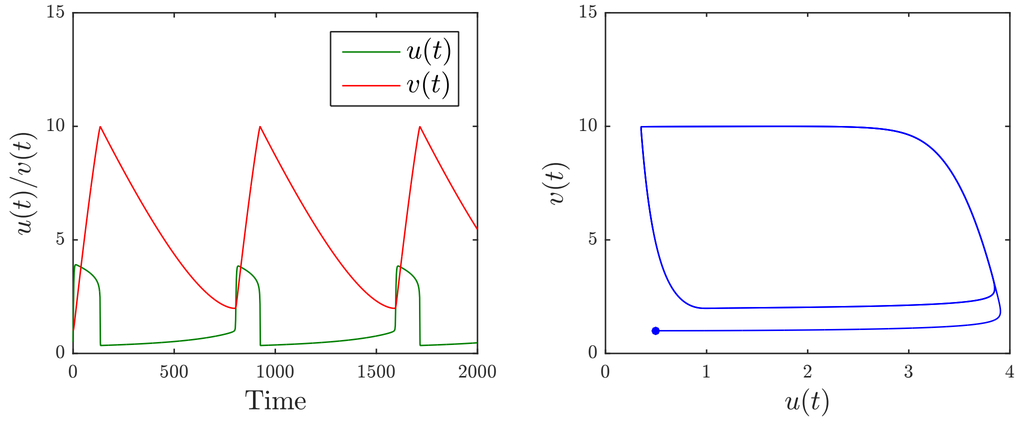


Figure 4.4: Numerical solutions of the local system (4.45) with initial conditions $(u_0, v_0) = (0.5, 1)$ and $\sigma = 1$.

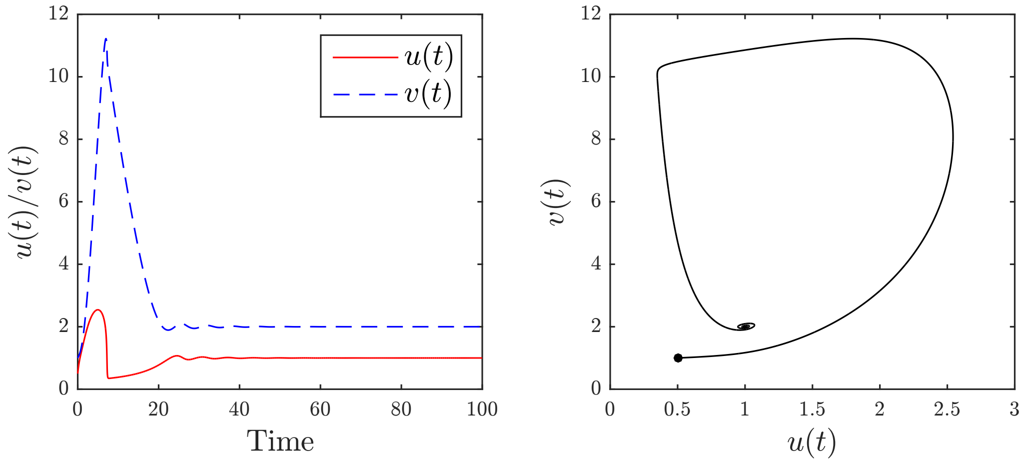


Figure 4.5: Numerical solutions of the local system (4.45) with initial conditions $(u_0, v_0) = (0.5, 1)$ and $\sigma = 0.02$.

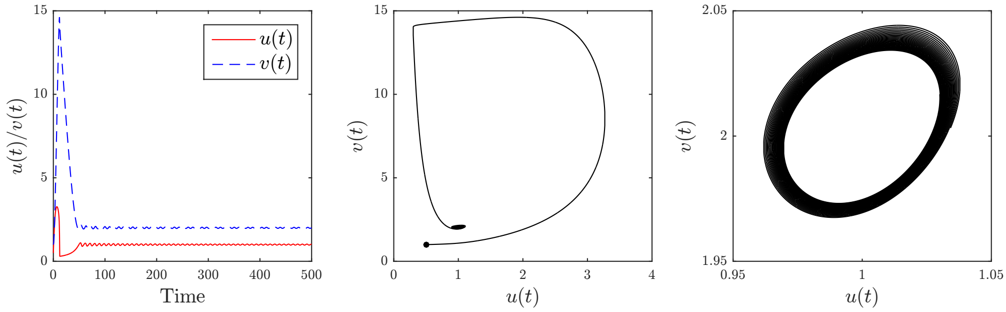


Figure 4.6: Numerical solutions of the local system (4.45) with initial conditions $(u_0, v_0) = (0.5, 1)$ and $\sigma = 0.505$.

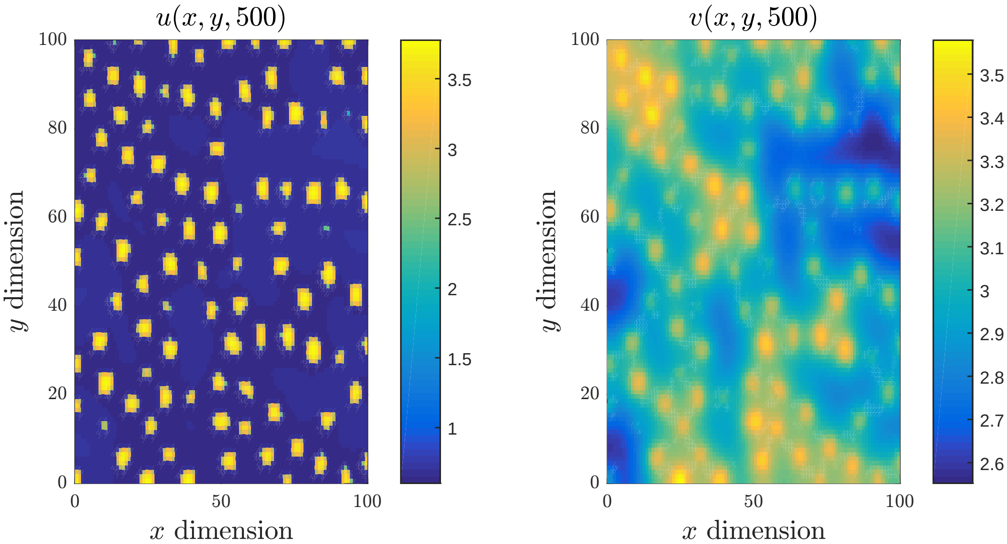


Figure 4.7: Numerical solutions of the diffusive system corresponding to ( 4.45) in the two spatial dimensions scenario with $\sigma = 0.505$.

# Conclusion

FINALLY, this two articles cover resultes, which the first, we have considered a time–fractional version of the Lengyel–Epstein system modeling the chlorite–iodide malonic acid (CIMA) chemical reaction. The Lengyel–Epstein model is well known for exhibiting Turing patterns, which makes it of interest to researchers in mathematics, chemistry, and biology. Introducing fractional time derivatives has recently been shown to model natural phenomena more accurately especially in chemical reactions. We have established sufficient conditions for the local asymptotic stability of the system's unique equilibrium in the ODE and PDE senses through the linearization method. In addition, we have employed the direct Lyapunov method to establish the global asymptotic stability of the steady state solution. Through numerical investigation, we have seen that a periodic solution in the standard case, which corresponds to pattern formation, became asymptotically stable when the differentiation order decreased below 1. This is an important observation that requires closer investigation and analysis as it provides a new perspective into the control and applications of the Lengyel–Epstein system. We have also seen that the presence of diffusion alters the stability conditions of the system, which is not at all unlike the standard case. Furthermore, we saw that the type of patterns that form as a result of the diffusion–driven instability changes as the fractional order is varied. In the second article,one deals with the dynamics of a reaction–diffusion system that generalizes the famous Lengyel–Epstein model for the CIMA reaction. Previous results reported in the literature derived sufficient conditions for the asymptotic stability of the system's unique steady state as well as the non–existence of Turing patterns. The current study has extended the results to include sufficient conditions for the existence of Turing patterns in addition to examining the Hopf–bifurcation of the system in the diffusion–free and diffusive cases. The study has established that Turing patterns occur subject to the bifurcation parameter $\sigma$ lying within a short interval around the bifurcation point. Numerical results obtained by means of the finite difference method have been presented to confirm and validate the theoretical findings. We have seen the emergence of spots in a two–dimensional implementation of the system after a duration of $500s$.

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