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Sur des propriétés des solutions de certaines équations fonctionnelles provenant de la programmation dynamique

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Contents

Introduction

L'étude de l'existence, unicité et les approximations itératives des solutions de différentes équations fonctionnelles émanant de la programmation dynamique ont attiré beaucoup de chercheurs depuis les années 70. La programmation dynamique qui consiste à décomposer un problème d'optimisation en sous-problèmes et les résoudre ensuite par stockage des résultats intermédiaires.

En 1978, l'article de Bellman and Lee [9] est considéré comme le fil déclencheur d'une série de questionnements concernant le lien des équations fonctionnelles avec la programmation dynamique. En 1984, Bakhta et Mitra [11] ont mis en place la forme basique d'une équation fonctionnelle qui provient de la programmation dynamique. Dans les années qui suivirent, une quantité importante d'articles s'intéressant à l'existence et l'unicité ainsi qu aux problèmes de convergence de différentes équations fonctionnelles ont vu le jour, voir par exemple [39, 40, 41, 42, 44].

La résolution de plusieurs problèmes mathématiques se ramène souvent à la recherche de point fixe pour certaines fonctions non linéaires. De plus, plusieurs problèmes intervenant en physique, dans des disciplines aussi diverses que l'éonomie, la biologie, la physique, la mécanique, etc.... De nombreux phénomènes sont modélisables sous forme de problèmes mathématiques. Cette mise en forme permet d'utiliser les ressources des mathématiques appliquées : programmation mathématique, recherche de points fixes. L'origine de la théorie du point fixe est une méthode des approximations successives utilisée pour prouver l'existence des solutions d'équations différentielles introduites indépendamment par Liouville en 1837 et Picard en 1890. En 1922, S. Banach a établi le théorème du point fixe qui porte son nom ou connu sous le nom du principe de contraction de Banach qui est un outil important en théorie des espaces métriques et en analyse en général, il garantit l'existence et l'unicité de points fixes pour certaines applications qui diminuent les distances et donne une méthode constructive pour trouver ces points fixes. Ce principe est donné comme suit:

Soit (X, d) un espace métrique. L'application $T : X \to X$ est dite une contraction sur X s'il existe $k \in [0, 1[$ telle que

$$d\left(T(x), T(y)\right) \leq kd(x, y)$$

pour tout $x, y \in X$.

Soit (X, d) un espace métrique complet, et $T : X \to X$ une contraction. Alors T admet un point fixe unique u dans X. De plus, pour tout élément arbitraire $x_0 \in X$, la suite $\{x_n\}_{n \geq 1}$ définie par $x_n = T(x_{n-1}) = T^n(x_0)$ converge vers u dans (X, d).

Ce théorème de Banach fournit aussi un algorithme d'approximation de ce point fixe comme limite d'une suite itérée, contrairement à d'autres théorèmes du point fixe (Brouwer, Schauder,...), qui nous assurent seulement l'existence de points fixes sans indiquer comment les déterminer.

Mais d'une part, montrer que l'application est contractante peut entraîner de laborieux calculs. D'autre part, les conditions sur l'application et l'espace étudié restreignent le nombre de cas aux quels on peut appliquer ce théorème. D'où la nécessité d'étendre et de généraliser le principe de contraction de Banach afin d'élargir son domaine d'applications, d'une part, et de récupérer ses avantages. L'une des généralisations les plus répandues des espaces métriques est donnée dans l'article [3, 20] à travers la notion d'espaces b-métriques. Dans l'article [82], une généralisation intéressante du principe de contraction de Banach est donnée en introduisant le concept de F-contraction. Après cela, la notion de F-contraction de Hardy-Rogers est introduite dans [18] comme une généralisation de la F-contraction dans un espace métrique complet. Une autre généralisation du théorème de Banach, dite, θ-contraction et établie par [30] a donné naissance à plusieurs résultats intéressants dans la théorie du point fixe et ses applications dans le cadre des équations fonctionnelles émanant de la programmation dynamique.

Cette thèse comprend quatre chapitres:

Dans le premier chapitre, on rappelle quelques outils nécessaires à notre étude menée dans cette thèse.

Dans le deuxième chapitre, on établi des généralisations de certains résultats existant dans la littérature concernant les équations fonctionnelles émanant de la programmation dynamique. D'une manière plus précise, des résultats d'existence, d'unicité et d'approximations itératives sont établis moyennant les théorème de Krasnoselskii, Boyd-Wong et Liu.

Le troisième chapitre est dédié essentiellement à la notion de F-contractions de Hardy–Rogers dans le cadre d'espaces b-métriques. Il s'agit de généraliser et améliorer certains résultats dans [79] (respectivement, dans [55]). Une application sur les équations fonctionnelles provenant de la programmation dynamique dans le cadre des espaces b-métriques est donnée pour mettre en valeur l'efficacité de nos résultats.

Au chapitre 4, on donne une démonstration courte et différente pour le théorème de Lukács et Kajántó et nous établissons une forme équivalente basée sur les θ-contractions. Une application intéressante (suivie d'un exemple) est réalisée à la fin de ce chapitre pour les équations fonctionnelles émanant de la programmation dynamique dans le contexte des espaces b-métriques.

Chapter 1

Preliminaries

1.1 Definitions and auxiliary results

Throughout this thesis, we denote by $\mathbb{N}$, $\mathbb{R}$ the sets of positive integers and real numbers, respectively. We also write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Henceforth, X will denote a nonempty set and the Picard sequence of a self-mapping $T : X \to X$ based on an arbitrary $x_0 \in X$ is given by $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$, where T^n denotes the n^{th}-iterates of T.

For convenience, we set

$$\begin{aligned}
&\Delta_1 := \left\{\varphi,\ \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is right continuous at } 0\right\};\\
&\Delta_2 := \left\{\varphi,\ \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is nondecreasing}\right\};\\
&\Delta_3 := \{\varphi,\ \varphi \in \Delta_1,\ \varphi(0) = 0\};\\
&\Delta_4 := \{\varphi,\ \varphi \in \Delta_3,\ \varphi(t) < t,\ \forall t > 0\};\\
&\Delta_5 := \{\varphi,\ \varphi \in \Delta_1 \cap \Delta_2,\ \varphi(t) < t,\ \forall t > 0\};\\
&\Delta_6 := \{\varphi,\ \varphi \in \Delta_2,\ \varphi(t) < t,\ \forall t > 0\};\\
&\Delta_7 := \left\{\varphi,\ \varphi \in \Delta_6 \text{ and } \varphi \text{ is right continuous in } \mathbb{R}^+\right\}.
\end{aligned}$$

1.1.1 b-metric spaces

In 1989, Bakhtin [4] introduced the concept of b-metric spaces as a generalization of the metric spaces in the sense that the triangle inequality contains a suitable constant $s \geq 1$ (see also Czerwik [20]). Since then, several published papers have dealt with b-metric spaces and fixed point theory in the setting of b-metric spaces (see, e.g., [2, 3, 10, 12, 13, 14, 24, 64, 67, 75] and some related references therein). For more details concerning some technical and useful tools in the context of b-metric spaces, the reader may consult [2] and [64]. Note that the topological framework of a b-metric space with the topology induced by its convergence was studied in [3].

We will first recall the definition of a b-metric space.

Definition 1.1.1 ***(See [21])*** *Let X be a nonempty set and let $s \geq 1$ be a given real number. A mapping $\sigma : X \times X \to [0, \infty)$ is said to be a b-metric if, for all $x, y, z \in X$, the following conditions hold:*

(b_1) $\sigma(x, y) = 0$ if and only if $x = y$;

(b_2) $\sigma(x, y) = \sigma(y, x)$;

(b_3) $\sigma(x, z) \leq s[\sigma(x, y) + \sigma(y, z)]$.

The pair (X, σ) is called a b-metric space with constant (or coefficient) $s \geq 1$.

It is obvious from the above definition that the class of b-metric spaces is larger than that of metric spaces, since a b-metric space is a metric space when $s = 1$ but the converse is not true. The following classical examples illustrate this fact.

Example 1.1.1 ***(See [2], [75])*** *Let (X, d) be a metric space and let the mapping $\sigma_d : X \times X \to [0, \infty)$ be defined by*

$$\sigma_d(x, y) = (d(x, y))^p, \quad \textit{for all } x, y \in X,$$

where $p > 1$ is a fixed real number. Then (X, σ_d) is a b-metric space with $s = 2^{p-1}$.

In particular, if $X=\mathbb{R}$, $d(x,y)=|x-y|$ *is the usual Euclidean metric and*

$$\sigma_d(x,y)=(x-y)^2, \quad \text{for all } x,y\in\mathbb{R},$$

then $(\mathbb{R},\sigma_d)$ *is a b-metric with* $s=2$. *However,* $(\mathbb{R},\sigma_d)$ *is not a metric space on* $\mathbb{R}$ *since* (b_3) *does not hold. Indeed,*

$$\sigma_d(-2,2)=16>8=4+4=\sigma_d(-2,0)+\sigma_d(0,2).$$

Example 1.1.2 ***(See** [36]) *Let* X *be the set of Lebesgue measurable functions on* $[0,1]$ *such that*

$$\int_0^1 |f(x)|^2\,dx<\infty.$$

Define $D: X\times X\to[0,\infty)$ *by*

$$D(f,g)=\int_0^1 |f(x)-g(x)|^2\,dx.$$

Then D *satisfies the following properties*

1. $D(f,g)=0$ *if and only if* $f=g$,
2. $D(f,g)=D(g,f)$, *for any* $f,g\in X$,
3. $D(f,g)\le 2(D(f,h)+D(h,g))$, *for any points* $f,g,h\in X$.

Clearly, (X,D) *is a b-metric space with* $s=2$ *but is not a metric space. For example, take* $f(x)=0$, $g(x)=1$ *and* $h(x)=\frac{1}{2}$, *for all* $x\in[0,1]$. *Then*

$$D(0,1)=1>\frac{1}{2}=\frac{1}{4}+\frac{1}{4}=D(0,\frac{1}{2})+D(\frac{1}{2},1).$$

Example 1.1.3 *Let* $0<p<1$ *and let*

$$L^p([0,1])=\left\{u:[0,1]\to\mathbb{R}:\int_0^1 |u(t)|^p\,dt<\infty\right\},$$

together with the functional $\sigma: L^p([0,1])\times L^p([0,1])\to[0,\infty)$ *given by*

$$\sigma(u,v)=\left(\int_0^1 |u(t)-v(t)|^p\,dt\right)^{\frac{1}{p}} \quad \text{for all } u,v\in L^p([0,1]),$$

is a b-metric space with coefficient $s=2^{\frac{1}{p}-1}$.

Example 1.1.4 *The space $l_p(\mathbb{R})$ with $0 < p < 1$, where*

$$l_p(\mathbb{R}) = \left\{ (x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\},$$

together with function $\sigma : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to [0, \infty)$ defined by

$$\sigma(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}} \text{ for all } x = (x_n), y = (y_n) \in l_p(\mathbb{R}),$$

is a b-metric space with coefficient $s = 2^{\frac{1}{p}-1}$.

We present now the concepts of convergence, Cauchy sequence and completeness in b-metric spaces.

Definition 1.1.2 ***(See** [12], [13], [14]) Let (X, σ) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:*

(a) convergent if and only if there exists $x \in X$ such that $\lim_{n\to\infty} \sigma(x_n, x) = 0$ and in this case we write $\lim_{n\to\infty} x_n = x$;

(b) Cauchy if and only if $\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0$.

Definition 1.1.3 ***(See** [12], [13], [14]) The b-metric space (X, σ) is said complete if every Cauchy sequence in X converges in X.*

Remark 1.1.1 ***(See** [12], [13], [14]) In a b-metric space, the following assertions hold:*

(i) a convergent sequence has a unique limit;

(ii) each convergent sequence is Cauchy.

Lemma 1.1.1 ***(See** [24, Lemma 2.1]) Let (X, σ) be a b-metric space with constant $s \geq 1$ and $\{x_n\}$, $\{y_n\}$ two sequences such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ in (X, σ). Then $\lim_{n\to\infty} \sigma(x_n, y_n) = 0$ if and only if $x = y$.*

It is worth recalling that a b-metric is generally not continuous (see, e.g., [27, Example 3.3]). The following lemmas are very useful to manage this problem.

Lemma 1.1.2 ***(See** [2], [55]) Let (X, σ) be a b-metric space with constant $s \geq 1$ and $\{x_n\}$ be a convergent sequence in X with $\lim x_n = x$. Then for each $y \in X$, we have*

$$\frac{1}{s}\sigma(x, y) \leq \liminf_{n\to\infty} \sigma(x_n, y) \leq \limsup_{n\to\infty} \sigma(x_n, y) \leq s\sigma(x, y).$$

Lemma 1.1.3 ***(See** [64, Lemma 1.7])* *Let (X,σ) be a b-metric space with constant $s \geq 1$ and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0. \tag{1.1.1}$$

If $\{x_n\}$ is not a Cauchy sequence in (X,σ), then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that the following items hold:

$$\varepsilon \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon;$$
$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)+1}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)+1}\right) \leq s^2\varepsilon;$$
$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)}\right) \leq s^2\varepsilon;$$
$$\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq s^3\varepsilon.$$

Inspired by the works in [62], we can state the following lemma

Lemma 1.1.4 *Let all the conditions of Lemma 1.1.3 be satisfied. Then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that the following items hold:*

$$\varepsilon^+ \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon^+;$$
$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)+1}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)+1}\right) \leq s^2\varepsilon;$$
$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)}\right) \leq s^2\varepsilon;$$
$$\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq s^3\varepsilon.$$

Proof. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and sequences $\left\{x_{m(k)}\right\}$ and $\left\{x_{n(k)}\right\}$ of positive integers such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and $\sigma\left(x_{m(k)}, x_{n(k)}\right) > \varepsilon$. Due to (1.1.1), this implies that $\sigma\left(x_{m(k)}, x_{n(k)-1}\right) \leq \varepsilon$ for all $k \geq 1$. Using the relaxed triangle inequality (b_3), we have

$$\begin{aligned}\sigma\left(x_{m(k)}, x_{n(k)}\right) &\leq s\sigma\left(x_{m(k)}, x_{n(k)-1}\right) + s\sigma\left(x_{n(k)-1}, x_{n(k)}\right)\\ &\leq s\varepsilon + s\sigma\left(x_{n(k)-1}, x_{n(k)}\right).\end{aligned}$$

This leads to

$$\frac{1}{s}\sigma\left(x_{m(k)}, x_{n(k)}\right) \leq \varepsilon + \sigma\left(x_{n(k)-1}, x_{n(k)}\right).$$

Since $\sigma\left(x_{n(k)-1}, x_{n(k)}\right) > 0$, then by taking limit superior as $k \to \infty$ with (1.1.1), we get

$$\frac{1}{s}\limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq \varepsilon^+,$$

or, equivalently,

$$\limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon^{+}. \tag{1.1.2}$$

On the other hand, we have

$$\frac{1}{k} + \sigma\left(x_{m(k)}, x_{n(k)}\right) > \frac{1}{k} + \varepsilon, \quad \text{for all } k \geq 1.$$

Taking the limit inferior as $k \to \infty$, we have

$$\liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \geq \varepsilon^{+}. \tag{1.1.3}$$

From (1.1.2) and (1.1.3), we obtain the first item of Lemma 1.1.4. Since the remaining items are the same as in Lemma 1.1.3, the proof is completed. ■

Remark 1.1.2 *Taking $s = 1$ (the case corresponding to a metric space (X, d)) in Lemma 1.1.4, we find Lemma 2.2 in [62]. More precisely, the above items become as follows*

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \varepsilon^{+}$$

and

$$\begin{aligned}\lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)}\right) &= \lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)+1}\right)\\ &= \lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) = \varepsilon.\end{aligned}$$

In 2015, An et al. [3] proved the following result:

Proposition 1.1.1 ***(See** [3, Proposition 3.11]) Let (X, σ) be a b-metric space with constant $s \geq 1$. If σ is continuous with respect in one variable, then σ is continuous in other variable.*

Obviously, we observe from the above result that if σ is not continuous with respect one variable, then σ is not continuous in each variable (refer to [3, Examples 3.9, 3.10].

We end this subsection by giving an example which illustrates some preceding properties concerning b-metric spaces.

Example 1.1.5 *Let $X = [0, \infty)$. Let $\sigma : X \times X \to [0, \infty)$ be a mapping defined by*

$$\sigma(x, y) = \begin{cases} d(x, y), & xy \neq 0, \\ 4d(x, y), & xy = 0, \end{cases}$$

where $d(x, y) = |x - y|$. Then the following hold:

(1) *(X, σ) is a complete b-metric space with constant $s = 4$.*

(2) *σ is not a metric on X.*

(3) *σ is not continuous in each variable.*

Proof. (1). We start to prove that (X,σ) is a b-metric space with constant $s=4$. Clearly, (b_1) and (b_2) are satisfied. For (b_3), we can easily observe that for any $x,y\in X$,

$$d(x,y)\leq \sigma(x,y)\leq 4d(x,y). \tag{1.1.4}$$

We consider then the following cases.

Case 1. Suppose that $xy\neq 0$. Then, using (1.1.4), for any $z\in X$, we obtain

$$\begin{aligned}\sigma(x,y)=d(x,y)&\leq d(x,z)+d(z,y)\\&\leq \sigma(x,z)+\sigma(z,y)\leq 4(\sigma(x,z)+\sigma(z,y)).\end{aligned}$$

Case 2. Assume that $xy=0$. Also, through (1.1.4), we have for any $z\in X$

$$\begin{aligned}\sigma(x,y)=4d(x,y)&\leq 4d(x,z)+4d(z,y)\\&\leq 4(\sigma(x,z)+\sigma(z,y)).\end{aligned}$$

Next, since (X,d) is a complete metric space, the completeness of (X,σ) follows immediately from (1.1.4).

(2). Indeed, σ is not a metric on X since we have

$$\sigma(0,2)=8>5=4+1=\sigma(0,1)+\sigma(1,2).$$

(3). Let $x_n=\dfrac{1}{n}$ for each $n\in\mathbb{N}$. We have

$$\lim_{n\to\infty}\sigma\left(\frac{1}{n},0\right)=\lim_{n\to\infty}\frac{4}{n}=0.$$

Then $\lim_{n\to\infty}x_n=0$ in (X,σ). On the other hand, we have

$$\lim_{n\to\infty}\sigma(x_n,1)=1\neq 4=\sigma(0,1).$$

This, together with Proposition 1.1.1, proves that σ is not continuous in each variable. ∎

1.1.2 F-contractions

Now, let us review some results concerning F-contractions related to the existing literature. In 2012, Wardowski [82] initiated one of an attractive and important generalization of the remarkable *Banach contraction principle* [8]. He considered a new type of contractions, the so-called *F-contraction.*

Definition 1.1.4 ***(See** [82]**)*** *Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be an F-contraction if there exist $F\in\mathcal{F}$ and $\tau>0$ such that for all $x,y\in X$,*

$$d(Tx,Ty)>0\Rightarrow \tau+F(d(Tx,Ty))\leq F(d(x,y)), \tag{1.1.5}$$

where $\mathcal{F}$ is the family of all functions $F:(0,\infty)\to\mathbb{R}$ satisfying the following conditions:

(F_1) *F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$, if $\alpha < \beta$, then $F(\alpha) < F(\beta)$.*

(F_2) *For each sequence $\{\alpha_n\}$ of positive numbers, the following holds:*

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty.$$

(F_3) *There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

Remark 1.1.3 ***(See** [82]**)*** *Let $\alpha > 0$. Let the following functions $F_1(\alpha) = \ln \alpha$, $F_2(\alpha) = \ln \alpha + \alpha$, $F_3(\alpha) = \dfrac{-1}{\sqrt{\alpha}}$ and $F_4(\alpha) = \ln(\alpha^2 + \alpha)$. Then, F_1, F_2, F_3 and $F_4 \in \mathcal{F}$.*

Remark 1.1.4 ***(See** [82]**)*** *Clearly, if F is an increasing function (not necessary strictly increasing), the inequality* (1.1.5) *implies that T is a contractive mapping, i.e.,*

$$d(Tx, Ty) < d(x, y), \quad \forall x, y \in X, \; x \neq y.$$

Hence, every F-contraction is a continuous mapping.

Wardowski proved that every self-map T on a complete metric space (X, d), which is an F-contraction is a Picard operator (see [82, Theorem 2.1]). Since then, the aforementioned result has been developed by many authors, see, for example, the survey [34] and the works [22, 85]. More precisely, these developments are mainly based on defining new kinds of contractions in different spaces or considering some weaker conditions on the mapping F.

Wardowski's result is given as follows:

Theorem 1.1.1 ***(See** [82, Theorem 2.1]**)*** *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.*

Remark 1.1.5 ***(See** [82]**)*** *Wardowski showed that T is a Banach contraction* [8] *by taking $F(\alpha) = \ln \alpha$ in* (1.1.5).

In [68], Secelean showed that condition (F_2) can be replaced by an equivalent and more easier one (noted (F_2'): $\inf F = -\infty$). Afterwards, Piri and Kumam [58] established Wardowski's theorem by using (F_2') and the continuity instead of (F_2) and (F_3), respectively. Later, Wardowski [83] proved a fixed point theorem concerning F-contractions when τ is taken as a function. In this work, the author used a relaxed version of (F_2) and dropped also condition (F_3). In 2018, Lukács and Kajántó [55] extended Wardowski's theorem in the setting of b-metric spaces and omitted condition (F_2). Very recently, some authors proved (in different ways) the original results of Wardowski without both conditions (F_2) and (F_3) (see, e.g., [56, Remark 3.7], [62, Corollary 3.21 and Theorem 4.1]). It is also worth mentioning that many others papers dealing with various types of F-contractions can be found in the literature (see, e.g., [17, 23, 25, 35, 56, 57, 59, 60, 61, 66, 69, 70, 71, 73, 74, 80, 81] and references therein).

1.1.3 Pseudometric spaces and some related results

Definition 1.1.5 *Let X be a nonempty set. A mapping $\rho : X \times X \to [0, \infty)$ is said to be a pseudometric if, for all $x, y, z \in X$, the following conditions hold:*

(d_1) $\rho(x, x) = 0$;

(d_2) $\rho(x, y) = \rho(y, x)$;

(d_3) $\rho(x, z) \leq [\rho(x, y) + \rho(y, z)]$.

The pair (X, ρ) is called a pseudometric space.

Lemma 1.1.5 *([41]) Let E be a nonempty set and $\{\rho_n\}_{n\in\mathbb{N}}$ a countable family of pseudometrics on E such that for any $x, y \in E$ with $x \neq y$, there exists $i \in \mathbb{N}$ satisfying $\rho_i(x, y) \neq 0$. Suppose that $d : E \times E \to \mathbb{R}^+$ satisfies that*

$$d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\rho_k(x, y)}{1 + \rho_k(x, y)}, \quad x, y \in E.$$

Then, d is a metric on E.

Remark 1.1.6 *The metric d in the above Lemma is said to be an induced metric by the countable family of pseudometrics $\{\rho_n\}_{n\in\mathbb{N}}$ and (E, d) is called an induced metric space by the countable family of pseudometrics $\{\rho_n\}_{n\in\mathbb{N}}$.*

- *A sequence $\{x_n\}_{n\in\mathbb{N}} \in E$ converges to a point $x \in E$ if and only if $\rho_k(x_n, x) \to 0$ for each $k \in \mathbb{N}$.*
- *$\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if $\rho_k(x_n, x_m) \to 0$ as $n, m \to \infty$ for each $k \in \mathbb{N}$.*

Definition 1.1.6 *([41]) Let (E, d) be an induced metric space by a countable family of pseudometrics $\{\rho_n\}_{n\in\mathbb{N}}$ and $T : (E, d) \to (E, d)$ be a mapping. For $x, y \in E, A \subseteq E$ and $k \in \mathbb{N}$, let us consider*

$$O_T(x) = \left\{T^i x, i \in \mathbb{N}_0\right\}, \quad O_T(x, y) = O_T(x) \cup O_T(y),$$
$$\delta_{\rho_k}(A) = \sup\{\rho_k(a, b), a, b \in A\}.$$

Then E is said to be $T-$orbitally complete iff every Cauchy sequence that is contained in $O_T(x)$ for $x \in E$ converges in E.

Theorem 1.1.2 *(Liu's fixed point theorem [39]) Let (E, d) be an induced metric space by a countable family of pseudometrics $\{\rho_n\}_{n\in\mathbb{N}}$, (E, d) is said to be $T-$orbitally complete space and $T : (E, d) \to (E, d)$ satisfy that*

(i) for any $(k, x) \in \mathbb{N}\times E$, there exists $p(k, x) > 0$ with $\delta_{\rho_k}(O_T(x)) \leq p(k, x)$;

(ii) there exists $\varphi \in \Delta_7$ such that

$$\rho_k(Tx, Ty) \leq \varphi(\delta_{\rho_k}(O_T(x, y))), \quad \forall (k, x, y) \in \mathbb{N}\times E^2.$$

Then, f has a unique fixed point $w \in E$ and the sequence $\{T^n x\}_{n\in\mathbb{N}_0}$ converges to w for each $x \in E$.

1.1.4 Some other useful fixed point theorems

Definition 1.1.7 *([15]) A metric space (X, d) is said to be metrically convex if for each $(x, y) \in M$, there exists a $z \neq x, y$ for which $d(x, y) = d(x, z) + d(z, y)$.*

Theorem 1.1.3 *(Boyd-Wong's fixed point theorem [15]) Suppose that (E, d) is a completely metrically convex metric space and that $T : (E, d) \to (E, d)$ satisfies*

$$d(Tx, Ty) \leq \varphi(d(x, y)), \quad \forall x, y \in E,$$

where $\varphi : \overline{P} \to \mathbb{R}^+$ satisfies $\varphi(t) < t$ for all $t \in \overline{P} - \{0\}$, $P = \{d(x, y), x, y \in E\}$ and $\overline{P}$ denotes the closure of P. Then T has a fixed point $u \in E$ and $\lim_{n\to\infty} T^n x = u$ for each $x \in E$.

Theorem 1.1.4 *(Krasnoselskii's fixed point theorem) Let F be a bounded closed convex subset of a Banach space E, P and $Q : F \to E$ satisfy that $Px + Qy \in F$ for every pair $x, y \in F$. If P is a contraction mapping and Q is completely continuous, then the equation $Px + Qx = x$ has a solution in F.*

Chapter 2

Solvability and iterative approximations of solutions for a functional equation originating from dynamic programming

Abstract

In this chapter, we present some results concerning a certain functional equations arising in dynamic programming. Roughly speaking, we provide some slight generalizations and extensions of the works [11, 39, 41, 46, 51, 52].

2.1 Preliminaries

Throughout this chapter, we consider the following:

- $\mathbb{R}^+ = [0, +\infty[$.
- $\mathbb{R}_- =]-\infty, 0]$.
- $(X, \|.\|)$ and $\left(Y, \|.\|'\right)$ be real Banach spaces.
- $S \subseteq X$ be the state space and $D \subseteq Y$ be the decision space.

For convenience, let us consider the following sets:

$B(S) := \{u,\ u : S \to \mathbb{R} \text{ is bounded}\}$;
$BC(S) := \{u,\ u \in B(S) \text{ is continuous}\}$;
$BB(S) := \{u,\ u : S \to \mathbb{R} \text{ is bounded on each bounded subsets of } S\}$;
$\chi(C, K) := \{h \in BC(S), |h(x_1)| \leq 2C, |h(x_2) - h(x_1)| \leq K \|x_1 - x_2\|, \forall x_1, x_2 \in S\}$,
$C, K \in \mathbb{R}^+ - \{0\}$.

Remark 2.1.1 *1. If we set*

$$d_k(f, g) = \sup\left\{|f(x) - g(x)|; x \in \overline{B}(0, k)\right\},$$

where

$$\overline{B}(0, k) = \{x; x \in S \text{ and } \|x\| \leq k\}$$

and

$$d(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(f, g)}{1 + d_k(f, g)}, \quad \forall (k, f, g) \in \mathbb{N} \times (BB(S))^2,$$

then, clearly $(BB(S), d)$ is the induced metric space by the countable family of pseudometrics $\{d_n\}_{n \in \mathbb{N}}$ and it is also complete.
2. If we consider the norme $\|h\|_1 = \sup\limits_{x \in S} |h(x)|$, it is easy to see that $(B(S), \|.\|_1)$ and $(BC(S), \|.\|_1)$ are Banach spaces.
3. Clearly, $\chi(C, K)$ is a bounded closed convex subset of the Banach space $BC(S)$.

2.2 Existence and iterative approximations for the functional equation

In this chapter, we study the following functional equation:

$$\begin{aligned} u(x) = {} & \text{opt}_{y \in D} \{p(x, y) [f(x, y) + G(x, y, u(a(x, y)))]\} \\ & + \text{opt}_{y \in D} \{q(x, y) [g(x, y) + H(x, y, u(b(x, y)))]\}, \quad x \in S, \end{aligned} \tag{2.2.1}$$

where "opt" denotes the sup or inf with $x \in S$ and $y \in D$. $u(x)$ is the optimal return function with initial state x and a, b represent the transformations of the process. The following lemma is useful in the sequel.

Lemma 2.2.1 *(See [46]) Let A be a nonempty set. Suppose that $J, K : A \to \mathbb{R}$ are two bounded functions. Then*

$$\left|opt_{x\in A} J(x) - opt_{x\in A} K(x)\right| \leq \sup_{x\in A} \left|J(x) - K(x)\right|.$$

Now, we are ready to state and prove our first result which deals with the existence of the solutions for the functional equation (2.2.1) in $\chi(C, K)$.

Theorem 2.2.1 *Let $C, K, \lambda_1, \lambda_2, L_1, L_2,$ and L_3 be positive constants and S be compact, $a, b : S \times D \to S$, $f, g, p, q : S \times D \to \mathbb{R}$ and $G, H : S \times D \times \mathbb{R} \to \mathbb{R}$ be mappings such that*

(H_1)

$$(i) \sup_{(x,y)\in S\times D} |p(x,y)| \leq \lambda_1, \sup_{(x,y)\in S\times D} |q(x,y)| \leq \lambda_2 \quad \text{with } \lambda_1 + \lambda_2 = 1,$$

$$(ii) \sup_{(x,y)\in S\times D\times\mathbb{R}} \max\left\{|f(x,y)|, |g(x,y)|, |G(x,y,t)|, |H(x,y,t)|\right\} \leq C;$$

(H_2)

$$\begin{aligned} &\sup_{y\in D} \max\{|p(x_2,y) - p(x_1,y)|, |q(x_2,y) - q(x_1,y)|, \\ &\quad |f(x_2,y) - f(x_1,y)|, |g(x_2,y) - g(x_1,y)|\} \\ \leq\ & L_1 \|x_2 - x_1\|, \quad x_1, x_2 \in S; \end{aligned}$$

$$\sup_{y\in D} \max\left\{\|a(x_2,y) - a(x_1,y)\|, \|b(x_2,y) - b(x_1,y)\|\right\} \leq L_2 \|x_2 - x_1\|, \quad x_1, x_2 \in S;$$

(H_3)

$$\begin{aligned} &\sup_{y\in D} \max\{|G(x_2,y,z_2) - G(x_1,y,z_1)|, |H(x_2,y,z_2) - H(x_1,y,z_1)|\} \\ \leq\ & L_3 \max(\|x_2 - x_1\|, |z_2 - z_1|), \quad x_1, x_2 \in S, z_1, z_2 \in \mathbb{R}; \end{aligned}$$

(H_4)

$$L_1(1 + 2C) + \max\{1, KL_2\} \leq K \quad \text{and } 0 < \lambda_1 L_3 < 1.$$

Then, the functional equation (2.2.1) has a solution $\omega \in \chi(C, K)$.

Proof. Let us define the mappings: $T_1, T_2 : \chi(C, K) \to BC(S)$ as follows:

$$T_1 u(x) = \text{opt}_{y\in D}\left\{p(x,y)\left[f(x,y) + G(x,y,u(a(x,y)))\right]\right\}, \quad x \in S, u \in \chi(C,K)$$

and

$$T_2 v(x) = \text{opt}_{y\in D}\left\{q(x,y)\left[g(x,y) + H(x,y,v(b(x,y)))\right]\right\}, \quad x \in S, v \in \chi(C,K).$$

By (H_1) and Lemma 2.2.1, on can get, for $x \in S$ and $u, v \in \chi(C, K)$,

$$\begin{aligned} |T_1 u(x)| &\leq \sup_{y\in D} |p(x,y)| \sup_{y\in D} \{|f(x,y)| + |G(x,y,u(a(x,y)))|\} \\ &\leq 2C \sup_{(x,y)\in S\times D} |p(x,y)| \leq 2C\lambda_1. \end{aligned}$$

and

$$\begin{aligned}|T_2 v(x)| &\leq \sup_{y\in D}|q(x,y)| \sup_{y\in D}\{|g(x,y)|+|H(x,y,v(b(x,y)))|\}\\ &\leq 2C \sup_{(x,y)\in S\times D}|q(x,y)| \leq 2C\lambda_2.\end{aligned}$$

Again by (H_1), the above inequalities lead to

$$\begin{aligned}|T_1 u(x)+T_2 v(x)| &\leq |T_1 u(x)|+|T_2 v(x)|\\ &\leq 2C, \quad \forall x\in S, u,v\in\chi(C,K).\end{aligned}$$

On the other hand and in view of $(H_1)-(H_4)$ with Lemma 2.2.1, we get

$$\begin{aligned}|T_1 u(x_2)-T_1 u(x_1)| \leq\;& \Big(\sup_{y\in D}|p(x_2,y)f(x_2,y)-p(x_1,y)f(x_1,y)|\\ &+\sup_{y\in D}|p(x_2,y)G(x_2,y,u(a(x_2,y)))-p(x_1,y)G(x_1,y,u(a(x_1,y)))|\Big)\\ \leq\;& \sup_{y\in D}|p(x_2,y)|\Big(\sup_{y\in D}|f(x_2,y)-f(x_1,y)|\\ &+\sup_{y\in D}|G(x_2,y,u(a(x_2,y)))-G(x_1,y,u(a(x_1,y)))|\Big)\\ &+\sup_{y\in D}|p(x_2,y)-p(x_1,y)|\Big(\sup_{y\in D}|f(x_1,y)|+\sup_{y\in D}|G(x_1,y,u(a(x_1,y)))|\Big)\\ \leq\;& \sup_{(x,y)\in S\times D}|p(x_2,y)|\big(L_1\|x_2-x_1\|\\ &+L_3\max\Big\{\|x_2-x_1\|,\sup_{y\in D}|u(a(x_2,y))-u(a(x_1,y))|\Big\}\Big)\\ &+2L_1C\|x_2-x_1\|\\ \leq\;& \lambda_1\big(L_1\|x_2-x_1\|\\ &+L_3\max\Big\{\|x_2-x_1\|,K\sup_{y\in D}\|a(x_2,y)-a(x_1,y)\|\Big\}\Big)\\ &+2L_1C\|x_2-x_1\|\\ \leq\;& \lambda_1(L_1(1+2C)+\max\{1,KL_2\})\|x_2-x_1\|\\ \leq\;& \lambda_1 K\|x_2-x_1\|, \quad \forall x_1,x_2\in S, u\in\chi(C,K)\end{aligned}$$

and

$$
\begin{aligned}
|T_2v(x_2) - T_2v(x_1)| \;\le\;& \Bigg(\sup_{y\in D} |q(x_2,y)\,g(x_2,y) - q(x_1,y)\,g(x_1,y)| \\
&+ \sup_{y\in D} |q(x_2,y)\,H(x_2,y,v(b(x_2,y))) - q(x_1,y)\,H(x_1,y,v(b(x_1,y)))|\Bigg) \\
\le\;& \sup_{y\in D} |q(x_2,y)| \Bigg(\sup_{y\in D} |g(x_2,y) - g(x_1,y)| \\
&+ \sup_{y\in D} |H(x_2,y,v(b(x_2,y))) - H(x_1,y,v(b(x_1,y)))|\Bigg) \\
&+ \sup_{y\in D} |q(x_2,y) - q(x_1,y)| \left(\sup_{y\in D} |g(x_1,y)| + \sup_{y\in D} |H(x_1,y,v(b(x_1,y)))|\right) \\
\le\;& \sup_{(x,y)\in S\times D} |q(x_2,y)| \,(L_1 \|x_2 - x_1\| \\
&+ L_3 \max\left\{\|x_2 - x_1\|, \sup_{y\in D} |v(b(x_2,y)) - v(b(x_1,y))|\right\}\Bigg) \\
&+ 2L_1 C \|x_2 - x_1\| \\
\le\;& \lambda_2 (L_1 \|x_2 - x_1\| \\
&+ L_3 \max\left\{\|x_2 - x_1\|, K \sup_{y\in D} \|b(x_2,y) - b(x_1,y)\|\right\}\Bigg) \\
&+ 2L_1 C \|x_2 - x_1\| \\
\le\;& \lambda_2 (L_1 (1 + 2C) + \max\{1, KL_2\}) \|x_2 - x_1\| \\
\le\;& \lambda_2 K \|x_2 - x_1\|, \quad \forall x_1, x_2 \in S, u \in \chi(C,K)
\end{aligned}
$$

The above inequalities through (H_1) yield that

$$
\begin{aligned}
& |T_1u(x_2) + T_2v(x_2) - T_1u(x_1) - T_2v(x_1)| \\
\le\;& |T_1u(x_2) - T_1u(x_1)| + |T_2v(x_2) - T_2v(x_1)| \\
\le\;& K\lambda_1 \|x_2 - x_1\| + K\lambda_2 |q(x,y)| \|x_2 - x_1\| \\
\le\;& K \|x_2 - x_1\|, \quad \forall x_1, x_2 \in S, v \in \chi(C,K)
\end{aligned}
$$

Actually, from above, we have obtained that, $T_1, T_2 : \chi(C,K) \to BC(S)$ and $T_1u + T_2v \in \chi(C,K)$, for any $u, v \in \chi(C,K)$.

Again, using Lemma 2.2.1 and (H_4), we have

$$
\begin{aligned}
& |T_1u(x) - T_1v(x)| \\
\le\;& \sup_{(x,y)\in S\times D} |p(x,y)| \sup_{y\in D} |G(x,y,u(a(x,y))) - G(x,y,v(a(x,y)))| \\
\le\;& \lambda_1 \sup_{y\in D} \{L_3 |u(a(x,y)) - v(a(x,y))|\} \\
\le\;& \lambda_1 L_3 \|u - v\|_1, \quad \forall x \in S, u, v \in \chi(C,K).
\end{aligned}
$$

Using (H_4), the last inequality ensures that T_1 is a contraction mapping in $\chi(C,K)$.

Now, we claim that T_2 is completely continuous. Let $\{v_n\} \subset \chi(C,K)$ and $v \in \chi(C,K)$ with $\lim_{n\to\infty} v_n = v$. Let $\varepsilon > 0$, thus, there exists $n_0 \in \mathbb{N}$ such that

$$\|v_n - v\|_1 < \varepsilon (1 + L_3)^{-1}, \quad \forall n \geq n_0,$$

which implies with $(H_1), (H_4)$ and Lemma 2.2.1

$$\begin{aligned} & |T_2 v_n(x) - T_2 v(x)| \\ \leq\ & \sup_{(x,y)\in S\times D} |q(x,y)| \sup_{y\in D} |H(x,y,v_n(b(x,y))) - H(x,y,v(b(x,y)))| \\ \leq\ & \lambda_2 \sup_{y\in D} \{L_3 |v_n(b(x,y)) - v(b(x,y))|\} \\ \leq\ & L_3 \|v_n - v\|_1 < \varepsilon, \quad \forall x \in S, u, v \in \chi(C,K). \end{aligned}$$

Hence

$$\|T_2 v_n - T_2 v\|_1 < \varepsilon$$

and T_2 is continuous in $\chi(C,K)$. Next, given $\varepsilon > 0$ and we put $\alpha = \varepsilon(1+K)^{-1}$
From preceding calculations, we have obtained for all $x_1, x_2 \in S, v \in \chi(C,K)$ with $\|x_2 - x_1\| < \alpha$

$$|T_2 v(x_2) - T_2 v(x_1)| \leq \lambda_2 K \|x_2 - x_1\| \leq K \|x_2 - x_1\|.$$

Hence, by theorem of Ascoli-Arzela, $T_2(\chi(C,K))$ is relatively compact in S. Therefore, T_2 is a completely continuous operator. In view of Theorem 1.1.4, there exists $\omega \in \chi(C,K)$ such that $(T_1 + T_2)\omega = \omega$ which is bounded continuous solution of the functional equation (2.2.1). This concludes the proof. ■

Theorem 2.2.2 *Let $(\varphi, \psi) \in \Delta_3 \times \Delta_5$ and S be compact, $a, b : S\times D \to S$, $f, g, p, q : S\times D \to \mathbb{R}$ and $G, H : S \times D \times \mathbb{R} \to \mathbb{R}$ be mappings such that*

(H_5) f, g are bounded on $S \times D$ and G, H are bounded on $S \times D \times \mathbb{R}$ and

$$\sup_{(x,y)\in S\times D} \{|p(x,y)| + |q(x,y)|\} \leq 1;$$

(H_6) for each $(x_0, \eta) \in S \times \{p, q, f, g, a, b\}$,

$$\lim_{x\to x_0} \eta(x,y) = \eta(x_0,y)$$

uniformly for $y \in D$, respectively;

(H_7)

$$\begin{aligned} & \max\{|G(x,y,z) - G(x_0,y,z)|, |H(x,y,z) - H(x_0,y,z)|\} \\ \leq\ & \varphi(\|x - x_0\|), \quad x, x_0 \in S, y \in D, z \in \mathbb{R}; \end{aligned}$$

(H_8)

$$\begin{aligned} & \max\{|G(x,y,z) - G(x,y,z_0)|, |H(x,y,z) - H(x,y,z_0)|\} \\ \leq\ & 2^{-1}\psi(|z - z_0|), \quad x, \in S, y \in D, z, z_0 \in \mathbb{R}. \end{aligned}$$

Then, the functional equation (2.2.1) *possesses a unique solution* $\omega \in BC(S)$ *and the sequence* $\{T^n h\}$ *converges to* ω *for each* $h \in BC(S)$, *where*

$$\begin{aligned} Th(x) &= opt_{y\in D}\{p(x,y)[f(x,y)+G(x,y,h(a(x,y)))]\} \\ &+ opt_{y\in D}\{q(x,y)[g(x,y)+H(x,y,h(b(x,y)))]\}, \quad x \in S. \end{aligned}$$

Proof. Let (x_0, h) be arbitrary in $S \times BC(S)$. First, it follows from assumption (H_5) that there exists $M \geq 1$ such that

$$\sup_{(x,y)\in S\times D\times\mathbb{R}} \max\{|p(x,y)|, |q(x,y)|, |f(x,y)|, |g(x,y)|, |G(x,y,t)|, |H(x,y,t)|\} \leq M. \tag{2.2.2}$$

From (2.2.2) and Lemma 2.2.1, it is easy to deduce that Th is bounded. Let $\varepsilon > 0$. By virtue of assumption (H_6), $(\varphi, \psi) \in \Delta_3 \times \Delta_5$ and the compactness of S, we obtain that there exists $\mu_1, \mu_2, \mu_3 > 0$ such that

$$\varphi(\|x - x_0\|) < \frac{\varepsilon}{8M}, \quad \forall x \in S \text{ with } \|x - x_0\| < \mu_1;$$

$$\psi(\mu_1) < \frac{\varepsilon}{8M};$$

$$\max\{|q(x,y) - q(x_0,y)|, |p(x,y) - p(x_0,y)|, |f(x,y) - f(x_0,y)|, |g(x,y) - g(x_0,y)|\} < \frac{\varepsilon}{8M},$$

for all $(x,y) \in S \times D$ with $\|x - x_0\| < \mu_1$;

$$|h(x) - h(x_0)| < \mu_1, \quad \forall x \in S \text{ with } \|x - x_0\| < \mu_2;$$

$$\max\{\|a(x,y) - a(x_0,y)\|, \|b(x,y) - b(x_0,y)\|\} \leq \mu_2,$$

for all $(x,y) \in S \times D$ with $\|x - x_0\| < \mu_3$.

From the above inequalities, one can get

$$\begin{aligned} &\psi\left(\sup_{y\in D}\max\{|h(a(x,y)) - h(a(x_0,y))|, |h(b(x,y)) - h(b(x_0,y))|\}\right) \\ \leq\ &\psi(\mu_1) < \frac{\varepsilon}{8M}, \quad \forall x \in S \times D \text{ with } \|x - x_0\| < \mu_3. \end{aligned}$$

Setting $\mu_4 = \min\{\mu_1, \mu_3\}$. Using (2.2.2), (H_7), (H_8) and Lemma 2.2.1, we obtain the following chain of inequalities:

$$\begin{aligned}
|Th(x)-Th(x_0)| &\le \sup_{y\in D}|p(x,y)|\Big(\sup_{y\in D}|f(x,y)-f(x_0,y)| \\
&\quad+\sup_{y\in D}|G(x,y,h(a(x,y)))-G(x_0,y,h(a(x_0,y)))|\Big) \\
&\quad+\sup_{y\in D}|p(x,y)-p(x_0,y)|\Big(\sup_{y\in D}|f(x,y)|+\sup_{y\in D}|G(x,y,h(a(x,y)))|\Big) \\
&\quad+\sup_{y\in D}|q(x,y)|\Big(\sup_{y\in D}|g(x,y)-g(x_0,y)| \\
&\quad+\sup_{y\in D}|H(x,y,h(b(x,y)))-H(x_0,y,h(b(x_0,y)))|\Big) \\
&\quad+\sup_{y\in D}|q(x,y)-q(x_0,y)|\Big(\sup_{y\in D}|g(x,y)|+\sup_{y\in D}|H(x,y,h(b(x,y)))|\Big) \\
&\le \frac{3\varepsilon}{4}+M\sup_{y\in D}\{|G(x,y,h(a(x,y)))-G(x,y,h(a(x_0,y)))| \\
&\quad+|G(x,y,h(a(x_0,y)))-G(x_0,y,h(a(x_0,y)))|\} \\
&\quad+M\sup_{y\in D}\{|H(x,y,h(b(x,y)))-H(x,y,h(b(x_0,y)))| \\
&\quad+|H(x,y,h(b(x_0,y)))-H(x_0,y,h(b(x_0,y)))|\} \\
&\le \frac{3\varepsilon}{4}+M\sup_{y\in D}\left\{\varphi(\|x-x_0\|)+2^{-1}\psi(|h(a(x,y))-h(a(x_0,y))|)\right\} \\
&\quad+M\sup_{y\in D}\left\{\varphi(\|x-x_0\|)+2^{-1}\psi(|h(b(x,y))-h(b(x_0,y))|)\right\} \\
&\le \frac{15\varepsilon}{16}<\varepsilon,\quad \forall x\in S \text{ with } \|x-x_0\|<\mu_4.
\end{aligned}$$

The above inequality implies that Th is continuous at an arbitrary x_0. Hence $Th\in BC(S)$. Given $\varepsilon>0$, $x\in S$ and $h,k\in BC(S)$. In light of (H_5), (H_8) and Lemma 2.2.1, we have

$$\begin{aligned}
|Th(x)-Tk(x)| &\le \sup_{y\in D}|p(x,y)|\sup_{y\in D}|G(x,y,h(a(x,y)))-G(x,y,k(a(x,y)))| \\
&\quad+\sup_{y\in D}|q(x,y)|\sup_{y\in D}|H(x,y,h(b(x,y)))-H(x,y,k(b(x,y)))| \\
&\le \sup_{(x,y)\in S\times D}\{|p(x,y)|+|q(x,y)|\}\left(2^{-1}\psi(|h(a(x,y))-k(a(x,y))|)\right. \\
&\quad\left.+2^{-1}\psi(|h(b(x,y))-k(b(x,y))|)\right) \\
&\le \psi(\|h-k\|_1),
\end{aligned}$$

which implies

$$\|Th-Tk\|_1\le\psi(\|h-k\|_1).$$

Consequently, all the assumptions of Theorem 1.1.3 are satisfied. Hence, T has unique fixed point $\omega\in BC(S)$, which is a unique solution of the functional equation (2.2.1) and the sequence $\{T^n h\}$ converges to ω for each $h\in BC(S)$. This concludes the proof. ■

Corollary 2.2.1 *Let $\psi \in \Delta_6$ and S be compact, $a, b : S \times D \to S$, $f, g, p, q : S \times D \to \mathbb{R}$ and $G, H : S \times D \times \mathbb{R} \to \mathbb{R}$ be mappings such that (H_5) and (H_8) are satisfied*

Then, the functional equation (2.2.1) *possesses a unique solution $\omega \in B(S)$ and the sequence $\{T^n h\}$ converges to ω for each $h \in B(S)$, where T given by*

$$\begin{aligned} Th(x) &= opt_{y\in D} \{p(x,y)[f(x,y) + G(x,y,h(a(x,y)))]\} \\ &+ opt_{y\in D} \{q(x,y)[g(x,y) + H(x,y,h(b(x,y)))]\}, \quad x \in S. \end{aligned}$$

Remark 2.2.1 *If $p(x,y) = p_0$ and $q(x,y) = q_0$ with $p_0 + q_0 = 1$ in Theorem 2.2.2 and Corollary 2.2.1, we recover Theorem 3.2 and Theorem 3.3 in [41] and its all consequences.*

Theorem 2.2.3 *Let $\varphi \in \Delta_7$, $a, b : S \times D \to S$, $f, g, p, q : S \times D \to \mathbb{R}$ and $G, H : S \times D \times \mathbb{R} \to \mathbb{R}$ be mappings such that*

(H_9) for each $(k,h) \in \mathbb{N} \times BB(S)$, there exists $p(k,h) > 0$ such that

$$\begin{aligned} & \max\{|f(x,y)| + |G(x,y,u(a(x,y)))|, \\ & |g(x,y)| + |H(x,y,u(b(x,y))), \delta_{d_k}(O_T(h))|\} \\ \le\ & p(k,h), \quad \forall (x,y) \in \overline{B}(0,k) \times D; \end{aligned}$$

(H_{10})

$$\begin{aligned} & \sup_{y\in D} \max\{|G(x,y,l_1(a(x,y))) - G(x,y,l_2(a(x,y)))|, \\ & |H(x,y,l_1(b(x,y))) - H(x,y,l_2(b(x,y)))|\} \\ \le\ & \varphi(\delta_{d_k}(O_T(l_1,l_2))), \quad \forall (k,x,l_1,l_1) \in \mathbb{N} \times \overline{B}(0,k) \times (BB(S))^2, \end{aligned}$$

where

$$\begin{aligned} Th(x) &= opt_{y\in D} \{p(x,y)[f(x,y) + G(x,y,h(a(x,y)))]\} \\ &+ opt_{y\in D} \{q(x,y)[g(x,y) + H(x,y,h(b(x,y)))]\}, \quad \forall (k,x,h) \in \mathbb{N} \times \overline{B}(0,k) \times (BB(S)). \end{aligned}$$

Assume that $(H_1) - (i)$ holds.Then, the functional equation (2.2.1) *possesses a unique solution $\omega \in BB(S)$ and the sequence $\{T^n h\}$ converges to ω for each $h \in BB(S)$.*

Proof. Through Lemma (2.2.1), assumptions (H_9) and $(H_1) - (i)$, we get

$$\begin{aligned} |Th(x)| &\le \sup_{y\in D} |p(x,y)| \sup_{y\in D} \{|f(x,y)| + |G(x,y,h(a(x,y)))|\} \\ &+ \sup_{y\in D} |q(x,y)| \sup_{y\in D} \{|g(x,y)| + |H(x,y,h(b(x,y)))|\} \\ &\le \lambda_1 p(k,h) + \lambda_2 p(k,h) = p(k,h) \end{aligned}$$

for all $(k,x,h) \in \mathbb{N} \times \overline{B}(0,k) \times BB(S)$. Hence, T maps $BB(S)$ into itself.

In view of (H_{10}) and again Lemma (2.2.1), we have

$$\begin{aligned}
|Tl_1(x) - Tl_1(x)| &\leq \sup_{y\in D}|p(x,y)|\sup_{y\in D}|G(x,y,l_1(a(x,y))) - G(x,y,l_1(a(x,y)))| \\
&\quad + \sup_{y\in D}|q(x,y)|\sup_{y\in D}|H(x,y,l_1(b(x,y))) - H(x,y,l_1(b(x,y)))| \\
&\leq \lambda_1\varphi(\delta_{d_k}(O_T(l_1,l_2))) + \lambda_2\varphi(\delta_{d_k}(O_T(l_1,l_2))) \\
&= \varphi(\delta_{d_k}(O_T(l_1,l_2)))
\end{aligned}$$

for all $(k,x,l_1,l_1) \in \mathbb{N}\times\overline{B}(0,k)\times(BB(S))^2$.

It follows that

$$d_k(Tl_1,Tl_2) \leq \varphi(\delta_{d_k}(O_T(l_1,l_2))), \quad (k,l_1,l_1) \in \mathbb{N}\times(BB(S))^2.$$

Consequently, all the assumptions of Theorem 1.1.2 are satisfied. Hence, T has unique fixed point $\omega \in BB(S)$, which is a unique solution of the functional equation and the sequence $\{T^n h\}$ converges to ω for each $h \in BB(S)$. This concludes the proof. ■

Remark 2.2.2 *In the case $p(x,y) = 1$ and $q(x,y) = 1$, in Theorem 2.2.3, we recover Theorem 3.1 in [39].*

Remark 2.2.3 *Through Examples 3.2, 3.2, 3.3 and 3.4 in [41], we can easily construct suitable examples to provide the usability of our obtained results (for example, taking $p(x,y) = f(x,y)$ and $q(x,y) = g(x,y)$.)*

Chapter 3

An application in dynamic programming via F-contractions of Hardy-Rogers type

Abstract

The purpose of this chapter is to introduce the notions of extended F-contraction of Hardy-Rogers-type and generalized F-weak contraction of Hardy-Rogers-type and to establish some new fixed point results for such kind of mappings in the setting of complete b-metric spaces. These fixed point results improve (and/or) extend those obtained in Refs. Vetro [Nonlinear Analysis: Modelling and Control **21(4)** (2016), 531–546] and Lukács and Kajántó [Fixed Point Theory **19(1)** (2018), 321–334] since some conditions made therein are removed or weakened. In addition, some illustrative examples are provided to show the usability of the obtained results. As an application of our results, we obtain the existence and uniqueness of solutions for certain functional arising in dynamic programming.

NB: This chapter is an extracted part from the following paper:
" D. Derouiche, H. Ramoul, *New fixed point results for F-contractions of Hardy–Rogers type in b-metric spaces with applications.* J. Fixed Point Theory Appl. **22**(86) (2020)."

3.1 F-contractions of Hardy-Rogers-type

We present here various types of F-contractions of Hardy-Rogers and some related works which will be needed for stating our results in the sequel.

In 2014, Wardowski and Dung [84] proved the following result.

Theorem 3.1.1 ***(See** [84, Corollary 2.5]**)*** *Let (X, d) be a complete metric space. Assume that there exist $\tau > 0$ and $F \in \mathcal{F}$ such that $T : X \to X$ satisfies*

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F\left(B_T^d(x, y)\right),$$

where

$$B_T^d(x, y) = ad(x, y) + bd(x, Tx) + cd(y, Ty) + e[d(x, Ty) + d(y, Tx)]$$

with $a, b, c, e \geq 0$ and $a + b + c + 2e < 1$. If T or F is continuous, then

(1) T has a unique fixed point $x^* \in X$.

(2) For all $x \in X$, the sequence $\{T^n x\}$ is convergent to x^*.

Afterwards, Cosentino and Vetro [18] introduced a new notion, namely, the notion of F-contraction of Hardy-Rogers-type given below.

Definition 3.1.1 ***(See** [18]**)*** *Let (X, d) be a metric space. A self-mapping T on X is called an F-contraction of Hardy-Rogers-type if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$,*

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F\left(Q_T^d(x, y)\right),$$

where

$$Q_T^d(x, y) = \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx)$$

with $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$.

The authors in [18] obtained the following fixed point result.

Theorem 3.1.2 ***(See** [18, Theorem 3.1]**)*** *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction of Hardy-Rogers-type. Then T has a fixed point. Moreover, if $\alpha + \delta + L \leq 1$, then the fixed point of T is unique.*

Later, Vetro [79] proved some new results about F-contraction of Hardy-Rogers-type. Before enunciating these results, we need to introduce some notations and definitions. Let us note $\mathbb{S}$ the family of all functions $\tau : (0, \infty) \to (0, \infty)$ satisfying the following property:

$$\liminf_{t \to \eta^+} \tau(t) > 0, \quad \text{for all } \eta \geq 0.$$

Let us consider also the following condition:

$(F_3') :$ F is continuous on $(0, \infty)$.

Henceforth, we denote by $\mathfrak{F}$ the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the conditions (F_1), (F_2) and (F_3') and by $\mathbb{F}$ the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the conditions (F_1) and (F_2).

Vetro [79] generalized the notion of F-contraction of Hardy-Rogers-type as follows:

Definition 3.1.2 ***(See** [79, Definition 3]**)*** *Let (X, d) be a complete metric space. A self-mapping T on X is called an F-contraction of Hardy-Rogers type if there exist $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that for all $x, y \in X$,*

$$d(Tx, Ty) > 0 \Rightarrow \tau(d(x, y)) + F(d(Tx, Ty)) \leq F\left(Q_T^d(x, y)\right),$$

where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$, $\gamma \neq 1$ and $\alpha + \delta + L \leq 1$.

The first Vetro's result is the following:

Theorem 3.1.3 ***(See** [79, Theorem 1]**)*** *Let (X, d) be a complete metric space. If T is an F-contraction of Hardy-Rogers type and F is continuous (i.e., $F \in \mathfrak{F}$), then T has a unique fixed point.*

Remark 3.1.1 *In [79, Remark 3], Vetro proved that if (F_3') is weaken to the condition that F is upper semicontinuous on $(0, \infty)$, then Theorem 3.1.3 holds for the strict inequality $\alpha+\delta+L < 1$.*

Vetro established also the following corollaries.

Corollary 3.1.1 ***(See** [79, Corollary 1]**)*** *Let (X, d) be a complete metric space and let T be a self-mapping on X. Assume that there exist an upper semicontinuous $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\tau(d(x, y)) + F(d(Tx, Ty)) \leq F(\beta d(x, Tx) + \gamma d(y, Ty)),$$

where $\beta, \gamma \in [0, \infty)$ satisfying $\beta + \gamma = 1$ and $\gamma \neq 1$. Then T has a unique fixed point in X.

Corollary 3.1.2 ***(See** [79, Corollary 3]**)*** *Let (X, d) be a complete metric space and let T be a self-mapping on X. Assume that there exist a continuous $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\tau(d(x, y)) + F(d(Tx, Ty)) \leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty)),$$

where $\alpha, \beta, \gamma \in [0, \infty)$ satisfying $\alpha + \beta + \gamma = 1$ and $\gamma \neq 1$. Then T has a unique fixed point in X.

Consistent with [17] and [55], what follows is needed to deal with more results concerning F-contraction of Hardy-Rogers-type.

In 2015, Cosentino et al. [17] introduced the following condition (noted (F_4) in [17, Definition 3.1]):

Let $s \geq 1$. If $\{\alpha_n\} \subset (0, \infty)$ is a sequence such that $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$, for all $n \in \mathbb{N}$ and some $\tau > 0$, then $\tau + F(s^n \alpha_n) \leq F\left(s^{n-1}\alpha_{n-1}\right)$, for all $n \in \mathbb{N}$.

In the same context, Lukács and Kajántó [55] defined a new class of functions (noted $\mathcal{F}_{s,\tau}$) satisfying an easier condition than (F_4). Their definition is given below.

Definition 3.1.3 ***(See** [55, Definition 2.7]**)*** *Let $s \geq 1$ and $\tau > 0$. We say that $F \in \mathbb{F}^*$ belongs to $\mathcal{F}_{s,\tau}$ if it is also satisfies*

$(F_{s,\tau})$ if $\inf F = -\infty$ *and $x, y, z \in (0, \infty)$ are such that $\tau + F(sx) \leq F(y)$ and $\tau + F(sy) \leq F(z)$ then*

$$\tau + F\left(s^2 x\right) \leq F(sy),$$

where $\mathbb{F}^*$ is the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the conditions (F_1) and (F_3).

Next, the authors in [55] introduced the notion of F-weak contraction of Hardy-Rogers-type in the setting of b-metric spaces as follows:

Definition 3.1.4 ***(See** [55, Definition 5.1]**)*** *Let (X, σ) be a b-metric space with constant $s \geq 1$, $a, b, c, e, f \geq 0$ real numbers and $T : X \to X$ an operator. If there exist $\tau > 0$ and $F \in \mathcal{F}_{s,\tau}$ such that for all $x, y \in X$ the inequality $\sigma(Tx, Ty) > 0$ implies*

$$\tau + F\left(s\sigma\left(Tx, Ty\right)\right) \leq F\left(A_T^{\sigma}\left(x, y\right)\right),$$

where

$$A_T^{\sigma}\left(x, y\right) = a\sigma\left(x, y\right) + b\sigma\left(x, Tx\right) + c\sigma\left(y, Ty\right) + e\sigma\left(x, Ty\right) + f\sigma\left(y, Tx\right),$$

then T is called an F-weak contraction of Hardy-Rogers-type.

In [55], Lukács and Kajántó showed that if F is an increasing function, then $(\mathrm{F}_{s,\tau})$ is equivalent to (F_4) (see [55, Proposition 2.8]) and proved the fixed point result below.

Theorem 3.1.4 ***(See** [55, Theorem 5.2]**)*** *Suppose that (X, σ) is a b-metric space with constant $s \geq 1$ and $T : X \to X$ is an F-weak contraction of Hardy-Rogers-type. If either $a + b + c + (s + 1)\, e < 1$ or $a + b + c + (s + 1)\, f < 1$ holds, then every $x_0 \in X$, the sequence $x_{n+1} = Tx_n$ converges to a fixed point of T. Moreover, if $a + e + f < s$ holds as well, then T has exactly one fixed point.*

3.2 Main results

In this section, we essentially improve (and/or) extend the aforementioned results: Theorem 3.1.3, Remark 3.1.1 and Theorem 3.1.4 in the setting of b-metric spaces. It is worth mentioning that in our following results, the b-metric need not to be continuous.

For convenience, we set

$$\mathcal{F}_c = \left\{F : (0, \infty) \to \mathbb{R} : \; F \text{ is nondecreasing continuous function}\right\}.$$

We denote by $\mathcal{S}_1$ the family of all functions $\tau : (0, \infty) \to (0, \infty)$ which satisfy the following condition:

$$\liminf_{t \to \eta^+} \tau\left(t\right) > 0, \quad \text{for all } \eta > 0. \tag{A_1}$$

Example 3.2.1 *Let the following function $G : (0, \infty) \to \mathbb{R}$, $G\left(x\right) = \dfrac{-1}{x + 1}$. Clearly, $G \in \mathcal{F}_c$ but it does not satisfy condition (F_2). Indeed, if $\alpha_n = \dfrac{1}{n}, n \in \mathbb{N}$, we have $\lim_{n \to \infty} \alpha_n = 0$ and $\lim_{n \to \infty} G\left(\alpha_n\right) = -1 \neq -\infty$. More precisely, we have $\mathfrak{F} \subset \mathcal{F}_c$.*

Example 3.2.2 ***(See** [73, Example 2.2]**)***

(a) *Let $\tau > 0$ be a fixed real number and $\tau_1\left(t\right) = \tau$ for all $t \in (0, \infty)$. Then $\tau_1 \in \mathcal{S}_1$.*

(b) *Let $\tau_2\left(t\right) = \ln\left(1 + t\right)$ for all $t \in (0, \infty)$. Then $\tau_2 \in \mathcal{S}_1$.*

(c) Let $\tau_3(t) = \varrho t$ for all $t \in (0,\infty)$, where $\varrho > 0$. Then $\tau_3 \in \mathcal{S}_1$.

Remark 3.2.1 *Since $\tau_2 \notin \mathbb{S}$, it is easy to see that $\mathbb{S} \subset \mathcal{S}_1$.*

Motivated by the works in [55] and [79], we refine the notions of F-contraction of Hardy-Rogers-type and F-weak contraction of Hardy-Rogers-type by introducing new notions in the context of b-metric spaces, namely, the notions of extended F-contraction of Hardy-Rogers-type, generalized F-weak contraction of Hardy-Rogers-type.

Before stating and proving our main results, we start to prove the following useful lemma (see also the works in [62] and [72]).

Lemma 3.2.1 *Let $\kappa \geq 1$ be a given real number. Let $\{t_n\} \subset (0,\infty)$ be a sequence and let $\phi, \psi : (0,\infty) \to \mathbb{R}$ be two functions satisfying the following conditions:*

(i) $\psi(\kappa t_n) \leq \phi(t_{n-1})$, for all $n \in \mathbb{N}$;

(ii) ψ is nondecreasing;

(iii) $\phi(t) < \psi(t)$, for all $t > 0$;

(iv) $\limsup_{t\to\eta^+} \phi(t) < \psi(\eta^+)$, for all $\eta > 0$.

Then $\lim_{n\to\infty} t_n = 0$.

Proof. First, we note that the right limit of ψ exists since ψ is nondecreasing. Through (i) and (iii), we have

$$\psi(\kappa t_n) \leq \phi(t_{n-1}) < \psi(t_{n-1}), \quad \text{for all } n \in \mathbb{N}.$$

Taking into account condition (ii), it follows that

$$\kappa t_n < t_{n-1}, \quad \text{for all } n \in \mathbb{N}.$$

As $\kappa \geq 1$, the last inequality implies that $\{t_n\}$ is a strictly decreasing sequence of positive numbers. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} t_n = r^+.$$

Now, we show that $r = 0$. Arguing by contradiction, we assume that $r > 0$. Again by (i) and (ii), we have

$$\psi(t_n) \leq \phi(t_{n-1}), \quad \text{for all } n \in \mathbb{N}. \tag{3.2.1}$$

Taking the upper limit as $n \to \infty$ in (3.2.1), we get

$$\psi(r^+) = \lim_{n\to\infty} \psi(t_n) \leq \limsup_{n\to\infty} \phi(t_{n-1}) \leq \limsup_{t\to r^+} \phi(t),$$

which contradicts (iv). Thus, $r = 0$, that is, $\lim_{n\to\infty} t_n = 0$. ■

We prove now the following proposition which plays an important role in the proofs of our results.

Proposition 3.2.1 *Let (X,σ) be a b-metric space with constant $s\geq 1$ and let λ be a given real number such that $1\leq\lambda\leq s$. Let $T:X\to X$ be a mapping and $\{x_n\}$ the Picard sequence of T based on an arbitrary $x_0\in X$. Assume that there exist a nondecreasing function F and $\tau\in\mathcal{S}_1$ such that for all $z\in X$ with $Tz\neq T^2z$,*

$$\begin{aligned}&\tau\left(\sigma\left(z,Tz\right)\right)+F\left(\lambda\sigma\left(Tz,T^2z\right)\right)\\&\leq F((d_1+d_2)\,\sigma\left(z,Tz\right)+d_3\sigma\left(Tz,T^2z\right)+d_4\sigma\left(z,T^2z\right)),\end{aligned}\tag{P}$$

where d_1,d_2,d_3,d_4 are nonnegative real numbers satisfying

$$d_1+d_2+d_3+2d_4s=\frac{\lambda}{s}\ \textit{and}\ d_3\neq\frac{\lambda}{s}.\tag{D}$$

Then $\lim\limits_{n\to\infty}\sigma\left(x_n,x_{n+1}\right)=0.$

Proof. Let us put $\sigma_n:=\sigma\left(x_n,x_{n+1}\right)$. If $x_n=x_{n+1}$ for some $n\in\mathbb{N}_0$, the proof is immediately finished. Hence we assume that

$$x_n\neq x_{n+1},\quad\text{for all } n\in\mathbb{N}_0.$$

This means that $Tx_{n-1}\neq T^2x_{n-1}$ for all $n\in\mathbb{N}$. Applying the inequality (P) with $z=x_{n-1}$, we have for all $n\in\mathbb{N}$

$$\begin{aligned}&\tau\left(\sigma_{n-1}\right)+F\left(\lambda\sigma_n\right)\\&\leq F\left(\left(d_1+d_2\right)\sigma_{n-1}+d_3\sigma_n+d_4\sigma\left(x_{n-1},x_{n+1}\right)\right).\end{aligned}\tag{3.2.2}$$

Using the relaxed triangle inequality (b_3), we get

$$\sigma\left(x_{n-1},x_{n+1}\right)\leq s\left(\sigma_{n-1}+\sigma_n\right),\quad\text{for all } n\in\mathbb{N}.$$

So, (3.2.2) turns into

$$\begin{aligned}&\tau\left(\sigma_{n-1}\right)+F\left(\lambda\sigma_n\right)\\&\leq F\left(\left(d_1+d_2+d_4s\right)\sigma_{n-1}+\left(d_3+d_4s\right)\sigma_n\right),\quad\text{for all } n\in\mathbb{N}.\end{aligned}\tag{3.2.3}$$

Since F is nondecreasing and $\tau\left(t\right)>0,\forall t>0$, it follows that

$$\lambda\sigma_n<\left(d_1+d_2+d_4s\right)\sigma_{n-1}+\left(d_3+d_4s\right)\sigma_n,\quad\text{for all } n\in\mathbb{N}.$$

This implies that

$$\left(\lambda-d_3-d_4s\right)\sigma_n<\left(d_1+d_2+d_4s\right)\sigma_{n-1},\quad\text{for all } n\in\mathbb{N}.\tag{3.2.4}$$

Since

$$\lambda-d_3-d_4s\geq\frac{\lambda}{s}-d_3-d_4s,$$

inequality (3.2.4) gives

$$\left(\frac{\lambda}{s}-d_3-d_4s\right)\sigma_n<\left(d_1+d_2+d_4s\right)\sigma_{n-1},\quad\text{for all } n\in\mathbb{N}.\tag{3.2.5}$$

Since $d_1 + d_2 + d_3 + 2d_4 s = \frac{\lambda}{s}$ and $d_3 \neq \frac{\lambda}{s}$, we get

$$\frac{\lambda}{s} - d_3 - d_4 s > 0.$$

Therefore, inequality (3.2.5) yields

$$\sigma_n < \frac{d_1 + d_2 + d_4 s}{\frac{\lambda}{s} - d_3 - d_4 s} \sigma_{n-1} = \sigma_{n-1}, \quad \text{for all } n \in \mathbb{N}. \tag{3.2.6}$$

As F is nondecreasing, then by substituting (3.2.6) into (3.2.3) and using again $d_1 + d_2 + d_3 + 2d_4 s = \frac{\lambda}{s}$ with $1 \leq \lambda \leq s$, we obtain

$$\begin{aligned} F(\lambda \sigma_n) &\leq F\left((d_1 + d_2 + d_4 s)\sigma_{n-1} + (d_3 + d_4 s)\sigma_{n-1}\right) - \tau(\sigma_{n-1}) \\ &= F\left(\frac{\lambda}{s}\sigma_{n-1}\right) - \tau(\sigma_{n-1}) \\ &\leq F(\sigma_{n-1}) - \tau(\sigma_{n-1}). \end{aligned} \tag{3.2.7}$$

This leads to

$$F(\lambda \sigma_n) \leq F(\sigma_{n-1}) - \tau(\sigma_{n-1}), \quad \text{for all } n \in \mathbb{N}. \tag{3.2.8}$$

Taking $\psi(t) = F(t)$ and $\phi(t) = F(t) - \tau(t)$ for all $t \in (0, \infty)$, the inequality (3.2.8) can be written in the following form:

$$\psi(\lambda \sigma_n) \leq \phi(\sigma_{n-1}), \quad \text{for all } n \in \mathbb{N}.$$

As F is nondecreasing, then in view of the last inequality and using the fact that $\tau \in \mathcal{S}_1$ (i.e., (A_1) holds), it is easy to see that all the conditions of Lemma 3.2.1 with ($\kappa = \lambda \geq 1$) are satisfied. Thus, $\lim_{n\to\infty} \sigma_n = 0$ and the proof is finished. ∎

Remark 3.2.2 *As in [79, Proposition 1, inequality (6)], Proposition 3.2.1 also furnishes that the sequence $\{\sigma_n\}$ is a strictly decreasing (see inequality (3.2.6)) when $\sigma_n > 0$, for all $n \in \mathbb{N}_0$.*

Remark 3.2.3 *Proposition 3.2.1 extends and improves [79, Proposition 1]. In fact, taking $s = 1$ (which yields $\lambda = 1$ as well) in Proposition 3.2.1 (it corresponds to the case of metric spaces), we find [79, Proposition 1]. Moreover, condition (F_2) from [79, Proposition 1] is omitted. Otherwise, for the function τ, we have used the condition that $\tau \in \mathcal{S}_1$ instead of the condition that $\tau \in \mathbb{S}$. This is a slightly weaker condition since $\mathbb{S} \subset \mathcal{S}_1$. In addition, we also change the condition that F is strictly increasing from [79, Proposition 1] into the weaker condition that F is nondecreasing (i.e., the strictness of the monotonicity of F is not necessary).*

3.2.1 Extended F-contraction of Hardy-Rogers-type

In this subsection and for the sake of readability, we present our results gradually in order to point out the different techniques used in some steps of our proofs in the case where the only omitted condition is (F_2) and in the case where we assume only the condition that F is nondecreasing.

Let (X, σ) be a b-metric space with constant $s \geq 1$. Throughout this subsection, we denote, for all $x, y \in X$,

$$Q_T^\sigma(x,y) = \alpha\sigma(x,y) + \beta\sigma(x,Tx) + \gamma\sigma(y,Ty) + \delta\sigma(x,Ty) + L\sigma(y,Tx),$$

where $\alpha, \beta, \gamma, \delta, L$ are nonnegative real numbers. If $s = 1$, we write $Q_T^d(x,y)$ instead of $Q_T^\sigma(x,y)$, where d is a metric on X.

We begin this subsection with the following definitions.

Definition 3.2.1 *Let (X, σ) be a b-metric space with constant $s \geq 1$. A mapping $T : X \to X$ is said to be an extended F-contraction of Hardy-Rogers-type if there exist $F : (0, \infty) \to \mathbb{R}$ and $\tau \in \mathcal{S}_1$ such that for all $x, y \in X$,*

$$\sigma(Tx,Ty) > 0 \Rightarrow \tau(\sigma(x,y)) + F(\sigma(Tx,Ty)) \leq F(Q_T^\sigma(x,y)). \tag{3.2.9}$$

Remark 3.2.4 *If F is nondecreasing, it is easy to see from Definition 3.2.1 that every T which is an extended F-contraction of Hardy-Rogers-type satisfies the following condition*

$$\sigma(Tx,Ty) < Q_T^\sigma(x,y), \tag{3.2.10}$$

for all $x, y \in X$ with $Tx \neq Ty$.

Now, we are ready to state and prove our main results. The following theorem is one of them and it is an extension and improvement of Theorem 3.1.3.

Theorem 3.2.1 *Let (X, σ) be a complete b-metric space with constant $s \geq 1$ and $T : X \to X$ an extended F-contraction of Hardy-Rogers-type with $F \in \mathcal{F}_c$. Suppose that either $(\mathcal{H}_s^1)$ or $(\mathcal{H}_s^2)$ holds, where*

$(\mathcal{H}_s^1)$ $\alpha + \beta + \gamma + 2\delta s = \dfrac{1}{s}$ *and* $\gamma \neq \dfrac{1}{s}$,

$(\mathcal{H}_s^2)$ $\alpha + \beta + \gamma + 2Ls = \dfrac{1}{s}$ *and* $\beta \neq \dfrac{1}{s}$.

Furthermore, we assume that $s^2\alpha + s^3(\delta + L) \leq 1$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.

Proof. First, we will show that T has at most one fixed point. Assume that x^* and y^* are two distinct fixed points of T, that is, $Tx^* = x^* \neq y^* = Ty^*$. Then

$$\sigma(Tx^*,Ty^*) = \sigma(x^*,y^*) > 0.$$

1. If $\alpha + \delta + L > 0$, from (3.2.9) (with $x = x^*$ and $y = y^*$), we obtain

$$\begin{aligned}\tau(\sigma(x^*,y^*)) + F(\sigma(x^*,y^*)) &\leq F((\alpha+\delta+L)\sigma(x^*,y^*)) \\ &\leq F((s^2\alpha + s^3(\delta+L))\sigma(x^*,y^*)) \\ &\leq F(\sigma(x^*,y^*)).\end{aligned}$$

The last inequality yields $\tau(\sigma(x^*,y^*)) \leq 0$, which is a contradiction.

2. If $\alpha+\delta+L=0$, from (3.2.10) (with $x=x^*$ and $y=y^*$), we have

$$\sigma\left(x^*,y^*\right)<Q_T^\sigma(x^*,y^*)=(\alpha+\delta+L)\,\sigma\left(x^*,y^*\right)=0,$$

which is a contradiction.

Thus, in both cases, we get a contradiction. Hence, T has at most one fixed point.

Next, we prove the existence of a fixed point. Let $\{x_n\}$ be the Picard sequence based on an arbitrary $x_0\in X$. If there exist $n_0\in\mathbb{N}_0$, such that $x_{n_0}=x_{n_0+1}$, then x_{n_0} is the fixed point of T and the proof is completed. If $x_n\neq x_{n+1}$, for all $n\in\mathbb{N}_0$, we have

$$\sigma_n:=\sigma\left(x_n,x_{n+1}\right)=\sigma\left(Tx_{n-1},Tx_n\right)>0,\quad\text{for all } n\in\mathbb{N}. \tag{3.2.11}$$

From the hypothesis of the theorem, we consider the following cases:

Case 1. If $\left(\mathcal{H}_s^1\right)$ holds. Owing to (3.2.11), we can apply the contractive condition (3.2.9) with $x=x_{n-1}$ and $y=x_n$. Hence, we get for all $n\in\mathbb{N}$

$$\begin{aligned}&\tau\left(\sigma\left(x_{n-1},x_n\right)\right)+F\left(\sigma\left(Tx_{n-1},Tx_n\right)\right)\\&\leq F((\alpha+\beta)\,\sigma\left(x_{n-1},x_n\right)+\gamma\sigma\left(x_n,Tx_n\right)+\delta\sigma\left(x_{n-1},Tx_n\right)).\end{aligned} \tag{3.2.12}$$

Putting $x_{n-1}=z$ in (3.2.12) and using the fact that

$$Tz=Tx_{n-1}=x_n\neq x_{n+1}=T^2x_{n-1}=T^2z,$$

the inequality (3.2.12) turns into (P) with $d_1=\alpha$, $d_2=\beta$, $d_3=\gamma$, $d_4=\delta$ and $\lambda=1$. Therefore, using the fact that $\tau\in\mathcal{S}_1$ and Proposition 3.2.1 with $\lambda=1$, we have $\lim\limits_{n\to\infty}\sigma_n=0$.

Case 2. If $\left(\mathcal{H}_s^2\right)$ holds. From (3.2.11), we can also apply (3.2.9) with $x=x_n$ and $y=x_{n-1}$. So, using the symmetry condition (b_2), we get for all $n\in\mathbb{N}$

$$\begin{aligned}&\tau\left(\sigma\left(x_{n-1},x_n\right)\right)+F\left(\sigma\left(Tx_{n-1},Tx_n\right)\right)\\&\leq F((\alpha+\gamma)\,\sigma\left(x_{n-1},x_n\right)+\beta\sigma\left(x_n,Tx_n\right)+L\sigma\left(x_{n-1},Tx_n\right)).\end{aligned} \tag{3.2.13}$$

Similarly, as in *Case 1*, the inequality (3.2.13) turns into (P) with $d_1=\alpha$, $d_2=\gamma$, $d_3=\beta$, $d_4=L$ and $\lambda=1$. Again, according to Proposition 3.2.1 with $\lambda=1$, we have $\lim\limits_{n\to\infty}\sigma_n=0$.

Consequently, in both cases, we obtain

$$\lim_{n\to\infty}\sigma_n=\lim_{n\to\infty}\sigma\left(x_n,x_{n+1}\right)=0. \tag{3.2.14}$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary, i.e., $\{x_n\}$ is not a Cauchy sequence. Then, from (3.2.14) and the first item of Lemma 1.1.4, there exist $\varepsilon>0$ and two sequences $\{m(k)\},\{n(k)\}$ of positive integers such that

$$\varepsilon^+\leq\liminf_{k\to\infty}\sigma\left(x_{m(k)},x_{n(k)}\right)\leq\limsup_{k\to\infty}\sigma\left(x_{m(k)},x_{n(k)}\right)\leq s\varepsilon^+.$$

Thus, we infer that there exists $k_0\in\mathbb{N}$ such that $\left\{\sigma\left(x_{m(k)},x_{n(k)}\right)\right\}$ is bounded for all $k\geq k_0$ and thereby it has a convergent subsequence. It follows that there exist a real number l and a subsequence $\{k\left(p\right)\}_{p\geq k_0}$ of $\{k\}_{k\geq k_0}$ such that

$$\lim_{p\to\infty}\sigma\left(x_{m(k(p))},x_{n(k(p))}\right)=l \tag{3.2.15}$$

with

$$0 < \varepsilon^{+} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq l \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon^{+}. \tag{3.2.16}$$

On the other hand, using (b_3), we get for all $p \geq k_0$

$$\begin{aligned}
&\sigma\left(x_{m(k(p))}, x_{n(k(p))}\right)\\
&\leq s\sigma\left(x_{m(k(p))}, x_{m(k(p))+1}\right) + s\sigma\left(x_{m(k(p))+1}, x_{n(k(p))}\right)\\
&\leq s\sigma\left(x_{m(k(p))}, x_{m(k(p))+1}\right) + s^2\sigma\left(x_{m(k(p))+1}, x_{n(k(p))+1}\right)\\
&+s^2\sigma\left(x_{n(k(p))}, x_{n(k(p))+1}\right)\\
&= s\sigma_{m(k(p))} + s^2\sigma\left(x_{m(k(p))+1}, x_{n(k(p))+1}\right) + s^2\sigma_{n(k(p))}.
\end{aligned}$$

This leads to

$$\begin{aligned}
&\sigma\left(x_{m(k(p))+1}, x_{n(k(p))+1}\right)\\
&\geq \frac{1}{s^2}\left(\sigma\left(x_{m(k(p))}, x_{n(k(p))}\right) - s\sigma_{m(k(p))} - s^2\sigma_{n(k(p))}\right),
\end{aligned} \tag{3.2.17}$$

for all $p \geq k_0$.

Taking the lower limit as $p \to \infty$ in (3.2.17) and using (3.2.14), we obtain

$$\liminf_{p\to\infty} \sigma\left(x_{m(k(p))+1}, x_{n(k(p))+1}\right) \geq \frac{l}{s^2}. \tag{3.2.18}$$

Consequently, there exists $N \geq k_0$ such that

$$\sigma\left(Tx_{m(k(p))}, Tx_{n(k(p))}\right) = \sigma\left(x_{m(k(p))+1}, x_{n(k(p))+1}\right) > 0, \quad \text{for all } p \geq N. \tag{3.2.19}$$

For convenience, we set

$$\begin{aligned}
a_p &= \sigma\left(x_{m(k(p))}, x_{n(k(p))}\right), & b_p &= \sigma\left(x_{m(k(p))+1}, x_{n(k(p))+1}\right),\\
c_p &= \sigma\left(x_{m(k(p))}, x_{n(k(p))+1}\right), & d_p &= \sigma\left(x_{n(k(p))}, x_{m(k(p))+1}\right).
\end{aligned}$$

Therefore, it follows from (3.2.19) that the contractive inequality (3.2.9) can be applied with $x = x_{m(k(p))}$ and $y = x_{n(k(p))}$. Hence, for all $p \geq N$, we have

$$\tau\left(a_p\right) + F\left(b_p\right) \leq F\left(\alpha a_p + \beta\sigma_{m(k(p))} + \gamma\sigma_{n(k(p))} + \delta c_p + L d_p\right).$$

Using (b_3), the monotonicity of F and $s^2\alpha + s^3(\delta + L) \leq 1$, we get

$$\begin{aligned}
&\tau\left(a_p\right) + F\left(b_p\right)\\
&\leq F\left(\left(\alpha + s\left(\delta + L\right)\right)a_p + \left(\beta + sL\right)\sigma_{m(k(p))} + \left(\gamma + s\delta\right)\sigma_{n(k(p))}\right)\\
&\leq F\left(\frac{1}{s^2}a_p + \left(\beta + sL\right)\sigma_{m(k(p))} + \left(\gamma + s\delta\right)\sigma_{n(k(p))}\right),
\end{aligned}$$

for all $p \geq N$.

Now, combining the above inequality with (3.2.15) and (3.2.18) through the fact that $F \in \mathcal{F}_c$, we obtain the following chain of inequalities

$$\begin{aligned}
&\liminf_{t\to l} \tau(t) + F\left(\frac{l}{s^2}\right) \\
&\leq \liminf_{p\to\infty} \tau(a_p) + F\left(\frac{l}{s^2}\right) \\
&\leq \liminf_{p\to\infty} \tau(a_p) + F\left(\liminf_{p\to\infty} b_p\right) \\
&= \liminf_{p\to\infty} \tau(a_p) + \liminf_{p\to\infty} F(b_p) = \liminf_{p\to\infty} [\tau(a_p) + F(b_p)] \\
&\leq \lim_{p\to\infty} F\left(\frac{1}{s^2} a_p + (\beta + sL)\,\sigma_{m(k(p))} + (\gamma + s\delta)\,\sigma_{n(k(p))}\right) \\
&= F\left(\frac{l}{s^2}\right).
\end{aligned}$$

Having in mind (3.2.16), we obtain a contradiction. This contradiction shows that $\{x_n\}$ is a Cauchy sequence. By completeness of (X, σ), $\{x_n\}$ converges to some point x^* in X, that is,

$$\lim_{n\to\infty} \sigma(x_n, x^*) = 0. \tag{3.2.20}$$

Finally, we show that x^* is a fixed point of T, that is, $Tx^* = x^*$. Assume on the contrary, i.e., $\sigma(x^*, Tx^*) > 0$. Then, through (3.2.20), there exists $n_0 \in \mathbb{N}$ such that

$$\sigma(x_n, x^*) \leq \frac{\sigma(x^*, Tx^*)}{2s}, \quad \forall n \geq n_0. \tag{3.2.21}$$

On the other hand, by (b_3), we have

$$\sigma(x^*, Tx^*) \leq s\sigma(x^*, Tx_n) + s\sigma(Tx_n, Tx^*). \tag{3.2.22}$$

Using (3.2.21), the inequality (3.2.22) yields

$$\begin{aligned}
\sigma(Tx_n, Tx^*) &\geq \frac{1}{s}\left(\sigma(x^*, Tx^*) - s\sigma(x^*, Tx_n)\right) \\
&= \frac{1}{s}\sigma(x^*, Tx^*) - \sigma(x^*, x_{n+1}) \\
&\geq \frac{\sigma(x^*, Tx^*)}{2s} > 0,
\end{aligned} \tag{3.2.23}$$

for all $n \geq n_0$.

Taking into account (3.2.23), we can apply (3.2.10) with $x = x_n$ and $y = x^*$. Hence, for all $n \geq n_0$, (3.2.22) gives

$$\begin{aligned}\sigma(x^*, Tx^*) &\leq s\sigma(x^*, Tx_n) + s\sigma(Tx_n, Tx^*). \\ &< s\sigma(x^*, Tx_n) + s\alpha\sigma(x_n, x^*) + s\beta\sigma(x_n, Tx_n) \\ &+ s\gamma\sigma(x^*, Tx^*) + s\delta\sigma(x_n, Tx^*) + sL\sigma(x^*, Tx_n) \\ &= s(1+L)\sigma(x^*, x_{n+1}) + s\alpha\sigma(x_n, x^*) \\ &+ s\beta\sigma(x_n, Tx_n) + s\delta\sigma(x_n, Tx^*) + s\gamma\sigma(x^*, Tx^*).\end{aligned}$$

The above inequality leads to

$$\begin{aligned}(1 - s\gamma)\sigma(x^*, Tx^*) &< s(1+L)\sigma(x^*, x_{n+1}) + s\alpha\sigma(x_n, x^*) \\ &+ s\delta\sigma(x_n, Tx^*) + s\beta\sigma(x_n, Tx_n),\end{aligned} \tag{3.2.24}$$

for all $n \geq n_0$.

Taking the limit superior as $n \to \infty$ in (3.2.24) and using Lemma 1.1.2, (3.2.14) and (3.2.20), we get

$$(1 - s\gamma)\sigma(x^*, Tx^*) \leq s^2\delta\sigma(x^*, Tx^*). \tag{3.2.25}$$

In a similar way, we can also apply (3.2.10) with $x = x^*$, $y = x_n$ and we obtain

$$(1 - s\beta)\sigma(x^*, Tx^*) \leq s^2 L\sigma(x^*, Tx^*). \tag{3.2.26}$$

Again, according to the hypothesis of the theorem, we consider the following cases:

Case 1. If $(\mathcal{H}_s^1)$ holds. In this case, we have $1 - s\gamma > 0$ and $\gamma + s\delta < \frac{1}{s}$. Consequently, (3.2.25) implies that

$$\sigma(x^*, Tx^*) \leq \frac{s^2\delta}{(1 - s\gamma)}\sigma(x^*, Tx^*) < \sigma(x^*, Tx^*),$$

which is a contradiction.

Case 2. If $(\mathcal{H}_s^2)$ holds. In this case, we have $1 - s\beta > 0$ and $\beta + sL < \frac{1}{s}$. Hence, (3.2.26) yields

$$\sigma(x^*, Tx^*) \leq \frac{s^2 L}{(1 - s\beta)}\sigma(x^*, Tx^*) < \sigma(x^*, Tx^*),$$

which is a contradiction. Therefore, whether $(\mathcal{H}_s^1)$ or $(\mathcal{H}_s^2)$ holds, we obtain a contradiction. So, we have $Tx^* = x^*$ and this completes the proof of the theorem. ■

In the sequel, $(\mathcal{H}_s^1)$ and $(\mathcal{H}_s^2)$ denote the hypotheses given in Theorem 3.2.1. Also, if $s = 1$, $(\mathcal{H}_s^1)$ and $(\mathcal{H}_s^2)$ are noted $(\mathcal{H}_1^1)$ and $(\mathcal{H}_1^2)$, respectively.

Remark 3.2.5 *Theorem 3.2.1 extends and greatly improves Theorem 3.1.3. Actually, by taking $s = 1$ in Theorem 3.2.1 with the hypothesis $(\mathcal{H}_1^1)$, we recover Theorem 3.1.3. In addition, we show that Theorem 3.1.3 can be proved also through the hypothesis $(\mathcal{H}_1^2)$. Moreover, condition (F_2) from Theorem 3.1.3 is omitted and the condition that $\tau \in \mathbb{S}$ is weaken to the condition that $\tau \in \mathcal{S}_1$. Beside these, we have shown implicitly from the proof of Theorem 3.2.1 that the strictness of the monotonicity of F and $\liminf_{t \to 0^+} \tau(t) > 0$ are superfluous conditions for all $s \geq 1$.*

If $s = 1$, then by taking $\delta = L = 0$ in Theorem 3.2.1, we obtain the following result.

Corollary 3.2.1 *Let (X, d) be a complete metric space and let T be a self-mapping on X. Assume that there exist $F \in \mathcal{F}_c$ and $\tau \in \mathcal{S}_1$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\tau\left(d\left(x,y\right)\right) + F\left(d\left(Tx,Ty\right)\right) \leq F\left(\alpha d\left(x,y\right) + \beta d\left(x,Tx\right) + \gamma d\left(y,Ty\right)\right),$$

where $\alpha, \beta, \gamma \in [0, \infty)$ satisfying $\alpha + \beta + \gamma = 1$ and $\gamma \neq 1$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.

Remark 3.2.6 *Corollary 3.2.1 improves Corollary 3.1.2. Indeed, condition (F_2) from Theorem 3.1.3 is deleted and the condition that $\tau \in \mathbb{S}$ is weaken to the condition that $\tau \in \mathcal{S}_1$. In addition, Corollary 3.1.2 remains true without the strictness of the monotonicity of F.*

Remark 3.2.7 *Corollary 3.2.1 generalizes and improves [58, Theorem 2.1]. In fact, by taking $\alpha = 1, \beta = \gamma = 0$ in Corollary 3.2.1 and $\tau(t) = \tau > 0$ for all $t \in (0, \infty)$, we find Theorem 2.1 of Piri and Kumam [58]. Corollary 3.2.1 shows that condition (F_2') can be omitted from [58, Theorem 2.1]. Besides these, the strictness of the monotonicity of F is not necessary.*

Remark 3.2.8 *Using $F(t) = -\dfrac{1}{t+1}$ and $\tau(t) = \dfrac{t}{294}$, the trivial example (see the details in [22, Example 3.19]) $T : X \to X$ (where $X = [0, 7]$) given by*

$$Tx = \begin{cases} 7, & \text{if } x \in \left]0,7\right], \\ 6, & \text{if } x = 0, \end{cases}$$

shows that all the conditions of Theorem 3.2.1 (with $\left(\mathcal{H}_s^1\right)$) are satisfied. Notice that F does not satisfy condition (F_2) and $\tau \notin \mathbb{S}$.

In the following remarks, we omit the details.

Remark 3.2.9 *The trivial example [22, Example 3.20] shows that Theorem 3.2.1 greatly improves Theorem 3.1.3.*

Remark 3.2.10 *The trivial example [22, Example 3.21] shows that Theorem 3.2.1 greatly improves Corollary 3.1.2.*

Our second result extends and greatly improves the result stated in Remark 3.1.1. In the following theorem, we prove a fixed point result concerning an extended F-contraction of Hardy-Rogers-type in the setting of b-metric spaces without both conditions (F_2) and "F is upper semicontinuous".

Theorem 3.2.2 *Let (X, σ) be a complete b-metric space with constant $s \geq 1$ and $T : X \to X$ satisfying the contractive condition (3.2.9) with $F : (0, \infty) \to \mathbb{R}$ a nondecreasing function and $\tau \in \mathcal{S}_1$. Suppose that either $\left(\mathcal{H}_s^1\right)$ or $\left(\mathcal{H}_s^2\right)$ holds. Furthermore, we assume that $s^2\alpha + s^3(\delta + L) < 1$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.*

Proof. In this proof, for the sake of avoiding repetition, many details are omitted here and readers are referred essentially to the proof of Theorem 3.2.1.

The uniqueness part is obtained similarly as in Theorem 3.2.1. Let $\{x_n\}$ be the Picard sequence based on an arbitrary $x_0 \in X$. Also, as in th proof of Theorem 3.2.1, without loss of generality, we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. Hence, we have

$$\sigma_n = \sigma\left(x_n, x_{n+1}\right) > 0.$$

Following the same steps as those used in the proof of Theorem 3.2.1, we obtain

$$\lim_{n\to\infty} \sigma_n = 0. \tag{3.2.27}$$

Now we prove that $\{x_n\}$ is a Cauchy sequence. Arguing by contradiction, we assume that $\{x_n\}$ is not a Cauchy sequence. By (3.2.27) and recalling again the process of proof of Theorem 3.2.1, there exist $\varepsilon > 0$, $k_0 \in \mathbb{N}$ and two subsequences $\left\{x_{m(k(p))}\right\}_{p\geq k_0}$, $\left\{x_{n(k(p))}\right\}_{p\geq k_0}$ of positive integers such that

$$\lim_{p\to\infty} \sigma\left(x_{m(k(p))}, x_{n(k(p))}\right) = l \tag{3.2.28}$$

where, $0 < \varepsilon^+ \leq l \leq s\varepsilon^+$.

Again, as in the proof of Theorem 3.2.1, we have

$$\liminf_{p\to\infty} \sigma\left(x_{m(k(p))+1}, x_{n(k(p))+1}\right) \geq \frac{l}{s^2}. \tag{3.2.29}$$

Let us put

$$\mu_s = \frac{l\left(1 - A_s\right)}{s^2 B_s}, \tag{3.2.30}$$

where

$$A_s = s^2\alpha + s^3\left(\delta + L\right) \tag{3.2.31}$$

and

$$B_s = 1 + \alpha + \beta + \gamma + 2s\left(\delta + L\right). \tag{3.2.32}$$

From the fact that $s^2\alpha + s^3\left(\delta + L\right) < 1$, we get $A_s < 1$ and $\mu_s > 0$. This implies, using (3.2.27), that there exist $j_1, j_2 \geq k_0$ such that

$$\begin{aligned} \sigma_{m(k(p))} &= \sigma\left(x_{m(k(p))}, x_{m(k(p))+1}\right) \leq \mu_s, \quad \text{for all } p \geq j_1, \\ \sigma_{n(k(p))} &= \sigma\left(x_{n(k(p))}, x_{n(k(p))+1}\right) \leq \mu_s, \quad \text{for all } p \geq j_2. \end{aligned} \tag{3.2.33}$$

On the other hand, by virtue of (3.2.28) and $\mu_s > 0$, it follows that there exists $j_3 \geq k_0$ such that

$$\sigma\left(x_{m(k(p))}, x_{n(k(p))}\right) \leq l + \mu_s, \quad \text{for all } p \geq j_3. \tag{3.2.34}$$

Since $B_s > 1$ (otherwise, if $B_s = 1$, we get $\alpha = \beta = \gamma = \delta = L = 0$, which contradicts $\left(\mathcal{H}_s^1\right)$ or $\left(\mathcal{H}_s^2\right)$), we have $\mu_s < \frac{l}{s^2}$. Then, in view of (3.2.29) and $\mu_s > 0$, there exists $j_4 \geq k_0$ such that

$$\sigma\left(Tx_{m(k(p))}, Tx_{n(k(p))}\right) > \frac{l - s^2\mu_s}{s^2} > 0, \quad \text{for all } p \geq j_4. \tag{3.2.35}$$

Using (3.2.35), the relaxed triangle inequality (b_3) and the monotonicity of F with keeping the same notations as those used in the proof of Theorem 3.2.1, the contractive inequality (3.2.9) with $x = x_{m(k(p))}$ and $y = x_{n(k(p))}$ gives for all $p \geq j_4$

$$\begin{aligned} &\tau(a_p) + F(b_p) \\ &\leq F\left((\alpha + s(\delta + L))\, a_p + (\beta + sL)\, \sigma_{m(k(p))} + (\gamma + s\delta)\, \sigma_{n(k(p))}\right). \end{aligned} \tag{3.2.36}$$

Setting $j = \max\{j_1, j_2, j_3, j_4\}$ and using (3.2.30)-(3.2.36) through again the monotonicity of F, we arrive at

$$\begin{aligned} &\tau(a_p) + F\left(\frac{l - s^2\mu_s}{s^2}\right) \\ &\leq F\left(\frac{A_s}{s^2} a_p + (\beta + sL)\, \sigma_{m(k(p))} + (\gamma + s\delta)\, \sigma_{n(k(p))}\right) \\ &\leq F\left(\frac{A_s}{s^2} (l + \mu_s) + (\beta + sL)\, \mu_s + (\gamma + s\delta)\, \mu_s\right) \\ &= F\left(\frac{lA_s}{s^2} + (B_s - 1)\, \mu_s\right) \\ &= F\left(\frac{l - s^2\mu_s}{s^2}\right), \end{aligned}$$

for all $p \geq j$.

The above inequality implies that $\tau(a_p) \leq 0$, for all $p \geq j$, which is a contradiction. In other words, $\{x_n\}$ is a Cauchy sequence. By completeness of (X, σ), $\{x_n\}$ converges to some point x^* in X, that is,

$$\lim_{n \to \infty} \sigma(x_n, x^*) = 0. \tag{3.2.37}$$

Following the same method as the one used in the proof of Theorem 3.2.1, we obtain also that x^* is a fixed point, i.e., $Tx^* = x^*$. This completes the proof of the theorem. ■

Remark 3.2.11 *Theorem 3.2.2 extends and greatly improves the result stated in Remark 3.1.1 on several sides. Firstly, by taking $s = 1$ in Theorem 3.2.2 with the hypothesis $(\mathcal{H}_1^1)$, we recover the result given in Remark 3.1.1. Secondly, Theorem 3.2.2 shows that both conditions (F_2) and "F is upper semicontinuous" can be omitted from the result stated in Remark 3.1.1. Thirdly, we show that the result given in Remark 3.1.1 can be proved also through the hypothesis $(\mathcal{H}_1^2)$. Fourthly, the condition that $\tau \in \mathbb{S}$ is weaken to the condition that $\tau \in \mathcal{S}_1$ and the condition that F is strictly increasing is changed into the weaker condition that F is nondecreasing (i.e., the strictness of the monotonicity of F is not necessary).*

Putting $\alpha = \delta = L = 0$, Theorem 3.2.2 reduces to the following corollary.

Corollary 3.2.2 *Let (X, σ) be a complete b-metric space with constant $s \geq 1$ and let T be a self-mapping on X. Assume that there exist a nondecreasing function $F : (0, \infty) \to \mathbb{R}$ and $\tau \in \mathcal{S}_1$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\tau(d(x, y)) + F(d(Tx, Ty)) \leq F(\beta d(x, Tx) + \gamma d(y, Ty)),$$

where $\beta, \gamma \in [0, \infty)$ satisfying $\beta + \gamma = \frac{1}{s}$ and $\gamma \neq \frac{1}{s}$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.

Remark 3.2.12 *Corollary 3.2.2 is a proper extension and an improvement of Corollary 3.1.1. In fact, by taking $s = 1$ in Corollary 3.2.2, we recover Corollary 3.1.1. Moreover, we show that both conditions (F_2) and "F is upper semicontinuous" from Corollary 3.1.1 can be removed. In addition, the condition that $\tau \in \mathbb{S}$ is changed into the slightly weaker condition that $\tau \in \mathcal{S}_1$ and Corollary 3.1.1 remains valid without the strictness of the monotonicity of F.*

Remark 3.2.13 *Using $\tau(t) = \frac{t}{17}$ and $F : (0, \infty) \to \mathbb{R}$ defined by*

$$F(t) = \begin{cases} \ln(t+1), & \text{if } t \in]0,1], \\ t + \frac{1}{t}, & \text{if } t > 1, \end{cases}$$

the trivial example (see [22, Example 3.32]) $T : X \to X$ (where $X = [0, 4]$) given by

$$Tx = \begin{cases} 3, & \text{if } x \in]0,4], \\ \frac{5}{2}, & \text{if } x = 0, \end{cases} \tag{3.2.38}$$

shows that all the conditions of Corollary 3.2.2 are satisfied. Notice that F is not upper semicontinuous at $t = 1$ and does not satisfy condition (F_2).

Remark 3.2.14 *The trivial example [22, Example 3.33] shows that Corollary 3.2.2 generalizes Corollary 3.1.1.*

3.2.2 Generalized F-weak contraction of Hardy-Rogers-type

In this subsection, we do several improvements in Theorem 3.1.4. For the sake of readability, we keep some notations used in [55]. Throughout this subsection, (X, σ) represents a b-metric space with constant $s \geq 1$. We recall again (see Definition 3.1.4), for all $x, y \in X$

$$A_T^\sigma(x,y) = a\sigma(x,y) + b\sigma(x,Tx) + c\sigma(y,Ty) + e\sigma(x,Ty) + f\sigma(y,Tx),$$

where a, b, c, e, f are nonnegative real numbers. If $s = 1$, we write $A_T^\sigma(x,y) = A_T^d(x,y)$, where d is a metric on X.

Before sating our result, we introduce the following definition.

Definition 3.2.2 *Let (X, σ) be a b-metric space with constant $s \geq 1$. A mapping $T : X \to X$ is said to be a generalized F-weak contraction of Hardy-Rogers-type if there exist a nondecreasing function $F : (0, \infty) \to \mathbb{R}$ and $\tau \in \mathcal{S}_1$ such that for all $x, y \in X$,*

$$\sigma(Tx,Ty) > 0 \Rightarrow \tau(\sigma(x,y)) + F(s\sigma(Tx,Ty)) \leq F(A_T^\sigma(x,y)). \tag{3.2.39}$$

Remark 3.2.15 *It is easy to see from Definition 3.2.2 that every T which is a generalized F-weak contraction of Hardy-Rogers-type satisfies the following condition*

$$\sigma(Tx, Ty) < \frac{1}{s} A_T^{\sigma}(x, y), \tag{3.2.40}$$

for all $x, y \in X$ with $Tx \neq Ty$.

Now, we are ready to state and prove our fourth result.

Theorem 3.2.3 *Let (X, σ) be a complete b-metric space with constant $s \geq 1$ and $T : X \to X$ a generalized F-weak contraction of Hardy-Rogers-type. Suppose that either $(\mathcal{A}_s^1)$ or $(\mathcal{A}_s^2)$ holds, where*

$(\mathcal{A}_s^1)$ $a + b + c + (s+1)e < 1$,

$(\mathcal{A}_s^2)$ $a + b + c + (s+1)f < 1$.

Furthermore, we assume that $sa + s^2(e + f) < 1$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.

Proof. The uniqueness part is obtained similarly as in Theorem 3.2.1. Let $\{x_n\}$ be the Picard sequence based on an arbitrary $x_0 \in X$. Again, as in Theorem 3.2.1 and without loss of generality, we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. Hence, we have

$$\sigma_n = \sigma(x_n, x_{n+1}) > 0.$$

Following the same steps as those used in Theorem 3.1.4 with (3.2.39) and (3.2.40), we obtain analogously

$$F(s\sigma_n) \leq F(\sigma_{n-1}) - \tau(\sigma_{n-1}), \quad \text{for all } n \in \mathbb{N}. \tag{3.2.41}$$

Note that the above part of the proof is proved without conditions (F_3) and $(\mathrm{F}_{s,\tau})$ (see Definition 3.1.3).

Now, by taking $\psi(t) = F(t)$ and $\phi(t) = F(t) - \tau(t)$ for all $t \in (0, \infty)$, the inequality (3.2.41) can be written in the following form:

$$\psi(s\sigma_n) \leq \phi(\sigma_{n-1}), \quad \text{for all } n \in \mathbb{N}.$$

Since F is nondecreasing, then in view of the above inequality and using the fact that $\tau \in \mathcal{S}_1$, it is easy to see that all the conditions of Lemma 3.2.1 are satisfied for $\kappa = s \geq 1$. Thus, $\lim_{n\to\infty} \sigma_n = 0$.

Now we prove that $\{x_n\}$ is a Cauchy sequence. Arguing by contradiction, we assume that $\{x_n\}$ is not a Cauchy sequence. Again, by the process of proof of Theorem 3.2.2 and using

$$\mu_s^* = \frac{l(1 - A_s^*)}{sB_s^*},$$

where

$$A_s^* = sa + s^2(e + f), \; B_s^* = a + b + c + s[1 + 2(e + f)],$$

instead of $\mu_s = \dfrac{l(1-A_s)}{s^2 B_s}$, we get

$$\tau(a_p) + F\left(\frac{l - s^2\mu_s^*}{s}\right) \le F\left(\frac{l - s^2\mu_s^*}{s}\right), \tag{3.2.42}$$

for infinitely many values of p.

This is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. By completeness of (X, σ), $\{x_n\}$ converges to some point x^* in X, that is,

$$\lim_{n\to\infty} \sigma(x_n, x^*) = 0. \tag{3.2.43}$$

The rest of the proof still the same as in Theorem 3.1.4 and the fact that x^* is a fixed point of T is proven in a similar way using (3.2.40). Thus, the proof of the theorem is finished. ∎

Remark 3.2.16 *It is worth noticing that* (3.2.42) *is well defined since* $l - s^2\mu_s^* > 0$. *This last fact comes from* $B_s^* > s$ *(otherwise, if* $B_s^* = s$*, we get* $\alpha = \beta = \gamma = \delta = L = 0$ *which contradicts inequality* (3.2.40)*).*

Remark 3.2.17 *Compared with Theorem 3.1.4, it is clear that Theorem 3.2.3 gives some improvements. Actually,* τ *is taken as a function in Theorem 3.2.3. Moreover, Theorem 3.2.3 shows that both conditions* (F_3) *and* $(F_{s,\tau})$ *from Theorem 3.1.4 are dropped and replaced by the condition that* $sa + s^2(e+f) < 1$. *This latter condition is quite simple and ensures simultaneously, with the remaining common hypotheses of Theorem 3.1.4 and Theorem 3.2.3, the existence and uniqueness of the fixed point. However, Theorem 3.2.3 does not cover totally Theorem 3.1.4, since the condition that* $a + e + f < s$ *(in Theorem 3.1.4) which is only used in the uniqueness part is slightly weaker than the condition that* $sa + s^2(e+f) < 1$. *Besides, the strictness of the monotonicity of* F *is not necessary.*

Remark 3.2.18 *By inspecting the proofs of Theorem 3.2.3 and Theorem 3.1.4, we can also obtain* $\lim_{n\to\infty} \sigma_n = 0$ *in a straightforward manner using an adapted version of Proposition 3.2.1 ((D) is changed into* $d_1 + d_2 + d_3 + (s+1)d_4 < \dfrac{\lambda}{s}$*). The desired result is obtained by taking* $\lambda = s$.

By taking $s = 1$ and $\tau(t) = \tau > 0$, $t \in (0, \infty)$ in Theorem 3.2.3, we obtain the following result.

Corollary 3.2.3 *Let* (X, d) *be a complete metric space and let* T *be a self-mapping on* X. *Assume that there exist a nondecreasing function* $F : (0, \infty) \to \mathbb{R}$ *and* $\tau > 0$ *such that for all* $x, y \in X$ *with* $Tx \neq Ty$,

$$\tau + F(d(Tx, Ty)) \le F\left(A_T^d(x, y)\right).$$

Suppose that either $(\mathcal{A}_1^1)$ *or* $(\mathcal{A}_1^2)$ *holds, where*

$(\mathcal{A}_1^1)$ $a + b + c + 2e < 1$,

$(\mathcal{A}_1^2)$ $a + b + c + 2f < 1$.

Furthermore, we assume that $a+e+f<1$. *Then* T *has a unique fixed point* x^* *and for every* $x_0 \in X$ *the sequence* $\{T^n x_0\}_{n\in\mathbb{N}}$ *converges to* x^*.

Remark 3.2.19 *Corollary 3.2.3 generalizes and greatly improves Theorem 3.1.1 in the following sense.*

1. *By taking* $f=e$ *in Corollary 3.2.3, we recover Theorem 3.1.1.*
2. *The assumption that* T *or* F *is continuous is removed.*
3. *Both conditions* (F_2) *and* (F_3) *are omitted.*
4. *The strictness of the monotonicity of* F *is not necessary.*

Remark 3.2.20 *Using* $\tau = \dfrac{22}{9}$ *and* $F:(0,\infty)\to\mathbb{R}$ *defined by*

$$F(t)=\begin{cases} -\dfrac{1}{t^2+1+(-1)^q}, & \text{if } 0<t\leq 1, \quad q\in\mathbb{N}_0, \\[2mm] t+\dfrac{1}{t}, & \text{if } t>1, \end{cases}$$

the trivial example given in [22, Example 3.43] $T:X\to X$ *(where* $X=[0,5]$*) defined by*

$$Tx=\begin{cases} 5, & \text{if } x\in\left]0,5\right], \\ \dfrac{9}{2}, & \text{if } x=0, \end{cases}$$

shows that all the conditions of Corollary 3.2.3 are satisfied. In addition, Corollary 3.2.3, generalizes and greatly improves Theorem 3.1.1. Indeed, Theorem 3.1.1 can not be applied since neither T *nor* F *is continuous. Moreover,* F *does not satisfy condition* (F_2) *when* q *is even and does not satisfy condition* (F_3) *when* q *is odd. In other words, Corollary 3.2.3 is greatly superior to Theorem 3.1.1.*

In what follows, we give an another proof of Theorem 1-(a) of Hardy-Rogers [26] (see also Reich [63]).

Corollary 3.2.4 ***(See** [26, Theorem 1-(a)]**)*** *Let* (X,d) *be a complete metric space and* T *a self-mapping on* X *satisfying for all* $x,y\in X$,

$$d(Tx,Ty)\leq \theta_1 d(x,y)+\theta_2 d(x,Tx)+\theta_3 d(y,Ty)+\theta_4 d(x,Ty)+\theta_5 d(y,Tx), \tag{3.2.44}$$

where $\theta_i, i=1,...,5$ *are nonnegative numbers such that* $\theta=\sum_{i=1}^{5}\theta_i<1$. *Then,* T *has a unique fixed point.*

Proof. First, we prove that T has at most one fixed point. Assume that x^* and y^* are two fixed points of T, i.e., $Tx^* = x^* \neq y^* = Ty^*$. Using (3.2.44) with $x = x^*$ and $y = y^*$, we get when $\theta \neq 0$ (the case $\theta = 0$ is trivial)

$$0 < d(x^*, y^*) \leq \theta d(x^*, y^*) < d(x^*, y^*).$$

It is a contradiction. Accordingly, T has at most one fixed point.

If $d(Tx, Ty) > 0$ with $x, y \in X$, we have $\theta > 0$ (otherwise, $\theta_i = 0, \forall i = 1, ..., 5$ and from (3.2.44), this yields $d(Tx, Ty) = 0$, which is a contradiction). Thus, choosing $\rho \in]\theta, 1[$, we can write

$$d(Tx, Ty) \leq \rho\left[ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(x, Ty) + fd(y, Tx)\right], \tag{3.2.45}$$

where

$$a = \frac{\theta_1}{\rho}, b = \frac{\theta_2}{\rho}, c = \frac{\theta_3}{\rho}, e = \frac{\theta_3}{\rho}, f = \frac{\theta_4}{\rho}.$$

In addition, we have

$$a + b + c + e + f = \frac{\theta}{\rho} < 1. \tag{3.2.46}$$

By taking $F(t) = \ln(t)$, $t \in (0, \infty)$ and $\tau(t) = \ln\left(\frac{1}{\rho}\right) > 0$, (3.2.45) turns into

$$\begin{aligned}&\tau + F(d(Tx, Ty)) \\ &\leq F\left[ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(x, Ty) + fd(y, Tx)\right],\end{aligned}$$

for all $x, y \in X$ with $Tx \neq Ty$ and a, b, c, e, f are nonnegative real numbers satisfying (3.2.46). Then, we distinguish the following cases:

(i) If $e \leq f$, from (3.2.46), we obtain $a + b + c + 2e < 1$. Therefore, Corollary 3.2.3 with $\left(\mathcal{A}_1^1\right)$ ensures that T has a fixed point.

(ii) If $e > f$, (3.2.46) implies that $a+b+c+2f < 1$. Consequently, the desired result follows from Corollary 3.2.3 with $\left(\mathcal{A}_1^2\right)$.

■

General remark: In all previous results, we could have removed the functions F, τ and only study the contractions (3.2.10) and (3.2.40) (Since F is nondecreasing and τ is positive). Nevertheless, by keeping F and τ we have analysed the steps in which the monotonicity and A_1 are needed. Therefore, we can study the possibility to remove (or relax) condition F_1 in a near future.

3.3 Existence and uniqueness of bounded solutions of functional equations in dynamic programming

In this subsection, we study the existence and uniqueness of the bounded solution of the following functional equation occurring in dynamic programming of multistage decision processes (see, e.g., [9] and [11]):

$$u(x) = \sup_{y \in D} \{f(x, y) + G(x, y, u(\varphi(x, y)))\}, \qquad x \in W, \tag{3.3.1}$$

where $f : W \times D \to \mathbb{R}$ and $G : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded, $\varphi : W \times D \to W$. We assume that W and D are Banach spaces. In this framework, W (respectively, D) is called the state space (respectively, the decision space). Furthermore, φ is the transformation of process and $u(x)$ represents the optimal return function with initial state x.

Let $X = B(W)$ denotes the space of all bounded real-valued functions on W. Now, we endow X with σ defined by

$$\sigma(h, k) = \sup_{x \in W} |h(x) - k(x)|^p, \quad p \geq 1,$$

for all $h, k \in X$. Hence, (X, σ) is a complete b-metric space with $s = 2^{p-1} \geq 1$. Indeed, from Example 1.1.1, we can deduce that (X, σ) is a b-metric space with $s = 2^{p-1} \geq 1$. Also, it is easy to see that every Cauchy sequence $\{h_n\}$ in X converges uniformly to a bounded function h^*, which allows us to obtain the completeness of X.

We also define the mapping $T : X \to X$ by

$$(Tu)(x) = \sup_{y \in D} \{f(x, y) + G(x, y, u(\varphi(x, y)))\}, \tag{3.3.2}$$

for all $u \in X$ and $x \in W$. Since f and G are bounded, it is easy to see that T is well defined.

Let $p \geq 1$ and let $\Psi : (0, \infty) \to (0, \infty)$ be defined by

$$\Psi(t) = \begin{cases} \dfrac{(3t)^{\frac{1}{p}}}{2^{1+\frac{3}{p}}}, & \text{if } 0 < t \leq 1, \\ \dfrac{1}{2^{1-\frac{1}{p}}}, & \text{if } t > 1. \end{cases}$$

Let $h, k \in X$. Denote

$$\chi_p(h, k) := \xi M(h, k),$$

where $M(h, k) = \sigma(h, Th) + \sigma(k, Tk)$ and $\xi = \dfrac{1}{2^p}$, $p \geq 1$.

Now, we are ready to state and prove our next result.

Theorem 3.3.1 *Let $p \geq 1$. Let T be the self-mapping on X defined by* (3.3.2) *and assume that the following condition is satisfied:*

$(\mathcal{K})$: For all $h, k \in X$ with $Th \neq Tk$,

$$|G(x, y, h(z)) - G(x, y, k(z))| \leq \Psi(\chi_p(h, k)),$$

where $x, z \in W$ and $y \in D$. Then the functional equation (3.3.1) *has a unique bounded solution.*

Proof. Let λ be an arbitrary positive number, $x \in W$ and $h, k \in X$ with $Th \neq Tk$. Then there exist $y_1, y_2 \in D$ such that

$$(Th)(x) < f(x, y_1) + G(x, y_1, h(\varphi(x, y_1))) + \frac{\lambda^{\frac{1}{p}}}{2}, \tag{3.3.3}$$

$$(Tk)(x) < f(x, y_2) + G(x, y_2, k(\varphi(x, y_2))) + \frac{\lambda^{\frac{1}{p}}}{2}. \tag{3.3.4}$$

Again, by definition of T, we have

$$(Th)(x) \geq f(x, y_2) + G(x, y_2, h(\varphi(x, y_2))), \tag{3.3.5}$$

$$(Tk)(x) \geq f(x, y_1) + G(x, y_1, k(\varphi(x, y_1))). \tag{3.3.6}$$

Utilizing (3.3.3) and (3.3.6) together with ($\mathcal{K}$), one can get

$$\begin{aligned}(Th)(x) - (Tk)(x) &< G(x, y_1, h(\varphi(x, y_1))) - G(x, y_1, k(\varphi(x, y_1))) + \frac{\lambda^{\frac{1}{p}}}{2} \\ &\leq |G(x, y_1, h(\varphi(x, y_1))) - G(x, y_1, k(\varphi(x, y_1)))| + \frac{\lambda^{\frac{1}{p}}}{2} \\ &\leq \Psi(\chi_p(h, k)) + \frac{\lambda^{\frac{1}{p}}}{2}.\end{aligned} \tag{3.3.7}$$

Analogously, from (3.3.4) and (3.3.5) together with ($\mathcal{K}$), we have

$$(Tk)(x) - (Th)(x) < \Psi(\chi_p(h, k)) + \frac{\lambda^{\frac{1}{p}}}{2}. \tag{3.3.8}$$

Combining (3.3.7) and (3.3.8), we deduce

$$|(Th)(x) - (Tk)(x)| < \Psi(\chi_p(h, k)) + \frac{\lambda^{\frac{1}{p}}}{2}.$$

Using the following inequality

$$(a + b)^p \leq 2^{p-1}(a^p + b^p), \qquad a, b > 0,$$

it follows that

$$|(Th)(x) - (Tk)(x)|^p < 2^{p-1}[\Psi(\chi_p(h, k))]^p + \frac{\lambda}{2}.$$

The above inequality yields

$$\sigma(Th, Tk) < 2^{p-1}[\Psi(\chi_p(h, k))]^p + \frac{\lambda}{2}. \tag{3.3.9}$$

Now, we discuss the two possible cases:

Case 1. If $0 < \chi_p(h, k) \leq 1$. In this case, (3.3.9) turns into

$$\sigma(Th, Tk) < \frac{3\chi_p(h, k)}{16} + \frac{\lambda}{2}. \tag{3.3.10}$$

As (3.3.10) does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrarily, we have

$$\sigma(Th, Tk) \leq \frac{3\chi_p(h,k)}{16}, \tag{3.3.11}$$

which follows that

$$\sigma(Th, Tk) < 1. \tag{3.3.12}$$

On the other hand, by using (b_3), we have

$$\begin{aligned}
\sigma(h,k) &\leq s\sigma(h,Th) + s^2\sigma(Th,Tk) + s^2\sigma(Tk,k) \\
&\leq s^2\sigma(h,Th) + s^2\sigma(k,Tk) + s^2\sigma(Th,Tk) \\
&\leq s^2 M(h,k) + s^2\sigma(Th,Tk) \\
&\leq s^2\frac{\chi_p(h,k)}{\xi} + s^2\sigma(Th,Tk).
\end{aligned}$$

Keeping in mind $s = 2^{p-1}$ and $\xi = \dfrac{1}{2^p}$, the last inequality leads to

$$\sigma(h,k) \leq 2s^3\chi_p(h,k) + s^2\sigma(Th,Tk). \tag{3.3.13}$$

Owing to (3.3.10) and (3.3.13), we get

$$\begin{aligned}
\frac{\sigma(h,k)}{16s^3} &\leq \frac{\chi_p(h,k)}{8} + \frac{\sigma(Th,Tk)}{16s} \\
&< \frac{\chi_p(h,k)}{8} + \sigma(Th,Tk) \\
&< \frac{\chi_p(h,k)}{8} + \frac{3\chi_p(h,k)}{16} + \frac{\lambda}{2} \\
&= \frac{5\chi_p(h,k)}{16} + \frac{\lambda}{2}.
\end{aligned} \tag{3.3.14}$$

Taking into account (3.3.14) and using the following compound inequality

$$\frac{a}{1+a} < \ln(1+a) < a, \quad \text{for all } a > 0,$$

we obtain

$$\begin{aligned}
\frac{\sigma(h,k)}{16s^3} + \ln(1+\sigma(Th,Tk)) &< \frac{5\chi_p(h,k)}{16} + \frac{\lambda}{2} + \sigma(Th,Tk) \\
&< \frac{\chi_p(h,k)}{2} + \lambda \\
&\leq \frac{\chi_p(h,k)}{1+\chi_p(h,k)} + \lambda \\
&< \ln(1+\chi_p(h,k)) + \lambda.
\end{aligned}$$

Since the last inequality does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrarily, we get

$$\frac{\sigma(h,k)}{16s^3} + \ln(1+\sigma(Th,Tk)) \leq \ln(1+\chi_p(h,k)). \tag{3.3.15}$$

In addition, by virtue of (3.3.11) and (3.3.13) with the fact that $0 < \chi_p(h,k) \leq 1$, we get

$$\sigma(h,k) \leq 2s^3 + \frac{3s^2}{16} = 2^{3p-2} + 3 \times 2^{2p-6}. \tag{3.3.16}$$

Case 2. If $\chi_p(h,k) > 1$. In this case, (3.3.9) takes the form

$$\sigma(Th,Tk) < 1 + \frac{\lambda}{2}. \tag{3.3.17}$$

Again, as (3.3.17) does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrarily, we have

$$\sigma(Th,Tk) \leq 1. \tag{3.3.18}$$

From (3.3.17) and using the following inequalities

$$\ln(1+b) < b, \qquad b + \frac{1}{b} \geq 2, \quad \text{for all } b > 0,$$

one gets

$$\begin{aligned} 1 + \ln(1+\sigma(Th,Tk)) &< 1 + \sigma(Th,Tk) \\ &< 2 + \frac{\lambda}{2} \\ &< \chi_p(h,k) + \frac{1}{\chi_p(h,k)} + \lambda. \end{aligned} \tag{3.3.19}$$

Since (3.3.19) does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrarily, we have

$$1 + \ln(1+\sigma(Th,Tk)) \leq \chi_p(h,k) + \frac{1}{\chi_p(h,k)}. \tag{3.3.20}$$

Therefore, bearing in mind inequalities (3.3.12), (3.3.16) and (3.3.18), inequalities (3.3.15) and (3.3.20) allow us to obtain

$$\begin{aligned} \tau(\sigma(h,k)) + F(\sigma(Th,Tk)) &\leq F(\chi_p(h,k)) \\ &= F\left(\frac{1}{2^p} M(h,k)\right), \end{aligned}$$

for all $h,k \in X$ and $Th \neq Tk$ with $F:(0,\infty) \to \mathbb{R}$ given by

$$F(t) = \begin{cases} \ln(t+1), & \text{if } t \in \left]0,1\right], \\ t + \dfrac{1}{t}, & \text{if } t > 1 \end{cases}$$

and $\tau:(0,\infty) \to (0,\infty)$ given as follows

$$\tau(t) = \begin{cases} \dfrac{t}{2^{3p+1}}, & \text{if } t \in \left]0, C_p\right], \\ 1, & \text{otherwise,} \end{cases}$$

where $C_p = 2^{3p-2} + 3 \times 2^{2p-6}$.

Hence, all the conditions of Corollary 3.2.2 are satisfied with $\beta = \gamma = \frac{1}{2^p}$, $p \geq 1$. Consequently, T has a unique fixed point u^* in $X = B(W)$. Thus, the functional equation (3.3.1) has a unique bounded solution. ■

Chapter 4

Some remarks on the paper by Lukács and Kajántó with an application to functional equations arising in dynamic programming

Abstract

In this chapter, we give a short and new proof of the main result (Theorem 3.1) due to Lukács and Kajántó (Results Math 73(82), (2018)). Our proof is established through Jachymski et.al fixed point theorem (1995) in the setting of semimetric spaces, and some new results due to Suzuki (2018). Roughly speaking, we provide a very short proof which is completely different from the original one in the paper of Lukács and Kajántó. In addition, we derive a new equivalent form of Lukács and Kajántó's fixed point theorem which can be used to generalize and improve Jleli and Samet's fixed point theorem (2014) in the context of b-metric spaces. As an application, the existence and uniqueness for functional equations occurring in dynamic programming is established, followed by a suitable example.

4.1 Introduction and preliminaries

Throughout this chapter, we will write $\mathbb{R}_+ = [0, \infty)$.

In 2014, Jleli and Samet [30] introduced the family Θ of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(θ_1) θ is nondecreasing;

(θ_2) for each sequence $\{t_n\} \subset (0, \infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \Leftrightarrow \lim_{n\to\infty} t_n = 0;$$

(θ_3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that

$$\lim_{t\to 0^+} \frac{\theta(t) - 1}{t^r} = l.$$

The notion of *θ-contraction* is given as follows.

Definition 4.1.1 *(See [30]) Let (X, d) be a complete metric space. A mapping $T : X \to X$ is called a θ-contraction if there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) > 0 \Rightarrow \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k .$$

Theorem 4.1.1 *(See [30, Corollary 2.1]) On a complete metric space (X, d), every θ-contraction mapping is a Picard operator.*

In 2018, Lukács and Kajántó [56] introduced a new contractive condition involving the so-called *comparison functions* φ in the setting of b-metric spaces (see [56, Definition 2.6]). The authors pointed out that this *φ-contraction type* is closer in spirit to the notion of F-contraction and they proved a new fixed point theorem (see [56, Theorem 3.1]). Using this result, the authors improved and extended [82, Theorem 2.1] to b-metric spaces by omitting and relaxing some conditions imposed on the function F. It should be mention that the proof established for Theorem 3.1 by Lukács and Kajántó is long and quite technical.

In this chapter, we propose a short and different proof of the main result in [56] (Theorem 3.1) using Jachymski et.al fixed point theorem [29, Theorem 1] and some new results proved by Suzuki [76] in the framework of semimetric spaces. Furthermore, we extend and improve the fixed point theorem (dealing with θ-contractions) of Jleli and Samet (2014) ([30]) in the framework of b-metric spaces. This fact is obtained because the class of θ-contractions can be derived from a suitable *φ-contractive condition type* (deduced from the one stated in [56]) which thereby extends the scope of Theorem 3.1 in [56] to study a wider class of mappings. At the end of this note, we give an application dealing with the existence and uniqueness of functional equations occurring in dynamic programming, followed by a suitable example.

We start by recalling the definitions of a semimetric space.

Definition 4.1.2 *(See [38, 76]) Let X be a nonempty set and let $s \geq 1$. A mapping $d : X \times X \to \mathbb{R}_+$ is said to be a semimetric on X if, for all $x, y \in X$, the following conditions hold:*
(D_1) $d(x, y) = 0$ if and only if $x = y$;
(D_2) $d(x, y) = d(y, x)$.
The pair (X, d) is called a semimetric-metric space.

In what follows, we state some definitions in the framework of semimetric spaces (see [38] and [77]).

Definition 4.1.3 *([77, Definition 2.2]) Let (X, d) be a semimetric space, let $\{x_n\}$ be a sequence in X and let $x, y \in X$.*

1. *$\{x_n\}$ is said to converge to x if $\lim_{n\to\infty} d(x_n, x) = 0$.*
2. *$\{x_n\}$ is said to be Cauchy (or d-Cauchy) if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m, n \geq N \Rightarrow d(x_m, x_n) < \varepsilon$.*
3. *$\{x_n\}$ is said to be complete if every Cauchy sequence converges.*
4. *$\{x_n\}$ is said to be Hausdorff if $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$.*

Now, we need to present some notations and auxiliary results which will be used in the sequel.

Definition 4.1.4 *Let $I \subseteq \mathbb{R}_+$. Let Φ_I be the set of all functions $\varphi : I \to I$ satisfying the following conditions:*

1. *φ is nondecreasing;*
2. *$\lim_{n\to\infty} \varphi^n(t) = 0$ for every $t \in I$.*

Remark 4.1.1 *When $I = \mathbb{R}_+$, we will write Φ instead of $\Phi_{\mathbb{R}_+}$. It is worth noting that Φ is called the class of comparison functions (see [65]).*

Definition 4.1.5 *(See [28]) Let $J \subseteq \mathbb{R}_+$. We denote by $\mathbb{G}_J$ the set of all functions $G : J \to J$ satisfying the following property:*
For any sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ in J, we have

$$\lim_{n\to\infty} G(\alpha_n) = 0 \Leftrightarrow \lim_{n\to\infty} \alpha_n = 0.$$

When $J = \mathbb{R}_+$, we will write $\mathbb{G}$ instead of $\mathbb{G}_{\mathbb{R}_+}$.

Lemma 4.1.1 *(See [78, Lemma 2.2] and [76, Lemma 16]) Let $\eta \in \mathbb{G}$. Then $\eta^{-1}(\{0\}) = 0$ holds, that is, $\eta(\alpha) = 0 \Leftrightarrow \alpha = 0$.*

Lemma 4.1.2 *(See [76, Lemma 17]) Let (X, d) be a semimetric space and let $\tilde{G} \in \mathbb{G}$. Define a function $\rho : X \times X \to \mathbb{R}_+$ by $\rho = \tilde{G} o d$. Let $\{x_n\}$ be a sequence in X and let $x \in X$. Then the following holds:*

1. *(X, ρ) is a semimetric space.*
2. *$\{x_n\}$ converges to x in (X, d) iff $\{x_n\}$ converges to x in (X, ρ).*
3. *$\{x_n\}$ is Cauchy in (X, d) iff $\{x_n\}$ is Cauchy in (X, ρ).*

4. *(X, d) is complete iff (X, ρ) is complete.*

5. *(X, d) is Hausdorff iff (X, ρ) is Hausdorff.*

Lemma 4.1.3 *(See [76, Lemma 19]) Let (X, d) be a semimetric space. Assume that the following holds:*
(D_4) For any $\varepsilon > 0$, there exists $\delta > 0$ such that, given $x, y, z \in X$,

$$d(x,y) < \delta \text{ and } d(y,z) < \delta \text{ imply } d(x,z) < \varepsilon.$$

Let $\tilde{G} \in \mathbb{G}$. Then (D_4) holds for $\rho = \tilde{G} \, o \, d$.

Lemma 4.1.4 *([76, Lemma 20]) Let (X, d) be a semimetric space. Assume that (D_4) holds. Then (X, d) is Hausdorff.*

Lemma 4.1.5 *(See [76, Lemma 18]) Let (X, d) be a b-metric space with coefficient $s \geq 1$. Then (D_4) holds.*

Now we recall the splendid fixed point theorem, due to Jachymski et.al [29], which plays a crucial role to establish our own proof in Sect. 3.

Theorem 4.1.2 *(a consequence of Theorem 1 in [29] or [77, Theorem 1.1]) Let (X, d) be a Hausdorff, complete semimetric space satisfying the following property:*
(D_5) : There exist $\delta > 0$ and $\varepsilon > 0$ such that, given $x, y, z \in X$,

$$d(x,y) < \delta \text{ and } d(y,z) < \delta \text{ imply } d(x,z) < \varepsilon.$$

Moreover, if there exists $\varphi \in \Phi$ such the mapping $T : X \to X$ satisfies

$$d(Tx, Ty) \leq \varphi(d(x,y)) \quad \text{for all } x, y \in X,$$

then T is a Picard operator.

Remark 4.1.2 *(See [76]) Notice that (D_5) is weaker than (D_4).*

4.2 Fixed point results of Lukács and Kajántó

In the paper [56], Lukács and Kajántó proved a fixed point theorem (see [56, Theorem 3.1]) in the context of b-metric spaces via φ-contractions type. In this section, we review all the tools and auxiliary results used by the authors in [56] to prove the aforementioned fixed point theorem.

From now onwards, $\mathbb{R}_+^*$ denotes the interval $(0, \infty)$. In addition, we write Φ_* and $\mathbb{G}_*$ instead of $\Phi_{\mathbb{R}_+^*}$ and $\mathbb{G}_{\mathbb{R}_+^*}$, respectively. In the sequel, it is substantial to state the following definitions (see [56]).

Definition 4.2.1 *(See [56, Definition 2.5]) Let $\mathbb{G}_1$ be the set of all functions $G : \mathbb{R}_+^* \to \mathbb{R}_+^*$ satisfying* $\inf G > 0$.

Definition 4.2.2 *([56, Definition 2.6]) Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $G, \varphi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be two functions. We denote by $\mathbb{T}(X,G,\varphi)$ the set of all mappings $T : X \to X$ such that for all $x, y \in X$, the following φ-contractive condition holds*

$$(\mathcal{R}) \quad d(Tx,Ty) > 0 \Rightarrow G(d(Tx,Ty)) \leq \varphi(G(d(x,y))).$$

The main result of Lukács and Kajántó [56] is the following.

Theorem 4.2.1 *([56, Theorem 3.1]) Let (X,d) be a complete b-metric space with coefficient $s \geq 1$. Assume that there exist a function $G \in \mathbb{G}_1 \cup \mathbb{G}_*$ and a function $\varphi \in \Phi_*$ such that $T \in \mathbb{T}(X,G,\varphi)$. Then T is a Picard operator.*

The authors in [56] have divided the proof of Theorem 4.2.1 into many auxiliary results. In the case when $G \in \mathbb{G}_1$, Theorem 4.2.1 is simply proved through [56, Lemma 4.2].

In the case when $G \in \mathbb{G}_*$, the result is obtained using several technical lemmas (see [56, Section 4] for more details on the proof). For the sake of readability, we recall now all the lemmas used in [56] to establish the proof of Theorem 4.2.1.

Lemma 4.2.1 *(See [37], [56, Lemma 2.2]) Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $\{x_n\}$ be a convergent sequence in X with $\lim x_n = x$. Then for each $y \in X$, we have*

$$\frac{1}{s} d(x,y) \leq \liminf_{n\to\infty} d(x_n,y) \leq \limsup_{n\to\infty} d(x_n,y) \leq s d(x,y).$$

Lemma 4.2.2 *([56, Lemma 4.1]) If $\varphi \in \Phi_*$ then $\varphi(t) < t$ for all $t \in \mathbb{R}_+^*$.*

Lemma 4.2.3 *([56, Lemma 4.3]) Suppose that $G \in \mathbb{G}_*$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, we get*

- $\sup\limits_{\alpha<\varepsilon} G(\alpha) \leq 1$;
- $\inf\limits_{\alpha\geq\varepsilon} G(\alpha) > 0$.

Lemma 4.2.4 *([56, Lemma 4.4]) Let (X,d) be a complete b-metric space with coefficient $s \geq 1$, $G \in \mathbb{G}_*$, $\varphi \in \Phi_*$ and $T \in \mathbb{T}(X,G,\varphi)$ and construct ε_0 according to Lemma 4.2.3. Then for all $\varepsilon \in (0,\varepsilon_0)$, there exists $n_\varepsilon \in \mathbb{N} \cup \{0\}$ such that*

$$d(x,y) < \varepsilon \Rightarrow d(T^n x, T^n y) < \frac{\varepsilon}{2s}$$

for all $n \geq n_\varepsilon$ and $x, y \in X$.

Lemma 4.2.5 *([56, Lemma 4.5]) Let (X,d) be a complete b-metric space with coefficient $s \geq 1$, $G \in \mathbb{G}_*$, $\varphi \in \Phi_*$ and $T \in \mathbb{T}(X,G,\varphi)$. Then for all $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists m_ε such that*

$$d(T^n x_{mn}, x_{mn}) < \frac{\varepsilon}{2s}$$

for all $m \geq m_\varepsilon$.

Lemma 4.2.6 *([56, Lemma 4.6]) Let (X, d) be a complete b-metric space with coefficient $s \geq 1$, $G \in \mathbb{G}_*$, $\varphi \in \Phi_*$ such that $T \in \mathbb{T}(X, G, \varphi)$ and construct ε_0 according to Lemma 4.2.3. Then for every $\varepsilon \in (0, \varepsilon_0)$, there exist $n = n_\varepsilon, m = m_\varepsilon \in \mathbb{N}$ such that*

$$T^n(B(x_{mn}, \varepsilon)) \subseteq B(x_{mn}, \varepsilon),$$

where $B(x_{mn}, \varepsilon) = \{a \in X; d(a, x_{mn}) < \varepsilon\}$.

Lemma 4.2.7 *([56, Lemma 4.7]) Let $\{x_k\}_{k \in \mathbb{N} \cup \{0\}}$ be a sequence such that $\lim\limits_{n \to \infty} d(x_{k+1}, x_k) = 0$. Then for all $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $m_0 \in \mathbb{N} \cup \{0\}$ such that*

$$d(x_{mn}, x_{mn+p}) < \varepsilon$$

for all $m \geq m_0$ and $p \in \{0, \cdots, n-1\}$.

Lukács and Kajántó [56] obtained also the following equivalent version (in connection with F-contractions) of Theorem 4.2.1.

Theorem 4.2.2 *([56, Theorem 3.2]) Let $\psi : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function such that $\lim\limits_{n \to \infty} \psi^n(t) = -\infty$ for each $t \in \mathbb{R}$ and $F : \mathbb{R}_+^* \to \mathbb{R}$ be a function that satisfies one of the following two conditions:*

(H_1) $\inf F > -\infty$;

(H_2) F satisfies (F_2).

Assume that (X, d) is a complete b-metric space with coefficient $s \geq 1$ and let $T : X \to X$ be a mapping such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow F(d(Tx, Ty)) \leq \psi(F(d(x, y))).$$

Then T is a Picard operator.

4.3 A short proof of theorem of Lukács and Kajántó and some consequences

Firstly, let mention that the proof of Theorem 4.2.1 (Theorem 3.1 by Lukács and Kajántó [56]) is very long and quite technical. In this section, we present a very short, simple and different proof for Theorem 4.2.1 in the case when $G \in \mathbb{G}_*$. The key ingredient for establishing our proof is Theorem 4.1.2 that can be applicable using some new results of Suzuki [76] in the setting of semimetric spaces. In addition, we derive an equivalent result of Theorem 4.2.1 which allows us to extend and improve Jleli and Samet's fixed point theorem (see [30, Corollary 2.1]).

4.3.1 Proof of Theorem 4.2.1

Proof. If $G \in \mathbb{G}_1$, we keep the same proof as in [56]. Now we deal with the case when $G \in \mathbb{G}_*$. First, we consider the functions $\tilde{G}, \tilde{\varphi} : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\tilde{G}(t) = \begin{cases} 0, & \text{if } t = 0, \\ G(t), & \text{if } t > 0 \end{cases}$$

and

$$\tilde{\varphi}(t) = \begin{cases} 0, & \text{if } t = 0, \\ \varphi(t), & \text{if } t > 0. \end{cases}$$

Since $G \in \mathbb{G}_*$ and $\varphi \in \Phi_*$, we immediately deduce that $\tilde{G} \in \mathbb{G}$ and $\tilde{\varphi} \in \Phi$. Next, by the contractive condition $(\mathcal{R})$, one gets

$$\tilde{G}\, o\, d(Tx, Ty) \leq \tilde{\varphi}\left(\tilde{G}\, o\, d(x, y)\right) \tag{4.3.1}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$. It is easy to see that the above inequality holds also when $Tx = Ty$. Setting $\rho = \tilde{G}\, o\, d$, inequality (4.3.1) takes the form

$$\rho(Tx, Ty) \leq \tilde{\varphi}(\rho(x, y)) \text{ for all } x, y \in X.$$

Therefore, according to Lemmas 4.1.2, 4.1.3, 4.1.4 and 4.1.5, we deduce that (X, ρ) is a Hausdorff, complete semimetric space and satisfies the property (D_4). By Remark 4.1.2 (that is, (D_4) implies (D_5)) and the fact that $\tilde{\varphi} \in \Phi$, we obtain that all the assumptions of Theorem 4.1.2 are satisfied. Thus, T is a Picard operator and the proof is completed. ■

Remark 4.3.1 *The approach used in our paper allowed us to prove Theorem 4.2.1 without use of all Lemmas 4.2.1, 4.2.2, 4.2.3, 4.2.4, 4.2.5, 4.2.6 and 4.2.7.*

4.3.2 Some consequences of Theorem 4.2.1

For convenience, Θ_2 will denotes the set of all functions $\theta : \mathbb{R}_+^* \to (1, \infty)$ satisfying (θ_2).

Inspired by Theorem 4.2.2, one can deduce another equivalent version of Theorem 4.2.1 (in terms of θ-contractions) given below.

Theorem 4.3.1 *Let $\chi : (1, \infty) \to (1, \infty)$ be a nondecreasing function such that $\lim_{n\to\infty} \chi^n(t) = 1$ for each $t \in (1, \infty)$ and let $\theta : \mathbb{R}_+^* \to (1, \infty)$ be a function that satisfies one of the following two conditions:*

(i) $\inf \theta > 1$;

(ii) $\theta \in \Theta_2$.

Assume that (X, d) is a complete b-metric space with coefficient $s \geq 1$ and let $T : X \to X$ be a mapping such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \theta(d(Tx, Ty)) \leq \chi(\theta(d(x, y))).$$

Then T is a Picard operator.

Proof. The result follows immediately from the equivalence between Theorem 4.2.1 and Theorem 4.3.1 obtained by taking $G(t) = \ln(\theta(t))$ and $\varphi(t) = \ln\left(\chi\left(e^t\right)\right)$ for all $t \in \mathbb{R}_+^*$. ∎

Remark 4.3.2 *Theorem 4.2.1, Theorem 4.2.2 and Theorem 4.3.1 are equivalent.*

Taking $\chi(t) = t^k$ with $k \in (0,1)$ for all $t \in (1,\infty)$ in Theorem 4.3.1, we obtain the following result.

Corollary 4.3.1 ***(Jleli and Samet fixed point in*** b***-metric spaces).*** *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and let $\theta : \mathbb{R}_+^* \to (1,\infty)$ be a function that satisfies one of the following two conditions:*

(i) $\inf \theta > 1$;

(ii) $\theta \in \Theta_2$.

Suppose that there exists $k \in (0,1)$ and $T : X \to X$ be a mapping such that for all $x, y \in X$,

$$d(Tx,Ty) > 0 \Rightarrow \theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k.$$

Then T is a Picard operator.

Remark 4.3.3 *Corollary 4.3.1 extends and improves the fixed point theorem of Jleli and Samet (see Theorem 4.1.1). Indeed, if we take $s = 1$ in Corollary 4.3.1, we recover Theorem 4.1.1. Moreover, we have shown that both conditions (θ_1) and (θ_3) from Theorem 4.1.1 can be omitted. Besides, condition (θ_2) is relaxed.*

Remark 4.3.4 *Applying Theorem 4.3.1 with $\chi(t) = t^{e^{-\tau}}$ with $\tau > 0$ and $\theta(t) = e^{e^{F(t)}}$, where $F : \mathbb{R}_+^* \to \mathbb{R}$ satisfying either (H_1) or (H_2) (see Theorem 4.2.2), one can deduce an extension of Theorem 2.1 in [82] to b-metric spaces. Furthermore, we have obtained an improvement of [82, Theorem 2.1] by relaxing condition (F_2) and omitting both conditions (F_1) and (F_3) (see [56, Remark 3.4] as well).*

4.4 An application to functional equations arising in dynamic programming

For the sake of readability, we remind the settings of the functional equations arising in dynamic programming. Let us suppose that W and D are two nonempty sets. Also we consider the functions $f : W \times D \to \mathbb{R}$, $H : W \times D \times \mathbb{R} \to \mathbb{R}$ and $\varphi : W \times D \to W$.

In this section, we use Corollary 4.3.1 to provide sufficient conditions for the existence and uniqueness of bounded solutions of the following functional equation:

$$u(x) = \sup_{y \in D} \{f(x,y) + H(x,y,u(\varphi(x,y)))\}, \qquad x \in W. \tag{4.4.1}$$

Such a type of equations appears in the study of dynamic programming in connection with multistage decision processes. In this setting, W is called state space and D is the decision space. Moreover, φ represents the transformation of process and $u(x)$ represents the optimal return function with initial state x. The basic form of (4.4.1) has been initiated by Bellman and Lee [9]. Thereafter, several works for various kinds of (4.4.1) have been done, see, for example, [16, 31, 32, 39, 49].

Let $B(W)$ denotes the set of all bounded real-valued functions defined on W. For any $h \in B(W)$, we define the classical norm given by

$$\|h\| = \sup_{x \in W} |h(x)|.$$

It is well known that $(B(W), \|.\|)$ is a Banach space. Hence, $B(W)$ endowed with the *sup* distance d defined by

$$d(h,k) = \sup_{x \in W} |h(x) - k(x)| \quad \text{for all } h,k \in B(W),$$

is a complete metric space.

Moreover, for some $p > 1$, we can define

$$\sigma(h,k) = (d(h,k))^p = \sup_{x \in W} |h(x) - k(x)|^p \quad \text{for all } h,k \in B(W).$$

It is easy to see that $(B(W), \sigma)$ is a complete b-metric space with coefficient $s = 2^{p-1}$ (see, for example, [22, 31]).

We recall the following result (a particular case in [46]).

Lemma 4.4.1 *(See [16, Lemma 1]) Let A be a nonempty set. Suppose that $J, K : A \to \mathbb{R}$ are two bounded functions. Then*

$$\left| \sup_{x \in A} J(x) - \sup_{x \in A} K(x) \right| \leq \sup_{x \in A} |J(x) - K(x)|.$$

Lemma 4.4.2 *Let Y and Z be two nonempty sets. Suppose that $g : Y \times Z \to \mathbb{R}$ is a function bounded from above. Then*

$$\sup_{(x,y) \in Y \times Z} g(x,y) = \sup_{x \in Y} \left(\sup_{y \in Z} g(x,y) \right).$$

Proof. Obvious. ■

Now, we are ready to state our application.

Theorem 4.4.1 *Suppose that the following conditions hold:*

(i) $f : W \times D \to \mathbb{R}$ and $H(.,.,0) : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded functions;

(ii) there exists $\tau \in (\ln 2, \infty)$ such that, for some $p > 1$, for any $x \in W, y \in D$ and $t, s \in \mathbb{R}$,

$$|H(x,y,t) - H(x,y,s)| \leq \frac{|t-s|}{(1 + \tau |t-s|^p)^{\frac{1}{p}}}.$$

Then the functional equation (4.4.1) *has a unique solution in* $B(W)$.

Proof. Let $T : (B(W), \sigma) \to (B(W), \sigma)$ be the mapping defined as follows:

$$(Th)(x) = \sup_{y \in D} \{ f(x,y) + H(x,y,h(\varphi(x,y))) \},$$

for all $h \in B(W)$ and $x \in W$.

First, we show that T is well defined (that is, $Th \in B(W)$). Denote

$$\lambda = \sup \{ |f(x,y)| ; (x,y) \in W \times D \} \text{ and } \mu = \sup \{ |H(x,y,0)| ; (x,y) \in W \times D \}$$

(λ and μ exist by condition (i)).

Let $h \in B(W)$, then there exists $N > 0$ such that

$$|h(t)| \leq N \quad \text{for all } t \in W. \tag{4.4.2}$$

Let $h \in B(W)$ and $x \in W$. Through Lemma 4.4.1, Lemma 4.4.2 and condition (ii) with (4.4.2), we obtain the following chain of inequalities

$$\begin{aligned}
|(Th)(x)| &= \left| \sup_{y \in D} \{ f(x,y) + H(x,y,h(\varphi(x,y))) \} \right| \\
&\leq \sup_{y \in D} |f(x,y)| + \sup_{y \in D} |H(x,y,h(\varphi(x,y)))| \\
&\leq \sup_{y \in D} |f(x,y)| + \sup_{y \in D} |H(x,y,h(\varphi(x,y))) - H(x,y,0)| \\
&+ \sup_{y \in D} |H(x,y,0)| \\
&\leq \sup_{y \in D} |f(x,y)| + \sup_{y \in D} \frac{|h(\varphi(x,y))|}{(1 + \tau |h(\varphi(x,y))|^p)^{\frac{1}{p}}} \\
&+ \sup_{y \in D} |H(x,y,0)| \\
&\leq \sup_{x \in W} \left(\sup_{y \in D} |f(x,y)| \right) + \sup_{y \in D} |h(\varphi(x,y))| \\
&+ \sup_{x \in W} \left(\sup_{y \in D} |H(x,y,0)| \right) \\
&\leq M := \lambda + N + \mu.
\end{aligned}$$

The above inequality implies that $Th \in B(W)$.

Next, for any $x \in W$ and $h, k \in B(W)$ such that $Th \neq Tk$, we have

$$\begin{aligned}
|(Th)(x) - (Tk)(x)|^p &= \left| \sup_{y \in D} \{ f(x,y) + H(x,y,h(\varphi(x,y))) \} \right. \\
&\quad \left. - \sup_{y \in D} \{ f(x,y) + H(x,y,k(\varphi(x,y))) \} \right|^p \\
&\leq \sup_{y \in D} |H(x,y,h(\varphi(x,y))) - H(x,y,k(\varphi(x,y)))|^p \\
&\leq \sup_{y \in D} \frac{|h(\varphi(x,y)) - k(\varphi(x,y))|^p}{1 + \tau |h(\varphi(x,y)) - k(\varphi(x,y))|^p},
\end{aligned} \tag{4.4.3}$$

where we have used Lemma 4.4.1 and condition (ii). By (4.4.3) and the monotonicity of the function $\mathbb{R}_+ \ni t \mapsto \dfrac{t}{1+\tau t}$, it follows that

$$|(Th)(x) - (Tk)(x)|^p \leq \frac{\sigma(h,k)}{1+\tau\sigma(h,k)}.$$

Taking the supremum on $x \in W$ in the last inequality, we deduce that

$$\sigma(Th,Tk) \leq \frac{\sigma(h,k)}{1+\tau\sigma(h,k)}. \tag{4.4.4}$$

Let us consider the following two cases:

Case 1. If $\sigma(h,k) > e$. From (4.4.4), we have

$$\sigma(Th,Tk) \leq \frac{1}{\tau}. \tag{4.4.5}$$

This implies that

$$e^{e^{-\dfrac{1}{\sigma(Th,Tk)}}} \leq e^{e^{-\tau}} = \left(\sqrt{e}\right)^{2e^{-\tau}} \leq \left(\sqrt{\sigma(h,k)}\right)^{2e^{-\tau}}. \tag{4.4.6}$$

Notice that from (4.4.4) and the fact that $\tau > \ln 2$, we have $\sigma(Th,Tk) \leq e$.

Case 2. If $0 < \sigma(h,k) \leq e$. By (4.4.4) and after routine calculations, one can get

$$-\frac{1}{\sigma(Th,Tk)} \leq -\frac{1}{\sigma(h,k)} - \tau,$$

which further implies

$$e^{e^{-\frac{1}{\sigma(Th,Tk)}}} \leq \left[e^{e^{-\frac{1}{\sigma(h,k)}}}\right]^{e^{-\tau}} \leq \left[e^{e^{-\frac{1}{\sigma(h,k)}}}\right]^{2e^{-\tau}}. \tag{4.4.7}$$

Let us remark that from (4.4.4), we also have

$$\sigma(Th,Tk) \leq \sigma(h,k) \leq e.$$

Hence, inequalities (4.4.6) and (4.4.7) allow us to obtain

$$\theta(\sigma(Th,Tk)) \leq [\theta(\sigma(h,k))]^k,$$

where $\theta : \mathbb{R}_+^* \to (1,\infty)$ defined by

$$\theta(t) = \begin{cases} e^{e^{-\frac{1}{t}}}, & 0 < t \leq e, \\ \sqrt{t}, & t > e, \end{cases} \tag{4.4.8}$$

and $k = 2e^{-\tau} \in (0,1)$ (since $\tau \in (\ln 2, \infty)$).

Thus, all the conditions of Corollary 4.3.1 are fulfilled. Consequently, the functional equation (4.4.1) has a unique solution in $B(W)$. ■

Remark 4.4.1 *It worth noting that the function θ, given by (4.4.8), does not satisfy both conditions (θ_1) and (θ_3).*

Example 4.4.1 *Consider the following functional equation*

$$u(x) = \sup_{y\in\mathbb{R}} \left\{ arctg\left(\frac{x}{1+x|y|}\right) + 1 + x + \frac{1}{1+|y|} + \frac{\left|u\left(\frac{x}{1+x+|y|}\right)\right|}{1+(4\ln(2))u^2\left(\frac{x}{1+x+|y|}\right)} \right\}. \quad (4.4.9)$$

It is easy to see that equation (4.4.9) is a particular case of equation (4.4.1) with $W = [0,1]$, $D = \mathbb{R}$ and

- $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x,y) = arctg\left(\frac{x}{1+x|y|}\right);$$

- $H : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *defined by*

$$H(x,y,z) = 1 + x + \frac{1}{1+|y|} + \frac{|z|}{1+4\ln(2)z^2};$$

- $\varphi : [0,1] \times \mathbb{R} \to [0,1]$ *given by*

$$\varphi(x,y) = \frac{x}{1+x+|y|}.$$

We have immediately

$$|f(x,y)| \le \frac{\pi}{4} \quad \text{for any } (x,y) \in [0,1] \times \mathbb{R},$$

and

$$|H(x,y,0)| \le 3 \quad \text{for any } (x,y) \in [0,1] \times \mathbb{R}.$$

Hence condition (i) of Theorem 4.4.1 is satisfied.

Now, we are going to prove that condition (ii) of Theorem 4.4.1 is satisfied. For arbitrary $t, s \in \mathbb{R}$, we have

$$\begin{aligned}
|H(x,y,t) - H(x,y,s)| &= \left|\frac{|t|}{1+t^2 4\ln(2)} - \frac{|s|}{1+s^2 4\ln(2)}\right| \\
&\le \frac{|t-s|\,|1-|ts|\,4\ln(2)|}{(1+t^2 4\ln(2))(1+s^2 4\ln(2))} \\
&= \frac{|t-s|\,|1-|ts|\,4\ln(2)|}{\sqrt{(1+t^2 4\ln(2))^2(1+s^2 4\ln(2))^2}} \\
&\le \frac{|t-s|\,|1-|ts|\,4\ln(2)|}{|1+ts4\ln(2)|\sqrt{1+(t-s)^2 2\ln(2)}} \\
&\le \frac{|t-s|}{\sqrt{1+(t-s)^2 2\ln(2)}},
\end{aligned}$$

where we have used the following inequalities

$$\left(1+a^2\right)\left(1+b^2\right) \geq (1+ab)^2\,, \quad \textit{for any } a,b \in \mathbb{R}$$

and

$$\left(1+a^2\right)\left(1+b^2\right) \geq 1+\frac{1}{2}(a-b)^2\,, \quad \textit{for any } a,b \in \mathbb{R}.$$

Thus, condition (ii) *of Theorem 4.4.1 is satisfied for* $p=2$ *and* $\tau = 2\ln(2)$. *Hence all the conditions of Theorem 4.4.1 are fulfilled. Therefore, the functional equation* (4.4.9) *has a unique solution* u *in* $B([0,1])$.

Conclusion and perspectives

Dans cette thèse, on s'est focalisé sur l'étude de certaines équations fonctionnelles émanant de la programmation dynamique. L'aspect dominant pour réaliser nos résultats d'existence, d'unicité et d'approximation est la théorie du point fixe. Nous avons donc ustilisé, soit des théorèmes du point fixe existant dans la litérature adaptès à nos problèmes, soit de nouveaux résultats (que nous avions démontrés via des contractions généralisées dans des espaces adéquats pour les appliquer ensuite pour nos équations fonctionnelles choisies. Comme perspectives, on peut essayer de voir les éventuelles généralisations pour un système d'équations fonctionnelles provenant de la programmation dynamique via des thèoremes du point fixe commun.

Résumé

Dans cette thèse, on s'intéresse de manière approfondie à quelques propriétés des équations fonctionnelles émanant de la programmation dynamique. Il s'agit essentiellement d'établir quelques des résultats d'existence et d'unicité et d'approximations itératives via de nouveaux résultats dans la théorie du point fixe.

Abstract

In this thesis , we deeply focus on some properties of functional equations arising in dynamic programming. Roughly speaking, we establish some results on existence, uniqueness and iterative approximations via new results in fixed point theory.

ملخص

في هذه الرسالة، نهتم بصفة معمقة ببعض خصائص المعادلات التحليلية الموجودة في البرمجة الديناميكية .بصفة دقيقة، نقوم ببرهنة نتائج الوجود و الوحدانية و التقريبات التكرارية بواسطة نتائج جديدة في نظرية النقطة الثابتة.

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