Abbas Laghrour Khenchela University
Faculty of Science and Technology
Department of Electrical Engineering

Numerical Methods
2GE +2GP
Duration : 1h 30

Final exam

Exercise 1 (6 point)

We consider the equation $x^3 - \cos x = 1$

1. Solve graphically this equation.
2. Choose a very short interval contains the root.
3. Check that this interval really contains the root.
    a. Solve the previous equation by the Dichotomy method in the interval [1, 1.5], we give the number n=7 (0,1,2,...,7) of iterations and the precision $\varepsilon=10^{-3}$.

Exercise 2 (6 pts) :

Determine the number of points ***n*** (number of subintervals) that must be used in the Simpson method, general form, to approximate the integral $I = \int_1^2 x\ln(x)dx$ with a precision of $\mathbf{10^{-3}}$ ($\varepsilon=\mathbf{10^{-3}}$).

Exercise 3 (8 pts) **:**

Find the equation of a parabola $y = ax^2 + bx + c$ that passes through the points (1, 2), (2, 9) and (-1, 6) by Jordan's method (a, b and c are unknowns of a system of equations).

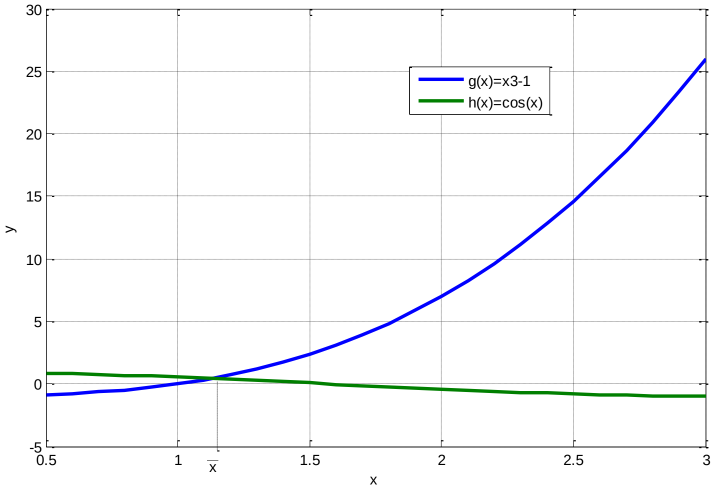
Good Luck

Typical Exam Answer Key

Exercise 1:

1. Locate graphically the root of this equation

We write the functions $f(x)=\left(x^3-1\right)-\cos x$ in the form of the difference of two known functions $f(x)=g(x)-h(x)=(x^3-1)-\cos x$.



From this graph we can easily notice that: $\bar{x}=1.127$

2. According to the graph the root belongs to the interval [1, 1.5].

3. Checking that the intervals obtained contain the roots

We have: $\left.\begin{array}{l} f(1) = -0.54 \\ f(1.5) = 2.3 \end{array}\right\}$, so $f(1)\times f(1.5)\langle 0$. According to the intermediate value theorem the equation f(x)=0 has a root in this interval.

4. Solving the previous equation by the dichotomy method, with $\varepsilon=10^{-3}$

The number of iterations n to satisfy this condition is: $n \geq \dfrac{\ln(\frac{b-a}{\varepsilon})}{\ln(2)} = 6.21$, we take n=7.

The solution by the Bisection method is: $x_n = \dfrac{a_n + b_n}{2}$ where n=0,1,2,..7. We then construct the following table:

i	*a_i*	*x_i*	*b_i*	*f(x_i)*
0	Θ1.000	⊕ 1.250	⊕1.5	0.637
1	Θ1.000	Θ 1.125	⊕1.250	-0.0073
2	Θ1.125	⊕1.188	⊕1.250	0.3006
3	Θ1.125	⊕1.156	⊕1.188	0.1443
4	Θ1.125	⊕1.140	⊕1.156	0.069
5	Θ1.125	⊕1.132	⊕1.140	0.0305
6	Θ1.125	⊕1.128	⊕1.132	0.012
7	Θ1.125	1.127	⊕1.128	0.0021

For $\varepsilon = 10^{-3}$, we retain the solution $\bar{x} = x_7 = 1.127$.

Exercise 2 :

We choose the number of sub-intervals n using the increase : $M_4 \frac{(b-a)^5}{180n^4} \le \varepsilon$ with

$$M_4 = \max_{x\in[1,2]} \left|f^{(4)}(x)\right| = \max_{x\in[1,2]} \left|\frac{2}{x^3}\right| = \frac{2}{1} = 2, \text{ so } n \ge \left(\sqrt[4]{M_4 \frac{(b-a)^5}{180\varepsilon}} = 1.8257\right)$$

We take $\mathbf{n=2}$.

The integration step is: $h = \frac{b-a}{n} = 0.5$

We apply Simpson's formula directly for n=2

$$I = \int_1^2 f(x)dx = \int_1^2 x\ln(x)dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right], \text{ with } x_1 = \frac{x_0 + x_2}{2} = 1.5$$

$$I = \int_1^2 f(x)dx = \int_1^2 x\ln(x)dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] = 0.6365$$

Note that we can integrate this function using the integral by parts.

$$I = \int_1^2 x\ln(x)dx = \left[\frac{x^2\ln(x)}{2} - \frac{x^2}{4}\right]_1^2 = 0.6363.$$

Exercise 3:

Determining the coefficients a, b and c leads to the resolution of the following system:

$$\begin{cases} a+b+c=2 \\ 4a+2b+c=9 \\ a-b+c=6 \end{cases} \quad \rightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \\ 6 \end{bmatrix}$$

Solving the system using Jordan's method

First step k=1

$$a_{11}^{(2)} = \frac{a_{11}^{(1)}}{a_{11}^{(1)}} = 1, a_{12}^{(1)} = \frac{a_{12}^{(1)}}{a_{11}^{(1)}}, \ldots, a_{1j}^{(1)} = \frac{a_{1j}^{(1)}}{a_{11}^{(1)}}, \quad j = 1, \ldots, 4$$

$$a_{22}^{(2)} = a_{22}^{(1)} - a_{21}^{(1)} a_{12}^{(2)} = -2 \,, \qquad a_{23}^{(2)} = a_{23}^{(1)} - a_{21}^{(1)} a_{13}^{(2)} = -3$$

$$b_{2}^{(2)} = b_{2}^{(1)} - a_{21}^{(1)} b_{2}^{(2)} = 1, \qquad a_{32}^{(2)} = a_{32}^{(1)} - a_{31}^{(1)} a_{12}^{(2)} = -2$$

$$a_{33}^{(2)} = a_{33}^{(1)} - a_{31}^{(1)} a_{13}^{(2)} = 0 \,, \qquad b_{3}^{(2)} = b_{3}^{(1)} - a_{31}^{(1)} b_{3}^{(2)} = 4$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 9 \\ 1 & -1 & 1 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & -3 & 1 \\ 0 & -2 & 0 & 4 \end{bmatrix}$$

$$\left[A^{(1)},\ b^{(1)} \right] \rightarrow \left[A^{(2)},\ b^{(2)} \right]$$

Second step k=2

$$a_{22}^{(3)} = \frac{a_{22}^{(2)}}{a_{22}^{(2)}} = 1, a_{23}^{(3)} = \frac{a_{23}^{(2)}}{a_{22}^{(2)}} = \frac{3}{2}, b_{2}^{(3)} = \frac{b_{2}^{(2)}}{a_{22}^{(2)}} = -\frac{1}{2}$$

$$b_{1}^{(3)} = b_{1}^{(2)} - a_{12}^{(2)} b_{1}^{(3)} = \frac{5}{2}, \qquad a_{13}^{(3)} = a_{13}^{(2)} - a_{12}^{(2)} a_{23}^{(3)} = -\frac{1}{2}$$

$$a_{33}^{(3)} = a_{33}^{(2)} - a_{32}^{(2)} a_{23}^{(3)} = 3, \qquad a_{34}^{(3)} = a_{34}^{(2)} - a_{32}^{(2)} a_{24}^{(3)} = 3$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & -3 & 1 \\ 0 & -2 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/2 & 5/2 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$$\left[A^{(2)},\ b^{(2)} \right] \rightarrow \left[A^{(3)},\ b^{(3)} \right]$$

Third step k=3

$$a_{33}^{(4)} = \frac{a_{33}^{(3)}}{a_{33}^{(3)}} = 1, b_{3}^{(4)} = \frac{b_{3}^{(3)}}{a_{33}^{(3)}} = 1$$

$b_1^{(4)} = b_1^{(3)} - a_{13}^{(3)} b_1^{(4)} = 3$, $b_2^{(4)} = b_2^{(3)} - a_{13}^{(3)} b_2^{(4)} = -2$

$$\begin{bmatrix} 1 & 0 & -1/2 & 5/2 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\left[A^{(3)},\ b^{(3)}\right] \rightarrow \left[A^{(4)},\ b^{(4)}\right]$$

After Jordan transformation we obtain: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

So the solution of this system is: $\begin{pmatrix} a = 3 \\ b = -2 \\ c = 1 \end{pmatrix}$

Then the equation of this polynomial is: $y = 3x^2 - 2x + 1$.