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ABSTRACT

We study the effects of Snyder–de Sitter commutation relations on relativistic bosons by solving analytically in the momentum space representation the Klein–Gordon oscillator in arbitrary dimensions. The exact bound state spectrum and the corresponding momentum space wave functions are obtained using Gegenbauer polynomials in the one-dimensional space and Jacobi polynomials in the D-dimensional case. Finally, we study the thermodynamic properties of the system in the high-temperature regime where we found that the corrections increase the free energy but decrease the energy, the entropy, and the specific heat that is no longer constant. This work extends the part concerning the Klein–Gordon oscillator for the Snyder–de Sitter case studied in two-dimensional space by Falek *et al.* [J. Math. Phys. **60**, 013505 (2019)].

I. INTRODUCTION

The relativistic harmonic oscillator (RHO) is one of the most used, among fundamental physics models, in experimental studies to explain confinement since it is characterized by the bound states with non-zero residual energy. For example, in the nuclear physics domain, RHO is the central potential of the nuclear shell model, and it has been used also as the confining two-body potential for quarks in particle physics. Since the article of Moshinsky and Szczepaniak,[1] the Dirac oscillator (DO) raised considerable attention by many researchers and thus has been studied intensively.[2–6] In addition, the study of this model recorded an extension to boson cases such as the Klein–Gordon Oscillator (KGO) for spin 0 bosons[7,8] and Duffin–Kemmer–Petiau Oscillator (DKPO) for spin 0 and 1 particles.[9–12] We also find the KGO in arbitrary dimensions, which is established in some series of articles.[13,14]

When talking about dimensions other than three, the harmonic oscillator is even more interesting because of the correspondence that exists between it and the Coulomb potential found in 2D systems[15,16] or between a 4D harmonic oscillator and a 3D Coulomb potential[17,18] (and the references therein). It is also used for modeling Landau levels in topological insulators,[19–21] and we mention here the recent realization of a 4D spinless topological insulator through the achievement of an explicit construction of a 2D electric circuit lattice;[22] this opens the way to realistic and ideal platforms to create higher-dimensional topological states in the laboratory.

On the other hand, there have been many attempts to study deformed quantum mechanical systems as they have a significant impact on the absorption of infinities that lie in the standard field theories. This was initially proposed in the Snyder model that has suggested that the measurement in noncommutative quantum mechanics can be governed by a generalized uncertainty principle (GUP).[23] Therefore, the fundamental length scale is supposed to be of the order of the Planck length,[24] which, in turn, leads to a minimum uncertainty in the position measurement. This approach is motivated by several physics theories such as noncommutative geometries,[25] doubly special relativity (DSR),[26] string theories,[27] and black hole physics.[28]

It is also legitimate to seek to incorporate the other pillar of modern physics, namely, the theory of gravitation, in this construction, through its most striking manifestation, which is curved space–time. Therefore, many attempts have been made to develop the equivalent of the Snyder model in the study of quantum mechanics in curved space–time by looking for some generalizations of the Heisenberg algebra by adding small corrections to the commutation relations such us the GUP[29] as well as the extended uncertainty principle (EUP),[30] with the main aim of integrating the combined effects of the non-commutative geometry and the theory of gravity in quantum mechanics. Recently, certain relativistic and non-relativistic problems have been solved within this framework of the curved Snyder model and we cite as examples: the Schrödinger oscillator system, the non-relativistic free particle, and the DO.[31,32]

The main purpose of this work is to investigate the deformed quantum formulation of the KGO in arbitrary dimensions in a deformed space obeying the Snyder–de Sitter (SdS) algebra with an emphasis on the determination of the thermodynamic functions that play a significant role in understanding the physical properties. We mention here the study of the thermodynamic properties of the ordinary DO that has received a great amount of attention for their description of the quark–gluon plasma model,[33–35] the deformed DO with a minimal length in a thermal bath,[36] where it was shown that there is an upper bound for the minimal length, and the one-dimensional (1-dim) DKPO with the SdS modelthat has significant importance in heavy quark systems.[37] We extend in this work the part concerning the KGO case in the SdS space studied in 2D in Ref. 38.

The outline of this paper is as follows: In Sec. II, we give a review of the SdS model. In Sec. III, we solve exactly the one-dimensional KG equation for the oscillator-like interaction with the deformed SdS algebra in the momentum space representation. We obtain the exact wave function and energy spectrum for this system. We discuss also some special cases and the corresponding numerical results. We extend the same study to an arbitrary dimension in Sec. IV where, by a straightforward calculation, we deduce the normalized wave functions and the energy spectrum. In the regime of high temperatures, the thermodynamic properties of the system are investigated and discussed numerically in Sec. V. The concluding remarks are given in Sec. VI.

II. REVIEW OF THE DEFORMED QUANTUM MECHANICS RELATION

In the non-relativistic SdS model, the deformed Heisenberg algebra in the three-dimensional case is defined by the following commutation relation:[31,32]

$$[X_i, P_j] = i\hbar\big(\delta_{ij} + \alpha_1 X_i X_j + \alpha_2 P_i P_j + \sqrt{\alpha_1\alpha_2}(X_i P_j + P_i X_j)\big), \tag{1}$$

$$[X_i, X_j] = i\hbar\alpha_2\xi_{ijk}L_k, [P_i, P_j] = i\hbar\alpha_1\xi_{ijk}L_k, \tag{2}$$

where α_1 and α_2 are the small positive parameters defining the deformations coming from the Snyder algebra and de Sitter space, respectively, and $L_k = \xi_{ijk}X_iP_j$ denotes the components of the angular momentum operator satisfying the usual algebra,

$$[L_i, X_j] = i\hbar\xi_{ijk}X_k, [L_i, P_j] = i\hbar\xi_{ijk}P_k, [L_i, L_j] = i\hbar\xi_{ijk}L_k. \tag{3}$$

In the same manner as in ordinary quantum mechanics, the commutation relation (1) gives rise to uncertainty Heisenberg relations [$\gamma_{ij} = \big(\sqrt{\alpha_1}\langle X_i\rangle + \sqrt{\alpha_2}\langle P_j\rangle\big)^2 \geqslant 0$]

$$\triangle X_i \triangle P_j \geq \frac{\hbar}{2}\Big(\delta_{ij} + \gamma_{ij} + \alpha_1(\Delta X_i)^2 + \alpha_2(\Delta P_i)^2 - 2\sqrt{\alpha_1\alpha_2}\Delta X_i\Delta P_j\Big), i = 1, 2, 3. \tag{4}$$

For its part, relation (2) implies the appearance of a nonzero minimal length in position and momentum uncertainties,

$$(\Delta X)_{\min} = \hbar\sqrt{\frac{\alpha_2(1+\gamma)}{1+2\sqrt{\alpha_1\alpha_2}}}, (\Delta P)_{\min} = \hbar\sqrt{\frac{\alpha_1(1+\gamma)}{1+2\sqrt{\alpha_1\alpha_2}}}. \tag{5}$$

The noncommutative operators X_i and P_i satisfying the SdS algebra (1) correspond to the rescaled uncertainty relation (4) in position and momentum space. In order to study the quantum mechanical problems, we represent these operators as functions of the operators x_i and p_i, satisfying the usual canonical commutation relations of the ordinary quantum mechanics. However, because of the noncommutative relations (2), there is no space or momentum representation. We write X_i and P_i according to p_i and ∂_{p_i} with the following transformations and we use the term "momentum" representation for these transformations:

$$X_i = i\hbar\sqrt{1-\alpha_2 p^2}\partial_{p_i} + \lambda\sqrt{\frac{\alpha_2}{\alpha_1}}\frac{p_i}{\sqrt{1-\alpha_2 p^2}}, \tag{6}$$

$$P_i = -i\hbar\sqrt{\frac{\alpha_2}{\alpha_1}}\sqrt{1-\alpha_2 p^2}\partial_{p_i} + (1-\lambda)\frac{p_i}{\sqrt{1-\alpha_2 p^2}}, \tag{7}$$

where p varies in the domain $]-1/\sqrt{\alpha_2}, 1/\sqrt{\alpha_2}[$ and λ is an arbitrary real constant.

III. ONE-DIMENSIONAL KLEIN–GORDON OSCILLATOR

The stationary equation describing the KGO in one-dimensional (1-dim) space is written à *la* Moshinsky:[1]

$$\big[c^2(P + im\omega X)(P - im\omega X) + m^2c^4 - E^2\big]\psi(p) = 0, \tag{8}$$

where m is the mass of the particle, ω is the classical oscillator frequency, and c is the speed of light.

Using the commutation relation (1) and the momentum space realization of the operators X and P, we get

$$\left[m^2\omega^2\left(1-\frac{\hbar\alpha_1}{m\omega}\right)X^2+(1-m\omega\hbar\alpha_2)P^2-m\omega\hbar\sqrt{\alpha_1\alpha_2}\,(PX+XP)-\varepsilon\right]\psi(p)=0, \tag{9}$$

with $\varepsilon = m\hbar\omega + \left(E^2 - m^2c^4\right)/c^2$.

We use the definition of SdS algebra from (6) and (7), and we rewrite this equation in the deformed momentum space,

$$\left\{\hbar^2\frac{\alpha_1}{\alpha_2}\gamma\gamma^*\left(\sqrt{1-\alpha_2p^2}\frac{\partial}{\partial p}\right)^2-\frac{2i\hbar\Omega}{\alpha_1\alpha_2}p\frac{\partial}{\partial p}-\frac{\eta\alpha_2p^2}{1-\alpha_2p^2}+\epsilon\right\}\psi(p)=0 \tag{10}$$

with

$$\gamma=\left(1+im\omega\sqrt{\frac{\alpha_2}{\alpha 1}}\right),\Omega=\alpha_1\left(\gamma\gamma^*\lambda-1\right),$$
$$\eta=\frac{1-\lambda}{\alpha_2}-m\omega\hbar+\left(\frac{\lambda}{\alpha_1\alpha_2}+\frac{i\hbar}{\sqrt{\alpha_1\alpha_2}}\right)\Omega,\epsilon=\varepsilon-\frac{i\hbar\Omega}{\sqrt{\alpha_1\alpha_2}}. \tag{11}$$

Now, in order to solve (10), we use the following change in the variable p:

$$p\to\rho=\frac{1}{k}\arcsin\left(\sqrt{\alpha_2}p\right)\text{ and }k=\hbar\sqrt{\alpha_1\gamma\gamma^*}. \tag{12}$$

Hence, the equation becomes ($\delta = -i\hbar\Omega/k\alpha_1\alpha_2$)

$$\left\{\frac{\partial^2}{\partial\rho^2}+2\delta\tan(k\rho)\frac{\partial}{\partial\rho}-\eta\tan^2(k\rho)+\epsilon\right\}\psi(\rho)=0. \tag{13}$$

We use the following transformation:

$$\psi(\rho)=\left(1-u^2\right)^{\frac{1}{2}\left(\nu+\frac{\delta}{k}\right)}f(u), \tag{14}$$

where ν is a constant to be determined later and $u = \sin(k\rho)$. By means of this substitution (14), the differential equation for $f(u)$ reduces to the following form:

$$\left[\left(1-u^2\right)\frac{\partial^2}{\partial u^2}-(2\nu+1)u\frac{\partial}{\partial u}+\frac{\epsilon}{k^2}-\left(\nu+\frac{\delta}{k}\right)\right]f(u)=0, \tag{15}$$

where we have chosen

$$\nu(\nu-1)=\frac{\delta}{k}\left(\frac{\delta}{k}+1\right)+\frac{\eta}{k^2}. \tag{16}$$

In order to avoid complex eigenvalues of ϵ, we must impose the condition $\Omega = 0$ to eliminate the imaginary part in (11); this fixes the value of the arbitrary parameter λ,

$$\lambda=\frac{1}{\gamma\gamma^*}=\frac{\alpha_1}{m^2\omega^2\alpha_2+\alpha_1}. \tag{17}$$

This also brings the differential equation (15) to the Gegenbauer form[39]

$$\left[\left(1-u^2\right)\frac{\partial^2}{\partial u^2}-(2\nu+1)u\frac{\partial}{\partial u}+n(n+2\nu)\right]f(u)=0, \tag{18}$$

where n and ν satisfy the relations

$$\frac{\varepsilon}{k^2}-\nu=n(n+2\nu)\text{ and }\nu(\nu-1)=\frac{\eta}{k^2}=\frac{1}{k^2}\left(\frac{m^2\omega^2}{m^2\omega^2\alpha_2+\alpha_1}-m\hbar\omega\right). \tag{19}$$

Solving the second relation for ν gives

$$\nu=\frac{1}{2}\left(1\pm\sqrt{1+\frac{4}{k^2}\left(\frac{m^2\omega^2}{m^2\omega^2\alpha_2+\alpha_1}-m\omega\hbar\right)}\right). \tag{20}$$

From expression (14), we see that $f(u)$ should be non-singular at $u = \pm 1$, and so the right value of ν is

$$\nu = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{k^2}\left(\frac{m^2\omega^2}{m^2\omega^2\alpha_2 + \alpha_1} - m\omega\hbar\right)}\right). \tag{21}$$

Now, the solution of (18) can be expressed in terms of Gegenbauer polynomials,

$$f(u) = \Lambda C_n^{\nu}(u). \tag{22}$$

Consequently, the expression of the wave function $\psi(\rho)$ is

$$\psi_n(\rho) = \Lambda\left(1 - u^2\right)^{\nu/2} C_n^{\nu}(u) = \Lambda\left(1 - \alpha_2 p^2\right)^{\nu/2} C_n^{\nu}\left(\sqrt{\alpha_2}p\right). \tag{23}$$

We can obtain the normalization constant Λ by applying the deformed normalization condition,

$$\int_{-\infty}^{+\infty} \frac{dp}{\left(1 - \alpha_2 p^2\right)^{1/2}} \psi^*(p)\psi(p) = 1. \tag{24}$$

Then, using the following identity:[39]

$$\int_{-1}^{+1} dq\left(1 - q^2\right)^{\nu - \frac{1}{2}}\left[C_n^{\nu}(q)\right]^2 = \frac{\pi 2^{1-2\nu}\Gamma(2\nu + n)}{n!(n + \nu)[\Gamma(\nu)]^2}, \tag{25}$$

we get

$$\Lambda = \frac{2^{\nu}\alpha_2^{1/4}}{\sqrt{2\pi}}\sqrt{\frac{n!(n + \nu)[\Gamma(\nu)]^2}{\Gamma(2\nu + n)}}. \tag{26}$$

This completes the determination of the wave functions (23).

Let us now check these solutions by studying the limits $\alpha_1 \to 0$ and $\alpha_2 \to 0$ from (21). Using the following relations:[39]

$$\lim_{\nu\to\infty} \nu^{-\frac{n}{2}} C_n^{\frac{\nu}{2}}\left(x\sqrt{\frac{2}{\nu}}\right) = \frac{1}{\sqrt{2^n}n!} H_n(x), \ \lim_{\nu\to\infty} \frac{\Gamma(\nu + a)}{\Gamma(\nu)}\nu^{-a} = 1 \tag{27}$$

with the doubling formula

$$\Gamma(2\nu) = \frac{2^{2\nu - 1}}{\sqrt{\pi}}\Gamma(\nu)\Gamma\left(\nu + \frac{1}{2}\right) \tag{28}$$

and limiting ourselves to the first order of α_2 $[\left(1 - \alpha_2 p^2\right)^{\nu/2} = \exp\left(-p^2/2m\omega\hbar\right)]$, we obtain directly the momentum space eigenfunction of the usual KGO (without deformation),

$$\psi_n(p) = \frac{1}{\sqrt{2^n}n!}\left(\frac{1}{\pi m\omega\hbar}\right)^{1/4} \exp\left(-\frac{p^2}{2m\omega\hbar}\right) H_n\left(\frac{p}{\sqrt{m\omega\hbar}}\right). \tag{29}$$

Now, to seek the energies, we employ Eq. (20) in the first relation of (18) and the expression of ε. It is straightforward to show that the deformed KGO energy spectrum E_n is

$$E_n = \pm mc^2\sqrt{1 + \frac{2\omega\hbar}{mc^2}n + \frac{\hbar^2\left(m^2\omega^2\alpha_2 + \alpha_1\right)}{m^2c^2}n^2}, n \in \mathbb{N}. \tag{30}$$

Due to the modification of the Heisenberg algebra, these expressions of the energy spectrum contain an additional correction term coming from the deformation, and its values increases with both parameters α_1 and α_2. Here, it should be noted that, according to the n^2 dependence of the energy levels that explains confinement at the high energy area, our result is similar to the energies of a spinless relativistic quantum particle moving in a square well potential whose boundaries are placed at $\pm\pi/2\sqrt{m^2\omega^2\alpha_2 + \alpha_1}$.

The shape of our energy spectrum can be tested as follows: Using the limit $\alpha_1 \to 0$, we obtain the same results as in the 1-dim KGO case in the presence of minimal length[13] (with $\beta = \alpha_2$ and $\beta' = 0$). Subsequently, when the deformed parameters are absent, i.e., $\alpha_1 = \alpha_2 = 0$, the result is strictly consistent with the KGO in 1-dim.[40]

Another interesting characteristic appears in our results when computing the energy levels spacing; this difference becomes constant for large values of n,

$$\lim_{n\to\infty} |\Delta E_n| = \hbar c\sqrt{m^2\omega^2\alpha_2 + \alpha_1} = \hbar\omega mc^2\sqrt{\theta} \text{ with } \theta = \frac{1}{c^2}\left(\frac{\alpha_1}{m^2\omega^2} + \alpha_2\right). \tag{31}$$

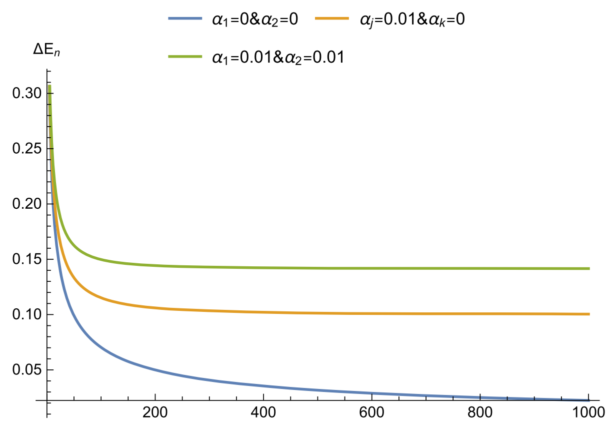

FIG. 1. Spectrum spacing ΔE_n with and without deformations.

In order to show this behavior, we have plotted the energy levels spacing vs the quantum number n for different values of the deformation parameters in Fig. 1 (we use the units $\hbar = c = m = 1$ and we put $\omega = 1$).

We see from Fig. 1 that when α_1 and α_2 tend to 0, the spacing ΔE_n between energy levels tends to zero for large values of n, i.e., the energy spectrum becomes continuous at this limit in the ordinary case (without deformation). In contrast, this continuous feature of the spectrum disappears and reduces to a bounded spectrum in the deformed case with a spacing value proportional to the deformation parameter θ. The smallness of this parameter θ makes that the spectrum appear almost continuous in the deformed case, and this also explains its continuous nature in the ordinary case.

Now, to obtain an upper bound on the parameters of deformation α_1 and α_2, we use (30) and we expand up to the first order in θ,

$$E_n = mc^2\sqrt{1+\frac{2\omega\hbar n}{mc^2}} + \frac{\hbar^2\omega^2 mc^2 n^2}{2\sqrt{1+\frac{2\omega\hbar n}{mc^2}}}\theta. \tag{32}$$

The deviation of the nth energy level from the usual case caused by the modified commutation relations (1) is given by

$$\frac{\Delta E_n}{\hbar\omega} = \frac{\hbar\omega mc^2 n^2}{2\sqrt{1+\frac{2\omega\hbar n}{mc^2}}}\theta. \tag{33}$$

We use the experimental results of the cyclotron motion of an electron in a Penning trap.[41] In the absence of deformations, the cyclotron frequency of an electron of mass m_e trapped in a magnetic field of strength B is $\omega_c = eB/m_e$; therefore, for a magnetic field of strength $B = 6T$, we have $m_e\hbar\omega_c = e\hbar B = 10^{-52}\ \mathrm{kg}^2\ \mathrm{m}^2\ \mathrm{s}^{-2}$. At this stage, if we assume that at the level $n = 10^{10}$, only deviations of the scale of $\hbar\omega_c$ can be detected, and by taking $\Delta E_n < \hbar\omega_c$ (no perturbation of the nth energy level is observed),[42] we can write the following constraint:

$$\theta < 10^{33}\,\mathrm{c}^{-2}\,\mathrm{kg}^{-2}\,\mathrm{m}^{-2}\,\mathrm{s}^{2}. \tag{34}$$

This leads to the following upper bounds according to the parameters of deformation α_1 and α_2; for $\alpha_1 \neq 0$ and $\alpha_2 = 0$, we get

$$\Delta X_{\min} = \hbar\sqrt{\alpha_2} < 3.33\times 10^{-18}\ \mathrm{m}, \tag{35}$$

and for $\alpha_1 = 0$ and $\alpha_2 \neq 0$, we have

$$\Delta P_{\min} = \hbar\sqrt{\alpha_1} < 3.17\times 10^{-36}\ \mathrm{kg\,m\,s}^{-1}. \tag{36}$$

For the non-relativistic limit, by setting $E = mc^2 + E_{nr}$ with the assumption that $mc^2 \gg E_{nr}$, we write the spectrum of the non-relativistic KGO in the deformed SdS space,

$$E_{nr} = n\hbar\omega\left(1 + n\frac{\hbar}{2m\omega}\left(m^2\omega^2\alpha_2 + \alpha_1\right)\right). \tag{37}$$

IV. D-DIMENSIONAL KLEIN–GORDON OSCILLATOR

In this section, we consider the stationary relativistic equation describing the D-dimensional (D-dim) KGO in the momentum representation,

$$\left[c^2(\mathbf{P} + im\omega\mathbf{X})(\mathbf{P} - im\omega\mathbf{X}) + m^2c^4 - E^2\right]\psi(\mathbf{P}) = 0, \tag{38}$$

which can be written with the help of commutation relation (1) as

$$\left[\left(m^2\omega^2 - m\omega\hbar\alpha_1\right)\mathbf{X}^2 + (1 - m\omega\hbar\alpha_2)\mathbf{P}^2 - m\omega\hbar\sqrt{\alpha_1\alpha_2}\,(\mathbf{PX} + \mathbf{XP}) - \varepsilon^{\grave{}}\right]\psi(\mathbf{P}) = 0, \tag{39}$$

where

$$\varepsilon^{\grave{}} = \frac{E^2 - m^2c^4}{c^2} + Dm\omega\hbar. \tag{40}$$

We use the following separation of the wave function into angular and radial parts:

$$\psi(\mathbf{P}) = Y^{m_l}_{l_{(D-1)},\ldots,l_2 l_1}(\mathbf{P})\varphi(p), \tag{41}$$

where $Y^{m_l}_{l_{(D-1)},\ldots,l_2 l_1}(\mathbf{P})$ are D-dim ultra-spherical harmonics; this enables us to make the following replacements in the momentum space (l is the orbital quantum number):

$$\sum_{i=1}^{D}\frac{\partial^2}{\partial p_i^2} = \frac{\partial^2}{\partial p^2} + \frac{D-1}{p}\frac{\partial}{\partial p} - \frac{L^2}{p^2}, \sum_{i=1}^{D}p_i\frac{\partial}{\partial p_i} = p\frac{\partial}{\partial p}, L^2 = l(l + D - 2). \tag{42}$$

Using the definition of the momentum space realization of the operators $\mathbf{X}$ and $\mathbf{P}$ in (6) and (7), we obtain the following equation in the deformed momentum space:

$$\left\{\begin{array}{c}\hbar^2\dfrac{\alpha_1}{\alpha_2}\gamma\gamma^*\left[\left(\sqrt{1-\alpha_2p^2}\dfrac{\partial}{\partial p}\right)^2 + \left(1-\alpha_2p^2\right)\left(\dfrac{D-1}{p}\dfrac{\partial}{\partial p} - \dfrac{L^2}{p^2}\right)\right] \\ -\dfrac{2i\hbar\Omega}{\sqrt{\alpha_1\alpha_2}}p\dfrac{\partial}{\partial p} - \dfrac{\eta\alpha_2p^2}{1-\alpha_2p^2} + \epsilon^{\grave{}}\end{array}\right\}\varphi(p) = 0, \tag{43}$$

where the expressions of γ, Ω, and η are given in (11) and ε' is defined by

$$\epsilon^{\grave{}} = \varepsilon^{\grave{}} - \frac{Di\hbar\Omega}{\sqrt{\alpha_1\alpha_2}}. \tag{44}$$

Now, in order to solve (43), we use the same transformation (12) of the 1-dim case to get

$$\left\{\begin{array}{c}\dfrac{\partial^2}{\partial\rho^2} + \left(k(D-1)\cot(k\rho) - \dfrac{2i\hbar\Omega}{k\sqrt{\alpha_1\alpha_2}}\tan(k\rho)\right)\dfrac{\partial}{\partial\rho} \\ -k^2l(l+D-2)\cot^2(k\rho) - \eta\tan^2(k\rho) + \epsilon^{\grave{}}\end{array}\right\}\varphi(\rho) = 0. \tag{45}$$

At this point, in order to avoid complex eigenvalues of the energies, we must also impose the same condition as in the 1-dim case (17) to eliminate the imaginary term; this transforms the master equation (45) to the following equation:

$$\left\{\begin{array}{c}\dfrac{\partial^2}{\partial\rho^2} + k(D-1)\cot(k\rho)\dfrac{\partial}{\partial\rho} - k^2\,l(l+D-2)\cot^2(k\rho) \\ -\left(\dfrac{1-\lambda}{\alpha_2} - m\omega\hbar\right)\tan^2(k\rho) + \varepsilon^{\grave{}}\end{array}\right\}\varphi(\rho) = 0. \tag{46}$$

In order to simplify this equation, we use the following ansatz:

$$\varphi(\rho) = \left(1-q^2\right)^{\mu/2}q^l f(q), \tag{47}$$

where μ is a constant to be determined and $q = sin(k\rho)$.

Doing so, the differential equation for $f(q)$ (46) is written as

$$\left\{\left(1-q^2\right)\frac{\partial^2}{\partial q^2} + \left[-(2\mu + 2l + D)q + \frac{2l+D-1}{q}\right]\frac{\partial}{\partial q} + \frac{\varepsilon^{\grave{}}}{k^2} - (2l-D)\mu - l\right\}f(q) = 0, \tag{48}$$

where μ is chosen to simplify this equation as

$$\mu(\mu-1) - \frac{1}{k^2}\left(\frac{1-\lambda}{\alpha_2} - m\omega\hbar\right) = 0. \tag{49}$$

From expression (47), we see that $f(q)$ should be non-singular when $q = \pm 1$, which implies

$$\mu = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{k^2}\left(\frac{m^2\omega^2}{m^2\omega^2\alpha_2 + \alpha_1} - m\omega\hbar\right)}\right). \tag{50}$$

We note that Eq. (48) possesses three singular points $q = 0, 1, -1$. In order to reduce this equation to a class of known differential equations with a polynomial solution, we use another change in the variable; we write $z = 2q^2 - 1$ and impose the following condition:

$$n_r(n_r + a + b + 1) = \frac{1}{4}\left[\frac{\varepsilon^{`}}{k^2} - (2l - D)\mu - l\right], \tag{51}$$

where n_r is a non-negative integer (radial quantum number) and we have also defined

$$a = \mu - 1/2, b = l - 1 + D/2. \tag{52}$$

Now, the differential equation (48) transforms to the following Jacobi form:

$$\left(1 - z^2\right)\frac{d^2 g}{dz^2} + [(b - a) - (a + b + 2)z]\frac{dg}{dz} + n_r(n_r + 1 + a + b)g(z) = 0. \tag{53}$$

The solutions are given by the Jacobi polynomials $g(z) = P_{n_r}^{(a,b)}(z)$, and so the radial wave function $\varphi_{n_r,l}(p)$ is

$$\varphi_{n_r,l}(p) = N\left(1 - \alpha_2 p^2\right)^{\mu/2}\left(\alpha_2 p^2\right)^{l/2} P_{n_r}^{(a,b)}\left(2\alpha_2 p^2 - 1\right). \tag{54}$$

To determine the normalization constant N, we use the deformed normalization condition in the D-dim space of radial wave functions,

$$\int_0^{1/\sqrt{\alpha_2}} \frac{Dp^{D-1}dp}{\left(1 - \alpha_2 p^2\right)^{1/2}}\psi^*(p)\psi(p) = 1, \tag{55}$$

and the Jacobi polynomial orthogonality relation[39]

$$\int_{-1}^{1} dy(1 - y)^{\alpha}(1 + y)^{\beta}\left[P_{n_r}^{(\alpha,\beta)}(y)\right]^2 = \frac{\pi 2^{\alpha+\beta+1}\Gamma(\alpha + n_r + 1)\Gamma(\beta + n_r + 1)}{n_r!(\alpha + \beta + 1 + 2n_r)\Gamma(\alpha + \beta + n_r + 1)}, \tag{56}$$

and so we get the expression

$$N = \frac{2\alpha_2^{D/4}}{\sqrt{2\pi}}\left[\frac{n_r!\left(2n_r + \mu + l + \frac{D-1}{2}\right)\Gamma\left(n_r + \mu + l + \frac{D-1}{2}\right)}{D\Gamma\left(n_r + \mu + \frac{1}{2}\right)\Gamma\left(n_r + l + \frac{D}{2}\right)}\right]^{1/2}. \tag{57}$$

Now to find the energies, we substitute relations (40) and (50) into the condition (51) and we define the principal quantum number n by $n = 2n_r + l$, and so we obtain the spectrum of the D-dim SdS KGO as follows:

$$E_{n,l} = \pm mc^2\left\{1 + \frac{2\hbar\omega}{mc^2}n + \frac{\hbar^2\left(\alpha_1 + m^2\omega^2\alpha_2\right)}{m^2c^2}\left[n^2 + (D - 1)n - l(l + D - 2)\right]\right\}^{1/2}. \tag{58}$$

The expression shows that we have the same term as in the 1-dim case with a small offset and a new correction containing the orbital quantum number l. It lifts the degeneracy of the energies, which remains, however, $(2l + 1)$. Relation (58) also shows that the contributions coming from the deformations increase with the dimension of the space D but decrease with the quantum number l. This shape of the energy spectrum can be tested by taking the limit $\alpha_1 \to 0$ and $\alpha_2 \to 0$ where we obtain the exact result of the D-dim KGO without deformation.[13,14] It also gives the same spectrum of the two-dimensional case obtained in Ref. 38 (when $B = 0$ in this case later).

We also mention that the energy levels spacing ΔE_n for large values of n in this case is given with the same limit of the 1-dim (31).

V. THERMAL PROPERTIES

In order to obtain the thermodynamic properties of the deformed D-dim KGO with SdS commutation relations, we consider the partition function of the system at finite temperature T according to the Boltzmann factor as follows:

$$Z = \sum_{n=0}^{\infty} e^{-\frac{E_{n,l}}{k_B T}} = \sum_{n=0}^{\infty} \exp\left(-\frac{mc^2}{k_B T}\sqrt{a_1 + a_2 n + a_3 n^2}\right), \tag{59}$$

where k_B is the Boltzmann constant and the other parameters of the expression are defined as follows:

$$a_1 = 1 - a_3 l(l + D - 2), a_2 = \frac{2\omega\hbar}{mc^2} + a_3(D - 1), \text{ and } a_3 = \frac{\hbar^2(\alpha_1 + m^2\omega^2\alpha_2)}{m^2c^2}. \tag{60}$$

To evaluate the function (59), we use the Euler–MacLaurin formula

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}f(0) + \int_0^{\infty} f(x)dx - \sum_{p=1}^{\infty} \frac{1}{(2p)!} B_{2p} f^{(2p-1)}(0), \tag{61}$$

where B_{2p} are the Bernoulli numbers and $f^{(2p-1)}$ designates the derivative of order $(2p-1)$. The integral in the expression is denoted as I and is given as follows:

$$I = \frac{2a_1}{\sqrt{a_2^2 - 4a_1a_3}} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \left(\frac{-4a_1a_3}{a_2^2 - 4a_1a_3}\right)^n \left[\frac{\Gamma(2n+2)}{\chi^{2n+2}} - \frac{e^{-\chi}}{2n+2}\Phi(1, 2n+2, \chi)\right], \tag{62}$$

where $\chi = \frac{mc^2}{k_BT}\sqrt{a_1}$, and we have used the new variable $y = \sqrt{1 + \frac{a_2}{a_1}n + \frac{\alpha_3}{a_1}n^2}$ and also the power series of the square root of the following integral:

$$I` = \frac{2a_1}{\sqrt{a_2^2 - 4a_1a_3}} \int_1^{\infty} dyy\left(1 + \frac{4a_1a_3}{a_2^2 - 4a_1a_3}y^2\right)^{-1/2} e^{-\chi y}. \tag{63}$$

At high temperatures, we note that the contributions of the first and the third terms in expression (61) and that of the second term in (62) become negligible compared to the term in $\chi^{-(2n+2)}$. Therefore, the partition function can be written as

$$Z \simeq \frac{2\left(\frac{k_BT}{mc^2}\right)^2}{\sqrt{a_2^2 - 4a_1a_3}} \sum_{n=0}^{\infty} (-1)^n \Gamma(2n+2) \frac{(2n-1)!!}{(2n)!!}\sigma^n \tag{64}$$

with

$$\sigma = \left(\frac{k_BT}{mc^2}\right)^2 \left(\frac{4a_3}{a_2^2 - 4a_1a_3}\right). \tag{65}$$

Keeping only the orders 0 and 1 contributions in both α_1 and α_2, we get the simplified form

$$Z \simeq \frac{(k_BT)^2}{m\omega\hbar c^2}\left[1 - \left(\frac{\alpha_1}{m^2\omega^2} + \alpha_2\right)\left[\frac{3(k_BT)^2}{c^2}\left(1 - \frac{1}{6}\left(\frac{mc^2}{k_BT}\right)^2\right) + \frac{(D-1)\hbar}{2m\omega}\right]\right]. \tag{66}$$

Considering that $k_BT \gg mc^2$ in the case of high temperatures, we finally obtain the expression of the partition function in this high-temperature regime,

$$Z \simeq \frac{(k_BT)^2}{m\omega\hbar c^2}\left[1 - \theta\left(3(k_BT)^2 + \frac{(D-1)\hbar c^2}{2m\omega}\right)\right], \tag{67}$$

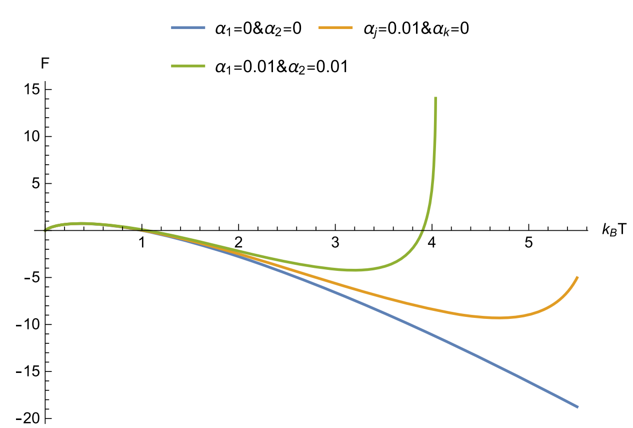

FIG. 2. Effects of the deformations on the free energy F.

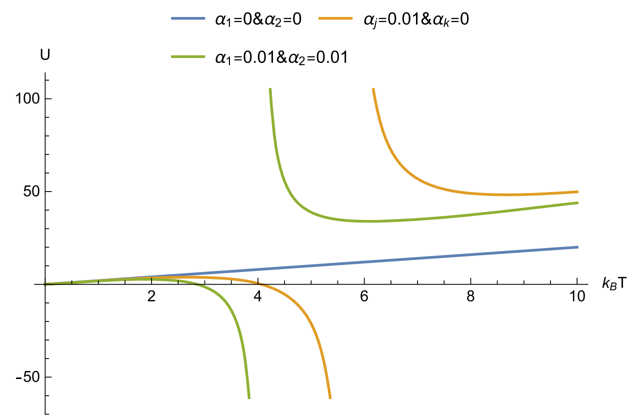

FIG. 3. Effects of the deformations on the energy U.

where the expression of θ is the same as in the 1-dim case (31).

According to their definitions, we can obtain the thermodynamic properties of our system, such as the free energy F, the mean energy U, the specific heat C, and the entropy S, as follows:

$$F = -k_B T \ln Z = -k_B T \ln\left[\frac{(k_B T)^2}{m\omega\hbar c^2}(1-\theta\delta)\right], \tag{68}$$

$$U = k_B T^2 \frac{\partial \ln Z}{\partial T} = 4k_B T\left[1 - \frac{2m\omega - \hbar c^2\theta(D-1)}{4m\omega - 2\theta\left(3m\omega(k_B T)^2 + m\omega\delta\right)}\right], \tag{69}$$

$$C = \frac{\partial U}{\partial T} = 4k_B\left[1 - \frac{\left(2m\omega - \hbar c^2\theta(D-1)\right)\left(4m\omega + 2\theta\left(9m\omega(k_B T)^2 - m\omega\delta\right)\right)}{\left(4m\omega - 2\theta\left(3m\omega(k_B T)^2 + m\omega\delta\right)\right)^2}\right], \tag{70}$$

$$S = -\frac{\partial F}{\partial T} = k_B\left[\frac{2\left(1-\theta\left(3(k_B T)^2+\delta\right)\right)}{(1-\theta\delta)} + \ln\left[\frac{(k_B T)^2}{m\omega\hbar c^2}(1-\theta\delta)\right]\right], \tag{71}$$

where $\delta = 3(k_B T)^2 + \frac{1}{2}(D-1)\hbar c^2 m^{-1}\omega^{-1}$.

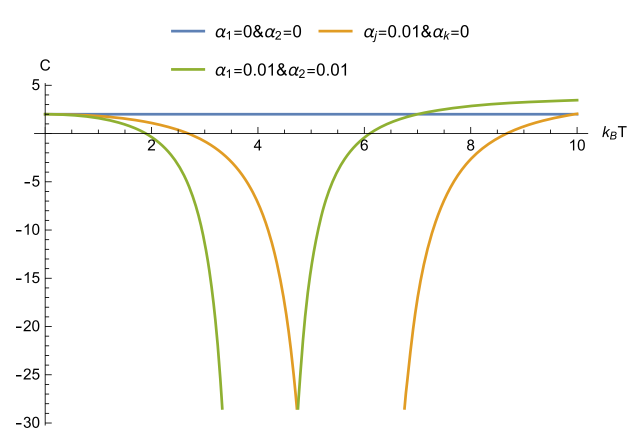

FIG. 4. Effects of the deformations on the heat capacity C.

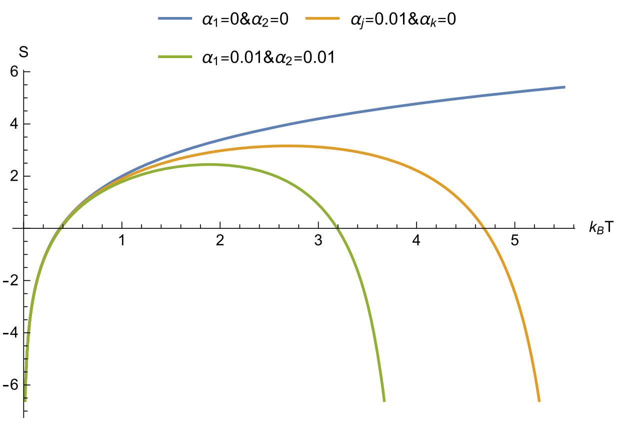

FIG. 5. Effects of the deformations on the entropy S.

We show the dependence of these thermodynamic properties with the temperature in the 3D case for a few indicative values of the parameters of deformations in Figs. 2–5. We see in these plots that the corrections increase the free energy F but decrease the other thermodynamic properties. We mention here that the space dimension does not affect the shape of these curves, but it just shifts them toward low temperatures.

Expressions (67)–(71) give the thermodynamic properties of the usual D-dim KGO when using the limit $\theta \to 0$ (or, equivalently, for $\alpha_1 \to 0$ and $\alpha_2 \to 0$).

VI. CONCLUSION

We studied, with an explicit calculation in the momentum space representation, the Klein–Gordon oscillator in an arbitrary dimension in the framework of deformed quantum mechanics with Snyder–de Sitter commutation relations; these deformations lead to a non-zero minimal uncertainty in the measurement of both position and momentum of the spinless particle.

In the one-dimensional space, the exact expression of the energy levels is obtained, and the wave function is expressed with the Gegenbauer polynomials. The associated energy spectrum is found with an additional correction to the usual one; this correction depends on the deformation parameters of both the Snyder algebra and de Sitter space. This correction grows quickly with n, and this can be associated with an additional confinement to the usual one representing the harmonic oscillator. Within large values of n, the corrections cause the spectrum to tend toward a discrete one with the spacing proportional to both parameters of the deformations; this explains the fact that the spectrum of the ordinary case (without deformations) is almost continuous in this limit.

For the case of arbitrary dimension, we obtain the normalized momentum space wave function in terms of the Jacobi polynomials. Concerning the associated energy levels, we found an additional term depending on the orbital quantum number l, which was absent in the non-deformed case. Therefore, the contributions of Snyder algebra and the effects of gravity lead to a lifting of the spectrum degeneracy found in the ordinary harmonic oscillator. The energy spacing for large values of n was found similar to the one-dimensional case.

Regarding the thermodynamic properties of our system, we showed that in the regime of high temperatures, they have been also affected by the deformation parameters. Comparing their values with those of the ordinary case, we found that all the thermodynamic properties decrease, except for the free energy F that increases.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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