Abbes Laghrour University of Khenchela | Academic year; 2024/2025
Faculty of Science and Technology | 1st SM
Department of Material sciences | Duration 1h 30

MATHEMATICS 2 EXAM

Exercise 1(4pt)

Let α , $\beta \in \mathbb{R}$,consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \alpha x + \beta & if \quad x \leq 1 \\ x^2 & if \quad x \succ 1 \end{cases}$

1) Determine the value of the real numbers α and β for which the function f is

a) continuous on $\mathbb{R}$. b) differentiable on $\mathbb{R}$.

Exercise 2 (4pt)

1) Prove that. $ch^2(x) - sh^2(x) = 1$

2) Solve the following equation in $\mathbb{R}$: $sh(x) = \frac{3}{4}$.

Exercise 3 (6pt

I) Let A be matrix given by $A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

1- Find $a,b \in \mathbb{R}$ such that $A^2 + aA + bI_2 = 0_{M_2(\mathbb{R})}$,

2- Deduce that A is inversible and determinate A^{-1}

II) Let A and C be two matrices given by:

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

1- Calculate A×C? Solve the system A×X=B by tow method such that $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$.

Exercise 4 (6pt)

I) a- Find $a,b \in \mathbb{R}$ a such that $\frac{1}{x(x-1)} = \frac{a}{x} + \frac{b}{x-1}$ for all $x \in \mathbb{R} - \{0,1\}$.

b-Deduce the primitive $\int \frac{1}{x(x-1)} dx$.

II) Calculate the following integrals:

① $\int_1^e \frac{(\ln x)^n}{x} dx$ (C.V) ② $\int \sin^3 x \cos^2 x dx$ (cv) ③ $\int x \arctan x dx$ (b p)

GOOD LUCK

Correction of the final exam

Exercise 1 (4pt)

Consider the function $f:\mathbb{R}\to\mathbb{R}, f(x)=\begin{cases}\alpha x+\beta & if \quad x\leq 1\\ x^2 & if \quad x\succ 1\end{cases}$ and $\alpha,\ \beta\in\mathbb{R}$,

a) f is continuous on $x_0\in\mathbb{R}\Leftrightarrow \lim_{\substack{x\to x_0\\ \succ}} f(x)=\lim_{\substack{x\to x_0\\ \prec}} f(x)=f(x_0)$

$x_0=1$ So $\lim_{\substack{x\to 1\\ \succ}} x^2=1$ and $\lim_{\substack{x\to 1\\ \prec}} \alpha x+\beta=\alpha+\beta$ therefore $\alpha+\beta=1$

b) f is differentiable on $x_0\in\mathbb{R}\Leftrightarrow \lim_{\substack{x\to x_0\\ \succ}}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{\substack{x\to x_0\\ \prec}}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0)$

$x_0=1$ so $\Leftrightarrow \lim_{\substack{x\to 1\\ \prec}}\frac{(\alpha x+\beta)-(\alpha+\beta)}{x-1}=\frac{\alpha(x-1)}{x-1}=\alpha$ and $\Leftrightarrow \lim_{\substack{x\to 1\\ \succ}}\frac{x^2-1}{x-1}=\lim_{\substack{x\to 1\\ \succ}}\frac{(x-1)(x+1)}{x-1}=2$

Therefore $\alpha=2$ and we have $\Rightarrow\beta=-1$

Exercise 2 (4pt)

1)We have $ch(x)=\frac{e^x+e^{-x}}{2}$ and $sh(x)=\frac{e^x-e^{-x}}{2}$

$ch(x)^2-sh(x)^2=\left[\frac{e^x+e^{-x}}{2}\right]^2-\left[\frac{e^x-e^{-x}}{2}\right]^2=\frac{e^{2x}+e^{-2x}+2}{4}-\frac{e^{2x}+e^{-2x}-2}{4}=1$

2) $sh(x)=\frac{3}{4}\Leftrightarrow\frac{e^x-e^{-x}}{2}=\frac{3}{4}$

$\Leftrightarrow 2e^x-2e^{-x}=3 \quad \Leftrightarrow e^x(2e^x-2e^{-x})=3e^x$

$2e^{2x}-2-3e^x=0$

We pose: $e^x=x$ we get: $2x^2-3x-2=0\Leftrightarrow(2x+1)(x-2)=0$

So: $(x=\frac{-1}{2})$ or $(x=2)$.

Since: $x\rangle 0$, then $x=2\Rightarrow e^x=2 \Rightarrow x=\ln 2$

Exercise3 (6pt)

I) Let A be matrix given by $A=\begin{pmatrix}2&0\\1&1\end{pmatrix}$

We calculate $A^2+aA+bI_2=0_{M_2(\mathbb{R})}\Leftrightarrow\begin{pmatrix}4&0\\3&1\end{pmatrix}+\begin{pmatrix}2a&0\\a&a\end{pmatrix}+\begin{pmatrix}b&0\\0&b\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$.

$\Leftrightarrow\begin{pmatrix}4+2a+b&0\\3+a&1+b+a\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}\Rightarrow\begin{cases}a=-3\\b=2\end{cases}$

2)We have $\det(A)=\begin{vmatrix}2 & 0\\ 1 & 1\end{vmatrix}=2\neq 0$ so A is inversible

$$A^2-3A+2I_2=0_{M_2(\mathbb{R})}\Leftrightarrow \frac{1}{2}A(A-3I)=-I \Leftrightarrow A^{-1}=-\frac{1}{2}(A-3I)=\begin{pmatrix}\frac{1}{2} & 0\\ -\frac{1}{2} & 1\end{pmatrix}$$

III- We calculate A×C=?

$$A\times C=\begin{pmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{pmatrix}\times\begin{pmatrix}0 & 1/2 & 1/2\\ 1/2 & 0 & 1/2\\ 1/2 & 1/2 & 0\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

We deduce that $A^{-1}=C$

We calculate $\det(A)=\begin{vmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=4\neq 0$

Now, we pass to solve the system $A\times X=B$ by using the inverse matrix method ; . The system takes the form $AX=B$, so its solution is given by $X=A^{-1}B$

$$A\times X=B\Leftrightarrow X=A^{-1}B=\begin{pmatrix}0 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & 0 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & 0\end{pmatrix}.\begin{pmatrix}4\\ 2\\ 0\end{pmatrix}=\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$$

Since det $\det(A)\neq 0$, so the system has a unique solution, given by method of Cramer:

$$x=\frac{\det(A_x)}{\det(A)}=\frac{\begin{vmatrix}4 & 1 & 1\\ 2 & -1 & 1\\ 0 & 1 & -1\end{vmatrix}}{\begin{vmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}}=\frac{4}{4}=1\ ;\ y=\frac{\det(A_y)}{\det(A)}=\frac{\begin{vmatrix}-1 & 4 & 1\\ 1 & 2 & 1\\ 1 & 0 & -1\end{vmatrix}}{\begin{vmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}}=\frac{8}{4}=2$$

$$z=\frac{\det(A_z)}{\det(A)}=\frac{\begin{vmatrix}-1 & 1 & 4\\ 1 & -1 & 2\\ 1 & 1 & -0\end{vmatrix}}{\begin{vmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}}=\frac{12}{4}=3$$. Therefore $(x, y, z)^t = (1,2,3)^{t}$.

Exercise 4 (6pt)

I) a- Let $x \in \mathbb{R}-\{0,1\}$

We have: $\frac{a}{x}+\frac{b}{x-1}=\frac{a(x-1)+bx}{x(x-1)}=\frac{x(a+b)-ax}{x(x-1)}$.

By matching we find : $a+b=0$ and $a=-1$; so : $a=-1$ and $b=1$.

b-Using the previous results:

$$\int \frac{1}{x(x-1)}dx=\int (\frac{-1}{x}+\frac{1}{x-1})dx$$

$$=\int \frac{-1}{x}+\int \frac{1}{x-1}dx=-\ln|x|+\ln|x-1|+C \ / \ (C \in \mathbb{R}) \ .$$

II) Calculate the integrals:

① We use the change of variable to determinate $\int_1^e \frac{(\ln x)^n}{x}dx$

We pose: $u=\ln x \Rightarrow du=\frac{dx}{x}$ so we have:

$$\int_1^e \frac{(\ln x)^n}{x}dx=\int_0^1 u^n dx=\left[\frac{1}{n+1}u^{n+1}\right]_0^1=\frac{1}{n+1}$$

② We use the change of variable to determinate $\int \sin^3 x \cos^2 x dx$

We pose $y=\cos x \Rightarrow dy=-\sin x dx$;by compensation we find

$$\int \sin^3 x \cos^2 x dx=-\int (1-y)y^2 dy=\int (y^2-y^4)dy=-\left[\frac{1}{3}y^3-\frac{1}{5}y^5\right]+C \ /C \in \mathbb{R}$$

③ We pose $\begin{cases} f=\arctan x \\ g=x \end{cases} \Rightarrow \begin{cases} f'=\frac{1}{1+x^2} \\ g=\frac{1}{2}x^2 \end{cases}$ Then

$$\int x \arctan x dx=\frac{1}{2}x^2 \arctan x-\int \frac{1}{1+x^2}\times\frac{1}{2}x^2 dx$$

$$=\frac{1}{2}x^2 \arctan x-\frac{1}{2}\int \frac{x^2+1-1}{1+x^2}dx$$

$$=\frac{1}{2}x^2 \arctan x-\frac{1}{2}\int dx+\frac{1}{2}\int \frac{1}{1+x^2}dx=\frac{1}{2}x^2 \arctan x-\frac{1}{2}x+\frac{1}{2}\arctan x+C \ /C \in \mathbb{R}$$