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Results in fixed point theory with applications

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Dedication

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We would like to dedicate this work to our parents, brothers, sisters and our friends; To the people who paved our way of science and knowledge; To all our distinguished teachers To every persons who supported us in our studies.

Thanks

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First we thank ALLAH for helping us to accomplish this work.
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We would like also to thank **M. Meftah Yacine** for accepting to be the President of this Jury.

Abstract

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This memoir is devoted to study some fixed point results with application to integral equations.

In chapter 2, we generalize Dass-Gupta fixed point in the context of b-metric spaces via F-contractions.

In chapter 3, we propose a new theorem concerning a fixed point result for expansive mappings and we give an application to rational integral equations to illustrate the validity of our results.

Contents

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Introduction

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In 1922, Banach proved a fixed point theorem which insures under appropriate conditions, the existence and uniqueness of a fixed point. This result of Banach is known as Banach's fixed point theorem or Banach contraction principle. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways.

In the paper [34], an interesting generalization of Banach contraction principle is given by introducing the concept of F-contraction. One of the most prevalent generalization of the metric spaces was given in the article [2] through the notion of b-metric spaces.

In this memoir, we establish some new fixed point results and we give applications to integral and differential equations. The main goal is to improve and generalize some results given in [24] and [28].

In chapter 2, a fixed point theorem is established and proved in the framework of b-metric spaces for Dass-Gupta contraction type through F-contractions with less conditions on the function F. An application to second-order differential equations is given at the end of this chapter.

Some fixed point results are given in the chapter 3 of this memoir which have been allowed us to generalize existing results in the paper [28].

The memoir is organized as follows:

In chapter 1, we recall some basic tools concerning b-metric spaces and F-contractions which will be used to prove our results.

In chapter 2, we generalize Dass-Gupta fixed point in the context of b-metric spaces via F-contractions and we give an application to second-order differential equations based on Green function and integral equations tools.

In chapter 3, we propose a new theorem concerning a fixed point result for expansive mappings and we give an application to rational integral equations to illustrate the validity of our results.

A conclusion and some prospects are given at the end of this memoir.

Chapter 1

Preliminaries and settings

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Abstract

In this chapter, we recall some basic tools concerning b-metric spaces and F-contractions which will be used to prove our results. Many arguments in this chapter are reproduced from the paper [11].

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}$ the sets of positive integers and real numbers, respectively. We also write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Henceforth, X will denote a nonempty set and the Picard sequence of a self-mapping $T : X \to X$ based on an arbitrary $x_0 \in X$ is given by $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$, where T^n denotes the n^{th}-iterates of T.

1.1 *b*-metric spaces

In 1989, Bakhtin [2] introduced the concept of *b*-metric spaces as a generalization of the metric spaces in the sense that the triangle inequality contains a suitable constant $s \geq 1$ (see also Czerwik [8]). Since then, several published papers have dealt with *b*-metric spaces and fixed point theory in the setting of *b*-metric spaces (see, e.g., [1, 4, 5, 6, 7, 13, 23, 24, 30] and some related references therein). For more details concerning some technical and useful tools in the context of *b*-metric spaces, the reader may consult [23]. Note that the topological framework of a *b*-metric space with the topology induced by its convergence was studied in [1].

We will first recall the definition of a *b*-metric space.

Definition 1.1.1. *Let X be a nonempty set and let $s \geq 1$ be a given real number. A mapping $\sigma : X \times X \to [0,\infty)$ is said to be a b-metric if, for all $x, y, z \in X$, the following conditions hold:*
(b_1) $\sigma(x,y) = 0$ *if and only if* $x = y$;
(b_2) $\sigma(x,y) = \sigma(y,x)$;
(b_3) $\sigma(x,z) \leq s\left[\sigma(x,y) + \sigma(y,z)\right]$.
The pair (X,σ) is called a b-metric space with constant $s \geq 1$.

Obviously, for $s = 1$ one obtains a metric on X.

Example 1.1.1. ***(See** [30]) Let (X,d) be a metric space and let the mapping $\sigma_d : X \times X \to [0,\infty)$ be defined by*

$$\sigma_d(x,y) = \left(d(x,y)\right)^p, \quad \text{for all } x,y \in X,$$

where $p > 1$ is a fixed real number. Then (X,σ_d) is a b-metric space with $s = 2^{p-1}$.

In particular, if $X = \mathbb{R}$, $d(x,y) = |x-y|$ is the usual Euclidean metric and

$$\sigma_d(x,y) = (x-y)^2, \quad \text{for all } x,y \in \mathbb{R},$$

then $(\mathbb{R},\sigma_d)$ is a b-metric with $s = 2$.

Example 1.1.2. *Let X be the set of Lebesgue measurable functions on $[0,1]$ such that*

$$\int_0^1 |f(x)|^2\, dx < \infty.$$

Define $D : X \times X \to [0,\infty)$ by

$$D(f,g) = \int_0^1 |f(x) - g(x)|^2\, dx.$$

Then, (X,D) is a b-metric space with $s = 2$

We present now the concepts of convergence, Cauchy sequence and completeness in *b*-metric spaces.

Definition 1.1.2. ***(See** [5], [6], [7]) Let (X,σ) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:*
(a) *convergent if and only if there exists $x \in X$ such that $\lim_{n\to\infty} \sigma(x_n,x) = 0$ and in this case we write $\lim_{n\to\infty} x_n = x$;*
(b) *Cauchy if and only if $\lim_{n,m\to\infty} \sigma(x_n,x_m) = 0$.*

Lemma 1.1.1. *(See [31]) Let (X,σ) be a b-metric space with constant $s \geq 1$ and let $\{x_n\}$ be a sequence in X. Assume that there exists $\lambda \in [0,1)$ satisfying*

$$\sigma(x_n, x_{n+1}) \leq \lambda\sigma(x_{n-1}, x_n)$$

for any $n \in \mathbb{N}$. Then, $\{x_n\}$ is a Cauchy sequence in X.

Definition 1.1.3. *(See [5], [6], [7]) The b-metric space (X,σ) is said complete if every Cauchy sequence in X converges in X.*

It is worth recalling that a *b*-metric is generally not continuous (see, e.g., [15, Example 3.3]). The following lemma is very useful to manage this problem.

Lemma 1.1.2. *Let (X,σ) be a b-metric space with constant $s \geq 1$ and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence in (X,σ), then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that the following items hold:

$$\varepsilon \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon;$$
$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)+1}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)+1}\right) \leq s^2\varepsilon;$$
$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)}\right) \leq s^2\varepsilon;$$
$$\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq s^3\varepsilon.$$

Proposition 1.1.1. *(See [1, Proposition 3.11]) Let (X,σ) be a b-metric space with constant $s \geq 1$. If σ is continuous with respect in one variable, then σ is continuous in other variable.*

We end this section by giving an example which illustrates some preceding properties concerning *b*-metric spaces (see [11])

Example 1.1.3. *Let $X = [0,\infty)$. Let $\sigma : X \times X \to [0,\infty)$ be a mapping defined by*

$$\sigma(x,y) = \begin{cases} d(x,y), & xy \neq 0, \\ 4d(x,y), & xy = 0, \end{cases}$$

where $d(x,y) = |x-y|$. Then the following hold:

(1) *(X,σ) is a complete b-metric space with constant $s = 4$.*

(2) *σ is not a metric on X.*

(3) *σ is not continuous in each variable.*

1.2 F-contractions

Now, let us review some results concerning F-contractions related to the existing literature. In 2012, Wardowski [34] introduced the notion of F-contraction as follows:

Definition 1.2.1. *(See [34]) Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$,*

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)), \tag{1.1}$$

where $\mathcal{F}$ is the family of all functions $F : (0,\infty) \to \mathbb{R}$ satisfying the following conditions:
(F_1) *F is strictly increasing.*
(F_2) *For each sequence $\{\alpha_n\}$ of positive numbers, the following holds:*

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty.$$

(F_3) *There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

Remark 1.2.1. *(See [34]) Let $\alpha > 0$. Let the following functions $F_1(\alpha) = \ln\alpha$, $F_2(\alpha) = \ln\alpha + \alpha$, $F_3(\alpha) = \dfrac{-1}{\sqrt{\alpha}}$ and $F_4(\alpha) = \ln(\alpha^2 + \alpha)$. Then, F_1, F_2, F_3 and $F_4 \in \mathcal{F}$.*

Remark 1.2.2. *(See [34]) Clearly, if F is an increasing function (not necessary strictly increasing), the inequality* (1.1) *implies that T is a contractive mapping, i.e.,*

$$d(Tx,Ty) < d(x,y), \quad \forall x,y \in X,\ x \neq y.$$

Hence, every F-contraction is a continuous mapping.

Wardowski's result is given as follows:

Theorem 1.2.1. *Let (X,d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.*

Remark 1.2.3. *(See [34]) Wardowski showed that T is a Banach contraction [3] by taking $F(\alpha) = \ln\alpha$ in* (1.1).

In [25], Secelean showed that condition (F_2) can be replaced by an equivalent and more easier one (noted (F_2'): $\inf F = -\infty$). Afterwards, Piri and Kumam [19] established Wardowski's theorem by using (F_2') and the continuity instead of (F_2) and (F_3), respectively. Later, Wardowski [35] proved a fixed point theorem concerning F-contractions when τ is taken as a function. In this work, the author used a relaxed version of (F_2) and dropped also condition (F_3). Very recently, some authors proved (in different ways) the original results of Wardowski without both conditions (F_2) and (F_3) (see, e.g., [18, Remark 3.7]. It is also worth mentioning that many others papers dealing with various types of F-contractions can be found in the literature (see, e.g., [12, 11, 14, 18, 20, 21, 22, 26, 27] and references therein).

1.3 Dass-Gupta contraction

In this section, we recall some results concerning Dass-Gupta contractions.

Theorem 1.3.1. *(Dass-Gupta contraction, see, [10]) Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping such that there exist* $k_1, k_2 \in [0, 1)$ *with* $k_1 + k_2 < 1$ *such that*

$$d\left(Tx, Ty\right) \leq k_1 d\left(x, y\right) + k_2 \frac{d\left(y, Ty\right)\left(1 + d\left(x, Tx\right)\right)}{1 + d\left(x, y\right)}$$

for all $x, y \in X$. *Then* T *has a unique fixed point* $x \in X$, *and the sequence* $\{T^n x\}$ *converges to the fixed point* x.

In the paper [24], Samet proved the following result in the framework of b-metric spaces.

Theorem 1.3.2. *Let* (X, σ) *be a b-metric space with constant* $s \geq 1$ *and* $T : X \to X$ *be a mapping such that there exist* $k_1, k_2 \in [0, 1)$ *with* $k_1 s + k_2 < 1$ *such that*

$$\sigma\left(Tx, Ty\right) \leq k_1 \sigma\left(x, y\right) + k_2 \frac{\sigma\left(y, Ty\right)\left(1 + \sigma\left(x, Tx\right)\right)}{1 + \sigma\left(x, y\right)}$$

for all $x, y \in X$. *Then* T *has a unique fixed point* $x \in X$, *and the sequence* $\{T^n x\}$ *converges to this fixed point.*

Chapter 2

F-Dass-Gupta-contraction mappings in b-metric spaces with application to second-order differential equations

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Abstract

In this chapter, we generalize Dass-Gupta fixed point in the context of b-metric spaces via F-contractions. An application to second-order differential equations is given to illustrate our results.

2.1 Fixed point theorem of F-Dass-Gupta-contraction in b-metric spaces

Let (X,σ) be a b-metric space with constant $s \geq 1$. Throughout this paper, for all $x, y \in X$, we denote

$$\mathscr{A}_\sigma^T(x,y) = \alpha\sigma(x,y) + \beta\sigma(y,Ty)\frac{1+\sigma(x,Tx)}{1+\sigma(x,y)},$$

where α, β are nonnegative real numbers.

Definition 2.1.1. *Let (X,σ) be a b-metric space with constant $s \geq 1$. A mapping $T: X \to X$ is called to be a F-Dass-Gupta-contraction if there exist a nondecreasing function $F:(0,\infty) \to \mathbb{R}$ and $\tau > 0$ such that for all $x, y \in X$,*

$$\sigma(Tx,Ty) > 0 \Rightarrow \tau + F(\sigma(Tx,Ty)) \leq F(\mathscr{A}_\sigma^T(x,y)). \tag{2.1}$$

Remark 2.1.1. *Immediately, we obtain from Definition 2.1.1 that every T which is a F-Dass-Gupta-contraction satisfies the following condition*

$$\sigma(Tx,Ty) < \mathscr{A}_\sigma^T(x,y), \tag{2.2}$$

for all $x, y \in X$ with $Tx \neq Ty$.

Theorem 2.1.1. *Let (X,σ) be a complete b-metric space with constant $s \geq 1$ and $T: X \to X$ be a F-Dass-Gupta-contraction with $\alpha, \beta \in [0,1)$ and $\alpha + \beta < 1$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.*

Proof. Now, we prove that T has a fixed point. Let $x_0 \in X$ be an arbitrary point and $\{x_n\}$ be the Picard sequence based on x_0. If there exist $n_0 \in \mathbb{N}_0$, such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is the fixed point of T. If $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}_0$, we get

$$\sigma_n := \sigma(x_n, x_{n+1}) = \sigma(Tx_{n-1}, Tx_n) > 0, \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

Applying inequality (2.1) with $x = x_{n-1}$ and $y = x_n$, we obtain for all $n \in \mathbb{N}$

$$\begin{aligned} &\tau + F(\sigma(x_n, x_{n+1})) \\ &\leq F\left(\alpha\sigma(x_{n-1},x_n) + \beta\sigma(x_n,x_{n+1})\frac{1+\sigma(x_{n-1},x_n)}{1+\sigma(x_{n-1},x_n)}\right) \\ &= F(\alpha\sigma(x_{n-1},x_n) + \beta\sigma(x_n,x_{n+1})). \end{aligned} \tag{2.4}$$

By the monotonicity of F, we get

$$\sigma_n < \alpha\sigma_{n-1} + \beta\sigma_n, \quad \text{for all } n \in \mathbb{N}. \tag{2.5}$$

Inequality (2.5) implies

$$\sigma_n < \frac{\alpha}{1-\beta}\sigma_{n-1}, \quad \text{for all } n \in \mathbb{N}. \tag{2.6}$$

Using Lemma 1.1.1 with $\lambda = \dfrac{\alpha}{1-\beta}$, we deduce that $\{x_n\}$ is a Cauchy sequence. Since (X,σ) is a complete b-metric space, $\{x_n\}$ converges to some point x^* in X, that is,

$$\lim_{n\to\infty} \sigma\left(x_n, x^*\right) = 0. \tag{2.7}$$

Now, we prove that x^* is a fixed point of T. Arguing by contradiction. Then, by virtue of (2.7), there exists $n_0 \in \mathbb{N}$ such that

$$\sigma\left(x_n, x^*\right) \leq \frac{\sigma\left(x^*, Tx^*\right)}{2s}, \quad \forall n \geq n_0. \tag{2.8}$$

On the other hand, by (b_3), we have

$$\sigma\left(x^*, Tx^*\right) \leq s\sigma\left(x^*, Tx_n\right) + s\sigma\left(Tx_n, Tx^*\right). \tag{2.9}$$

By (2.8), (2.9) yields

$$\begin{aligned}
\sigma\left(Tx_n, Tx^*\right) &\geq \frac{1}{s}\left(\sigma\left(x^*, Tx^*\right) - s\sigma\left(x^*, Tx_n\right)\right)\\
&= \frac{1}{s}\sigma\left(x^*, Tx^*\right) - \sigma\left(x^*, x_{n+1}\right)\\
&\geq \frac{\sigma\left(x^*, Tx^*\right)}{2s} > 0,
\end{aligned} \tag{2.10}$$

for all $n \geq n_0$.

Through (2.10), inequality (2.2) can be applied with $x = x^*$ and $y = x_n$. Hence, (2.9) takes the form

$$\begin{aligned}
\sigma\left(x^*, Tx^*\right) &\leq s\sigma\left(x^*, Tx_n\right) + s\sigma\left(Tx_n, Tx^*\right).\\
&< s\sigma\left(x^*, Tx_n\right) + s\alpha\sigma\left(x^*, x_n\right) + s\beta\sigma\left(x_n, Tx_n\right)\frac{1+\sigma\left(x^*, Tx^*\right)}{1+\sigma\left(x^*, x_n\right)}\\
&= s\sigma\left(x^*, x_{n+1}\right) + s\alpha\sigma\left(x^*, x_n\right) + s\beta\sigma_n\frac{1+\sigma\left(x^*, Tx^*\right)}{1+\sigma\left(x^*, x_n\right)}
\end{aligned} \tag{2.11}$$

for all $n \geq n_0$.

On the other hand, we have $\lim_{n\to\infty} \sigma_n = 0$ (since $\{x_n\}$ is a Cauchy sequence). Next, passing to the limit as $n \to \infty$ in (3.1) and using (2.7), we get

$$\sigma\left(x^*, Tx^*\right) \leq 0,$$

which is a contradiction. Therefore, x^* is a fixed point.

Now, we prove the uniqueness of the fixed point of T. Assume that x^* and y^* are two distinct fixed points of T, i.e., $Tx^* = x^* \neq y^* = Ty^*$. Then

$$\sigma\left(Tx^*, Ty^*\right) = \sigma\left(x^*, y^*\right) > 0. \tag{2.12}$$

We consider the following two cases.

1. If $\alpha > 0$, then using (2.12) and the fact that F is nondecreasing, the contractive inequality (2.1) with $x = x^*$ and $y = y^*$ yields

$$\begin{aligned} \tau + F\left(\sigma\left(x^*, y^*\right)\right) &\leq F(\alpha\sigma\left(x^*, y^*\right) \\ &\leq F(\sigma\left(x^*, y^*\right)), \end{aligned}$$

which contradicts the fact that $\tau > 0$.

2. If $\alpha = 0$, then again through (2.12) and using the inequality (2.2) with $x = x^*$ and $y = y^*$, we have

$$\sigma\left(x^*, y^*\right) < 0,$$

which is a contradiction. Thus, the fixed point of T is unique.

□

Remark 2.1.2. *Theorem 2.1.1 is proved without conditions* (F_2), (F_3) *and the strictness of the monotonicity of F.*

Remark 2.1.3. *Taking* $F(t) = \ln(t)$ *and* $s = 1$ *in Theorem 2.1.1, we recover Dass-Gupta theorem (see Theorem 1.3.1). Indeed, here* $k_1 = e^{-\tau}\alpha$ *and* $k_2 = e^{-\tau}\beta$.

Remark 2.1.4. *By using the same arguments that used in Remark 2.1.3, Theorem 2.1.1 represents an improvement of Theorem 1.3.2.*

2.2 Application to second-order differential equations

Motivated and inspired by the work [11], we discuss the existence and uniqueness of solutions of the following two-point boundary value problem for the second order differential equation:

$$\begin{cases} -\dfrac{d^2x}{dt^2} = f(t, x(t)), & t \in I, \\ x(0) = x(1) = 0, \end{cases} \tag{2.13}$$

where $I = [0,1]$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let $X = \mathscr{C}(I, \mathbb{R})$ be the space of all continuous functions $x : I \to \mathbb{R}$. It is well known that X endowed with

$$\sigma\left(x, y\right) = \sup_{t \in I}\left\{(x(t) - y(t))^2\right\}, \quad \text{for all } x, y \in X$$

is a complete b-metric space with constant $s = 2$.

Theorem 2.2.1. *Assume that the following condition is satisfied:*

(W) *for all* $z, \omega \in \mathbb{R}$ *and for all* $r \in I$, *we have*

$$\left|f(r,z) - f(r,w)\right| \leq \sqrt{\ln\left(\frac{(z-w)^2}{8} + 1\right)}.$$

Then, problem (2.13) *has a unique solution* $x^* \in \mathscr{C}^2(I, \mathbb{R})$

Proof. It is known that problem (2.13) is equivalent to the following integral equation

$$x(t)=\int_0^1 G(t,r)\,f(r,x(r))\,dr,\quad \forall t\in I, \tag{2.14}$$

where G is the Green function associated to problem (2.13), given by

$$G(t,r)=\begin{cases} t(1-r), & 0\le t\le r\le 1,\\ r(1-t), & 0\le r\le t\le 1.\end{cases}$$

Consequently, $x\in\mathscr{C}^2(I,\mathbb{R})$ is a solution of problem (2.13) if and only if $x\in\mathscr{C}(I,\mathbb{R})$ is a solution of the integral equation (2.14).

Now, we can define the mapping $T:X\to X$ as follows

$$Tx(t)=\int_0^1 G(t,r)\,f(r,x(r))\,dr,\quad \forall t\in I,\ \forall x\in X.$$

Therefore, to find a unique fixed point $x^*\in X$ of T is equivalent to establishing the existence and uniqueness of solutions of problem (2.13).

Let $x,y\in X$ such that $Tx\neq Ty$. Using assumption (W), we get

$$\begin{aligned}\left((Tx)(t)-(Ty)(t)\right)^2 &\le \left[\int_0^1 G(t,r)\sqrt{\ln\left(\frac{\left(x(r)-y(r)\right)^2}{8}+1\right)}dr\right]^2\\ &\le \ln\left(\frac{\sigma(x,y)}{8}+1\right)\left(\sup_{t\in I}\int_0^1 G(t,r)\,dr\right)^2\\ &=\frac{1}{64}\ln\left(\frac{\sigma(x,y)}{8}+1\right).\end{aligned}$$

In the last inequality, we have used that $\sup_{t\in I}\int_0^1 G(t,r)\,dr=\frac{1}{8}$. Hence,

$$\sigma(Tx,Ty)\le\frac{1}{64}\ln\left(\frac{\sigma(x,y)}{8}+1\right). \tag{2.15}$$

Utilizing (2.15), after routine calculations, we obtain

$$\sigma(Tx,Ty)\le\frac{\sigma(x,y)}{8}\le\mathscr{M}(x,y),$$

where

$$\mathscr{M}(x,y)=\frac{1}{8}\sigma(x,y)+\frac{1}{4}\sigma(y,Ty)\frac{1+\sigma(x,Tx)}{1+\sigma(x,y)},$$

The above inequality can be written as follows

$$\tau+F(\sigma(Tx,Ty))\le F(\mathscr{A}(x,y)),$$

where $\tau = \ln 2$, $F(t) = \ln(t)$, for all $t \in (0, \infty)$, and

$$\mathscr{A}(x,y) = \frac{1}{4}\sigma(x,y) + \frac{1}{2}\sigma(y,Ty)\frac{1+\sigma(x,Tx)}{1+\sigma(x,y)}.$$

Hence, all the conditions of Theorem 2.1.1 are satisfied with $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{2}$. Therefore, T has unique fixed point u^* in $X = \mathscr{C}(I, \mathbb{R})$. Then, the problem (2.13) has a unique solution u^* in $\mathscr{C}^2(I, \mathbb{R})$. $\square$

Chapter 3

Fixed point results of expansive mappings and application to integral equations

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Abstract

In this chapter, we propose a new theorem concerning a fixed point result for expansive mappings. More precisely, we generalize the work [28] and we give an application to rational integral equations to illustrate the validity of our results.

3.1 Preliminaries and auxiliary results

In this section, we recall some important tools and we prove some results in the fixed point theory which will be used in the sequel.

The following result is crucial to prove our results.

Theorem 3.1.1. *Let K be a nonempty closed convex subset of a Banach space E. Suppose that the mapping $S: K \to K$ is continuous and $S(K)$ resides in a compact subset of E. Then S has at least one fixed point in K.*

Let $C(X)$ denote the space of all continuous functions on a compact metric sapace X. In $C(X)$ we always regard the distance between functions f and g in $C(X)$ to be

$$d(f,g) = \max\{|f(x) - g(x)| : x \in X\}.$$

(a) $F \subset C(X)$ is bounded means that there exists a positive constant $M < \infty$ such that $|f(x)| \leq M$ for each $x \in X$ and each $f \in F$, and
(b) $F \subset C(X)$ is equicontinuous means that: for every $\varepsilon > 0$ there exists $\delta > 0$ (which depends only on ε) such that for $x, y \in X$:

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall f \in F.$$

Theorem 3.1.2. *(Ascoli-Arzela) Let X be a compact set. A subset F of $C(X)$ is compact if and only if it is closed, bounded, and equicontinuous.*

Lemma 3.1.1. *Let (X,d) be a metric space and $\{x_n\}$ a sequence in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy-sequence in (X,d), then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ such that the following equalities hold:

$$\begin{aligned}&\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right)\\ &= \lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)}\right) = \lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)+1}\right) = \varepsilon^+.\end{aligned}$$

Proof. If $\{x_n\}$ is not a Cauchy sequence and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, then there exist $\varepsilon > 0$ and sequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of positive integers such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$, $\sigma\left(x_{m(k)}, x_{n(k)}\right) > \varepsilon$ and $\sigma\left(x_{m(k)}, x_{n(k)-1}\right) \leq \varepsilon$ for all $k \geq 1$.

Using the triangle inequality, we have

$$\begin{aligned}\varepsilon < \sigma\left(x_{m(k)}, x_{n(k)}\right) &\leq \sigma\left(x_{m(k)}, x_{n(k)-1}\right) + \sigma\left(x_{n(k)-1}, x_{n(k)}\right)\\ &\leq \varepsilon + \sigma\left(x_{n(k)-1}, x_{n(k)}\right).\end{aligned}$$

Taking the limit as $k \to \infty$, we get

$$\lim_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) = \varepsilon^+.$$

On the other hand, through again the triangle inequality and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, we have

$$\begin{aligned} d\left(x_{m(k)}, x_{n(k)}\right) &\leq d\left(x_{m(k)}, x_{n(k)+1}\right) + d\left(x_{n(k)}, x_{n(k)+1}\right) \\ &\leq d\left(x_{m(k)}, x_{n(k)}\right) + 2d\left(x_{n(k)}, x_{n(k)+1}\right). \end{aligned}$$

Taking the limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)+1}\right) = \lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \varepsilon^+.$$

Using the same manner, we get also

$$\lim_{k\to\infty} d\left(x_{n(k)}, x_{m(k)+1}\right) = \lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \varepsilon^+.$$

By the triangle inequality, it is easy to see that

$$\left|d\left(x_{n(k)+1}, x_{m(k)+1}\right) - d\left(x_{m(k)}, x_{n(k)}\right)\right| \leq d\left(x_{n(k)}, x_{n(k)+1}\right) + d\left(x_{m(k)}, x_{m(k)+1}\right).$$

This implies

$$\lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) = \lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \varepsilon^+.$$

□

Let us note $\mathbb{S}$ the family of all functions $\tau : (0,\infty) \to (0,\infty)$ satisfying the following property:

$$\liminf_{t\to\eta^+} \tau(t) > 0, \quad \text{for all } \eta > 0.$$

Proposition 3.1.1. *Let (X,d) be a complete metric space and $T : X \to X$ a self-map. If there exist a nondecreasing function $F : (0,\infty) \to \mathbb{R}$ and $\tau \in \mathbb{S}$ such that for all $x, y \in X$,*

$$d(Tx,Ty) > 0 \Rightarrow \tau\left(d(Tx,Ty)\right) + F\left(d(Tx,Ty)\right) \leq F\left(d(x,y)\right), \tag{3.1}$$

then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.

Proof. Now, we prove that T has a fixed point. Let $x_0 \in X$ be an arbitrary point and $\{x_n\}$ be the Picard sequence based on x_0. If there exist $n_0 \in \mathbb{N}_0$, such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is the fixed point of T. If $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}_0$, we get

$$d_n := \sigma(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) > 0, \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Applying the contractive inequality (3.1) with $x = x_{n-1}$ and $y = x_n$, we obtain for all $n \in \mathbb{N}$

$$\tau(d(x_n, x_{n+1})) + F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) \tag{3.3}$$

Using the monotonicity of F and the fact that $\tau(t) > 0, \forall t > 0$, we get

$$d_n < d_{n-1}, \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

The above inequality implies that $\{\sigma_n\}$ is a strictly decreasing sequence of positive numbers. Hence, there exists $l \geq 0$ such that

$$\lim_{n\to\infty} d_n = l^+.$$

Now, we prove that $l = 0$. Arguing by contradiction, we assume that $l > 0$. Since F is nondeceasing, the right limit of F exists, that is,

$$\lim_{t \to r^+} F(t) = F(r+0) = F\left(r^+\right), \quad \text{for all } r \in (0, \infty). \tag{3.5}$$

Having in mind (3.5) and letting $n \to \infty$ in (3.3), one gets

$$\begin{aligned} \liminf_{t \to l^+} \tau(t) &\leq \liminf_{n \to \infty} \tau(d_n) \\ &\leq \lim_{n \to \infty} \left(F(d_{n-1}) - F(d_n)\right) \\ &= F\left(l^+\right) - F\left(l^+\right) \\ &= 0, \end{aligned}$$

which a contradiction. Hence,

$$\lim_{n \to \infty} d_n = 0^+. \tag{3.6}$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary, i.e., $\{x_n\}$ is not a Cauchy sequence.

Applying (3.1) with $x = x_{m(k)}$ and $y = x_{n(k)}$ and we get

$$\tau\left(d\left(Tx_{m(k)}, Tx_{n(k)}\right)\right) + F\left(d\left(Tx_{m(k)}, Tx_{n(k)}\right)\right) \leq F\left(d\left(x_{m(k)}, x_{n(k)}\right)\right).$$

By Lemma 3.1.1 with (3.6) and taking the limit inferior as $k \to \infty$, we obtain

$$\begin{aligned} \liminf_{t \to \varepsilon^+} \tau(t) &\leq \liminf_{k \to \infty} \tau\left(d\left(x_{m(k)+1}, x_{n(k)+1}\right)\right) \\ &\leq \liminf_{k \to \infty} \left[F\left(d\left(x_{m(k)}, x_{n(k)}\right)\right) - F\left(d\left(x_{m(k)+1}, x_{n(k)+1}\right)\right)\right] \\ &\leq F\left(\varepsilon^+\right) - F\left(\varepsilon^+\right) = 0, \end{aligned}$$

which is a contradiction.

In other words, $\{x_n\}$ is a Cauchy sequence. As (X, d) is complete, $\{x_n\}$ converges to some point x^* in X, that is,

$$\lim_{n \to \infty} d\left(x_n, x^*\right) = 0.$$

In view of Remark 1.2.2, T is continuous and we immediately get

$$Tx^* = T\left(\lim_{n \to \infty} x_n\right) = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_{n+1} = x^*$$

Therefore, x^* is a fixed point of T.

Now, we prove the uniqueness. Assume that x^* and y^* are two distinct fixed points of T, i.e., $Tx^* = x^* \neq y^* = Ty^*$. The contradiction follows from the fact that

$$\tau\left(d\left(x^*, y^*\right)\right) + F\left(d\left(x^*, y^*\right)\right) \leq F\left(d\left(x^*, y^*\right)\right),$$

implies $\tau\left(d\left(x^*, y^*\right)\right) \leq 0$. □

Let us note $\mathbb{L}$ the family of all functions $\beta : (0,\infty) \to (0,1)$ satisfying the following property:

$$\limsup_{t\to\eta^+} \beta(t) < 1, \quad \text{for all } \eta \in (0,\infty)$$

and

$$\limsup_{t\to\infty} \beta(t) < 1.$$

Corollary 3.1.1. *Let $\beta \in \mathbb{L}$. Let (X,d) be a complete metric space and $S : X \to X$ a self-map satisfying*

$$d(Sx,Sy) \le \frac{d(x,y)}{1 - d(x,y)\ln\left(\beta\left(d\left(Sx,Sy\right)\right)\right)}. \tag{3.7}$$

Then S has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{S^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.

Proof. Applying (3.1) with $F(t) = -\frac{1}{t}$ and $\tau(t) = -\ln(\beta(t))$ with $\beta : (0,\infty) \to (0,1)$ a decreasing function, it is easy to get (3.7). Indeed, F is a nondecreasing function and $\tau \in \mathbb{S}$ since

$$\begin{aligned} \lim\inf_{t\to\eta^+} \tau(t) &= \lim\inf_{t\to\eta^+} \left(-\ln\left(\beta(t)\right)\right) \\ &= -\lim\sup_{t\to\eta^+} \left(\ln\left(\beta(t)\right)\right) \\ &= -\ln\left(\lim\sup_{t\to\eta^+} \beta(t)\right) > 0 \end{aligned}$$

□

3.2 Main results

The goal of this section is to study existence and uniqueness of the fixed point for expansive mappings-types. Roughly speaking, we generalize some results existing in the paper [28].

For the given metric space (X,d) and subset $K \subset X$, the expansive mappings $T : K \to X$, i.e., satisfying the following inequality

$$d(Tx,Ty) \ge h d(x,y)$$

for all $x, y \in K$ and some $h > 1$.

The authors in [28] , proposed the following condition
(A) : T is injective and there exists $H > 0$ such that, for all $x, y \in K$, $x \ne y$, the following holds

$$H \le \frac{1}{d(x,y)} - \frac{1}{d(Tx,Ty)}. \tag{3.8}$$

In our work, we propose to generalize the form (A) by the following one

(A^*) : T injective and there exists $\beta \in \mathbb{L}$ such that, for all $x, y \in K$, $x \ne y$, the following condition holds;

$$\beta(d(x,y)) \ge e^{\frac{1}{d(Tx,Ty)} - \frac{1}{d(x,y)}}. \tag{3.9}$$

Remark 3.2.1. *Taking $\beta(t) = e^{-H}$ with $H > 0$ in 3.9, we obtain 3.8.*

Theorem 3.2.1. *Let K be a nonmpty closed subset of a complete metric space (X,d). If a mapping $T : K \to X$ satisfies (A^*) and $K \subset T(K)$ then T has a unique fixed point.*

Proof. The condition (A^*) implies that the inverse $T^{-1} : T(K) \to K$ of the mapping T exists. Taking $u, v \in T(K)$ such that $u \neq v$, one can find $x, y \in K, x \neq y$ with $Tx = u, Ty = v$. By (A^*), there exists $\beta \in \mathbb{L}$ such that

$$\beta(d(x,y)) \geq e^{\frac{1}{d(Tx,Ty)} - \frac{1}{d(x,y)}}, \tag{3.10}$$

or, equivalent to

$$-\ln(\beta\left(d\left(T^{-1}u, T^{-1}v\right)\right)) \leq \frac{1}{d(T^{-1}u, T^{-1}v)} - \frac{1}{d\,(u,v)}. \tag{3.11}$$

Due the fact that $K \subset T(K)$, the above inequality leads to

$$d\,(Su, Sv) \leq \frac{d\,(u,v)}{1 - d\,(u,v)\ln\left(\beta\,(d\,(Su,Sv))\right)}, \quad \forall u, v \in K. \tag{3.12}$$

with $S = T^{-1}$

From the closedness of K, we obtain that K is a complete subspace of X. Applying Corollary 3.1.1, we deduce the existence of a unique $x^* \in K$ and hence $Tx^* = x^*$. □

Example 3.2.1. *Let $a > 1$ and consider $X = [0,a]$ equipped with the Euclidean metric. Let $\lambda \in]1,a]$, $K = [0,1]$ and $Tx = \lambda x$, $x \in K$. Then T satisfies (A^*) and $K \subset T(K)$.*

Indeed, it is easy to see that $T : K \to X$ is well defined since $\lambda \in]1,a]$ and T is injective. Also, we get, $K = [0,1] \subset [0,\lambda] = T(K)$.

On the other hand, we have for all $x, y \in K$ and $x \neq y$;

$$\begin{aligned} \frac{1}{|Tx - Ty|} - \frac{1}{|x-y|} &= \frac{1}{\lambda|x-y|} - \frac{1}{|x-y|} \\ &= \left(\frac{1}{\lambda} - 1\right)\frac{1}{|x-y|} \\ &\leq \left(\frac{1}{\lambda} - 1\right)|x-y|. \end{aligned}$$

The last inequality holds through the fact that $|x-y| \leq 1$.

Thus, we obtain

$$\beta\left(|x-y|\right) \geq e^{\left(\frac{1}{|Tx-Ty|} - \frac{1}{|x-y|}\right)},$$

where $\beta : (0,\infty) \to (0,1)$ and $\beta(t) = e^{Ct}$ with $C = \left(\frac{1}{\lambda} - 1\right) < 0.$, and thus we easy check that $\beta \in \mathbb{L}$.

Remark 3.2.2. *let (X,d) be a metric space and let $\emptyset \neq K \subset X$. If $T : K \to X$ fulfils the condition (A^*), then K is bounded.*

Proof. Suppose on the contrary that K is unbounded, then $\sup_{(x,y)\in K} d\left(x,y\right)=\infty$, so there are $(x_n),\left(y_n\right)\subset K$; $x_n\neq y_n$ such that:

$$\lim_{n\to\infty} d\left(x_n,y_n\right)=\infty,$$

that is

$$\lim_{n\to\infty}\frac{1}{d\left(x_n,y_n\right)}=0.$$

Since T is injective, $Tx_n\neq Ty_n$ for all n and hence, by (A^*), one has:

$$\begin{aligned}\ln\left(\beta\left(d\left(x_n,y_n\right)\right)\right) &> \ln\left(\beta\left(d\left(x_n,y_n\right)\right)\right)-\frac{1}{d\left(Tx_n,Ty_n\right)}\\ &\geq -\frac{1}{d\left(x_n,y_n\right)},\quad \forall n,\end{aligned}$$

or, equivalently,

$$\beta\left(d\left(x_n,y_n\right)\right)>e^{-\frac{1}{d\left(x_n,y_n\right)}},\quad \forall n.$$

Taking the limit superior as $n\to\infty$, we obtain

$$\limsup_{t\to\infty}\left(\beta\left(t\right)\right)\geq\limsup_{n\to\infty}\left(d\left(x_n,y_n\right)\right)\geq 1,$$

a contradiction. □

Proposition 3.2.1.
Consider a normed space $(X,\|.\|)$, $\phi\neq K\subset X$ and let $T:K\to X$ satisfy (A^*) then the inverse : $(I-T)^{-1}:(I-T)(K)\to K$ exists and is continuous.

Proof. Take $x,y\in K, x\neq y$ and denote $F=I-T$. Via (A^*), we get for $\beta\in\mathbb{L}$

$$1+\left\|x-y\right\|\ln(\beta(\left\|x-y\right\|))\geq\frac{\left\|x-y\right\|}{\left\|Tx-Ty\right\|}>0,\tag{3.13}$$

On the other hand, we have :

$$0<\left\|Tx-Ty\right\|\leqslant\left\|Fx-Fy\right\|+\left\|x-y\right\|.$$

Combining (3.13) and the last inequality, we deduce

$$\frac{1}{\left\|Fx-Fy\right\|+\left\|x-y\right\|}\leq\frac{1}{\left\|Tx-Ty\right\|}\leq\frac{1+\left\|x-y\right\|\ln(\beta(\left\|x-y\right\|))}{\left\|x-y\right\|}.$$

Again, through (3.13), we obtain

$$\left\|Fx-Fy\right\|\geq\frac{-\left\|x-y\right\|^2\ln(\beta(\left\|x-y\right\|))}{1+\left\|x-y\right\|\ln(\beta(\left\|x-y\right\|))}.$$

Therefore for all $x, y \in K$ the following inequality holds

$$\|Fx - Fy\| \geq -\|x-y\|^2 \ln(\beta(\|x-y\|)). \tag{3.14}$$

F is immediately injective ; otherwise, for some $x \neq y$ and $Fx = Fy$, we get

$$-\|x-y\|^2 \ln(\beta(\|x-y\|)) \leq 0,$$

a contradiction.

Consequently, F is invertible. Moreover, through the fact that $\beta \in \mathbb{L}$, (3.14) and for every $x, y \in (I-T)(K)$, we see that there exists $C > 0$ such that

$$\|F^{-1}x - F^{-1}y\|^2 \leq C\|x-y\|, \tag{3.15}$$

which ensures the the continuity of F^{-1}. □

Theorem 3.2.2. *Let K be a nonempty closed convex subset of a Banach space E and let $T, S : K \to E$ be the mappings satisfying the following conditions:*

1. *S is continuous and $S(K)$ resides in a compact subset of E.*
2. *T satisfies (A^*)*
3. *for every $z \in S(K)$ the inclusion $K \subset z + T(K)$ holds.*

Then there exists $\bar{x} \in K$ such that $S\bar{x} + T\bar{x} = \bar{x}$

Proof.
First, observe that for every $z \in S(K)$ the mapping $T + z$, due to (2) and (3), satisfies: the assumptions of Theorem 3.2.1. Thus, the equation

$$Tx + z = x \tag{3.16}$$

has only one solution $x \in K$. Denote by v a function which for every $z \in S(K)$ assigns $x \in K$ such that the above equation holds. Therefore for every $z \in S(K)$ we have:

$$T(v(z)) + z = v(z) \tag{3.17}$$

Consider $z_1, z_2 \in S(K)$. Without reducing the generality of our considerations we can assume that $v(z_1) \neq v(z_2)$. Since T satisfies (A^*) and the fact that $\beta \in \mathbb{L}$, there exists $C > 0$ such that

$$C \leq \frac{1}{\|v(z_1) - v(z_2)\|} - \frac{1}{\|T(v(z_1)) - T(v(z_2))\|} \tag{3.18}$$

Hence,

$$\|v(z_1) - v(z_2)\| \leq \frac{\|T(v(z_1)) - T(v(z_2))\|}{1 + C\|T(v(z_1)) - T(v(z_2))\|}. \tag{3.19}$$

Moreover, by $T(v(z)) + z = v(z)$, the following holds:

$$\|T(v(z_1)) - T(v(z_2))\| \leq \|v(z_1) - v(z_2)\| + \|z_1 - z_2\|. \tag{3.20}$$

Thus, we obtain

$$\begin{aligned}\|v((z_1)) - v(z_2)\| &\leq \frac{\|v(z_1) - v((z_2))\| + \|z_1 - z_2\|}{1 + C\|v(z_1) - v(z_2)\| + C\|z_1 - z_2\|} \\ &= \frac{\|v(z_1) - v(z_2)\|}{1 + C\|v(z_1) - v(z_2)\| + C\|z_1 - z_2\|} \\ &+ \frac{\|z_1 - z_2\|}{1 + C\|v(z_1) - v(z_2)\| + C\|z_1 - z_2\|} \\ &\leq \frac{\|v(z_1) - v(z_2)\|}{1 + C\|v(z_1) - v(z_2)\|} + \frac{\|z_1 - z_2\|}{1 + C\|z_1 - z_2\|}\end{aligned}$$

In consequence, we have

$$\frac{C\|v(z_1) - v(z_2)\|^2}{1 + C\|v(z_1) - v(z_2)\|} \leq \frac{\|z_1 - z_2\|}{1 + C\|z_1 - z_2\|}, \tag{3.21}$$

which yields that $\|z_1 - z_2\| \to 0$ implies $\|v(z_1) - v(z_2)\| \to 0$, and thus v is continuous on $S(K)$. By the continuity of S, it follows the continuity of the mapping $vS : K \to K$ and also that $(vS)(K)$ resides in a compact subset of E. According to Theorem 3.1.1, there exists $\bar{x} \in K$ such that $v(S(\bar{x})) = \bar{x}$. Through $T(v(z)) + z = v(z)$ we get

$$T(v(S(\bar{x}))) + S(\bar{x}) = v(S(\bar{x})) \tag{3.22}$$

what finally yields $T\bar{x} + S\bar{x} = \bar{x}$. □

3.3 An application to nonlinear integral equation

Let us consider the following integral equation rational type.

$$x(t) = \frac{p(t)\,x(t)}{1 - q(t)} + \int_{-\infty}^{t} f(t-s)\,g(x(s))\,ds, \tag{3.23}$$

where the mappings $p, q : \mathbb{R} \to \mathbb{R}$ are continuous and periodic with period $P > 0$, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous. We assume that

$(C1)$ $q(t) \geq 3p(t) + 1$ for all $t \in \mathbb{R}$,

$(C2)$ $p_0 = \min_{t\in\mathbb{R}} p(t) > 0$, $q_0 = \max_{t\in\mathbb{R}} q(t) < 1$,

$(C3)$

$$M = \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \left|f(t-s)\right| ds < \infty,$$

and

$$M' = \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \left|f'(t-s)\right| ds < \infty,$$

for all $t \in \mathbb{R}$,

$(C4)$

$$MM_g \leq \frac{P_M + q_0 - 1}{1 - q_0},$$

where M from $(C3)$,

$$M_g = \max_{|t| \leq 1} \left| g(t) \right|$$

and

$$P_M = \max_{t \in \mathbb{R}} p(t).$$

In order to establish the existence problem for the equation (3.23), we will apply Theorem 3.2.2.

Consider the Banach space E of all continuous periodic self-mappings on $\mathbb{R}$ with period $P > 0$, where $x \in E$ we denote by $\|x\|$ the supremum norm, i.e.,

$$\|x\| = \sup_{t \in [0,P]} |x(t)|.$$

Let K be a subset of E of the form

$$K := \{x \in E; \|x\| \leq 1\}$$

Theorem 3.3.1. *If the conditions* $(C1)-(C4)$ *are fullfiled then equation* (3.23) *has a* P*−periodic solution.*

Proof. Let us put for $x \in K$;

$$(Tx)(t) = \frac{p(t)\, x(t)}{1 - q(t)}$$

and

$$(Sx)(t) = \int_{-\infty}^{t} f(t-s)\, g(x(s))\, ds.$$

Note that, by $(C2)$, we have $1 - q(t) > 0$ for all $t \in \mathbb{R}$. The problem (3.23) is equivalent to the fixed point of the equation

$$x = Tx + Sx.$$

Consider $x, y \in K$ such that $Tx \neq Ty$. For $t \in [0,P]$, we get

$$\begin{aligned} & \frac{1}{\left|x(t) - y(t)\right|} - \frac{1}{\left|Tx(t) - Ty(t)\right|} \\ = & \frac{1}{\left|x(t) - y(t)\right|} - \frac{1 - q(t)}{p(t)\left|x(t) - y(t)\right|} \\ = & \frac{p(t) + q(t) - 1}{p(t)\left|x(t) - y(t)\right|} \end{aligned}$$

Using $(C1)$, we obtain

$$\frac{1}{\left|x(t) - y(t)\right|} - \frac{1}{\left|Tx(t) - Ty(t)\right|} \geq \frac{4}{\left|x(t) - y(t)\right|} \geq \|x - y\|.$$

Therefore, we get

$$\begin{aligned} \frac{\|Tx - Ty\|}{1 + \|x - y\| \, \|Tx - Ty\|} & \geq \frac{\left|Tx(t) - Ty(t)\right|}{1 + \|x - y\| \left|Tx(t) - Ty(t)\right|} \\ & \geq \left|x(t) - y(t)\right| \end{aligned}$$

for all $t \in [0, P]$. Consequently, we have

$$\frac{\|Tx - Ty\|}{1 + \|x - y\| \|Tx - Ty\|} \geq \|x - y\|.$$

It is easy to see that T is an injective mapping on K. So, (A^*) is satisfied for $\beta(t) = e^{-t}$. Thus the condition (2) of Theorem 3.2.2 is satisfied.

Now we are going to prove the condition (1) of Theorem 3.2.2. In order to reach our goal, we follow the method developed in the paper [33].

If $x \in K$ then $\|x\| \leq 1$ and

$$\|(Sx)\| \leq \left\| \int_{-\infty}^{t} f(t-s)\, g(x(s))\, ds \right\| \leq MM_g. \tag{3.24}$$

On the other hand, a basic change of variable allows us to get :

$$\begin{aligned} (Sx)(t+P) &= \int_{-\infty}^{t+P} f(t+P-s)\, g(x(s))\, ds \\ &= \int_{-\infty}^{t} f(t-s)\, g(x(s))\, ds = (Sx)(t) \end{aligned} \tag{3.25}$$

(3.24) and (3.25) show that S maps K into E and $S(K)$ is uniformly bounded.

For $z \in K$, differentiating $(Sz)(t)$ (using Leibniz formula), we obtain

$$(Sz)'(t) = f(0)\, g(z(t)) + \int_{-\infty}^{t} f'(t-s)\, g(z(s))\, ds,$$

which yields

$$\|(Sz)'\| \leq \left(|f(0)| + M'\right) M_g$$

Through the mean value theorem, the above inequality implies that $S(K)$ is an equicontinuous subset of E Then using the Ascoli-Arzela Theorem (see Theorem 3.1.2), we obtain that S is a compact mapping.

On the other hand, as g is a continuous function, we have

$$\|Sx\| \leq C(M)\, \|x\|,$$

where $C(M)$ is a positive constant which depends on M. Thus, S is continuous and the condition (1) of Theorem 3.2.2 is satisfied.

Finally, it remains to check the inclusion (3) in Theorem 3.2.2. Let $z \in S(K)$. Hence, there exists $w \in K$ such that $z(t) = (Sw)(t)$ for all $t \in [0, P]$. We are going to show that for any $x \in K$, we have $x - z \in T(K)$. Take $x \in K$. In view of $(C4)$ we get

$$\begin{aligned} \|x - z\| &\leq \|x\| + \|Sx\| \\ &\leq 1 + MM_g \\ &\leq \frac{P_M}{1 - q_0}. \end{aligned} \tag{3.26}$$

Let us put

$$r := \frac{P_M}{1-q_0}$$

and

$$B_r := \{y \in E; \|y\| \le r\}.$$

First, we observe that for every $w \in K$,

$$\|Tw\| \le \frac{P_M}{1-q_0}\|w\| \le r,$$

which yields that $w \in B_r$.

On the other hand, taking any $y \in E$ such that $\|y\| \le r$, we can put for $t \in [0, P]$

$$w(t) = \frac{1-q(t)}{p(t)} y(t).$$

Using $(C1)$and $(C2)$, we easily see that

$$\|w\| = \frac{1-q(t)}{p(t)}\|y\| \le -3\|y\| \le r$$

and

$$Tw(t) = \frac{p(t)\frac{1-q(t)}{p(t)}y(t)}{1-q(t)} = y(t),$$

which implies that $w \in T(K)$. Consequently,

$$T(K) = B_r, r = \frac{P_M}{1-q_0}. \tag{3.27}$$

From (3.26) and (3.27), we have $x - z \in B_r = T(K)$ and the condition (3) of Theorem 3.2.2 is satisfied. Therefore equation (3.23) has a P-periodic solution and the proof is completed. □

Conclusion and prospects

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3.4 Chapter 2 and around

In this memoir, we have proved a fixed point result for Dass-Gupta contraction-type in the context of b-metric spaces via F-contraction and we improve some existing results in the literature (see chapter 2). We give also an application o second-order differential equations.

As a prospects, it will be interesting to improve the application given in the section 2.2 by constructing a suitable function F without conditions (F_2), (F_3) and the strictness of the monotonicity of F. It will be interesting as well to apply Theorem 2.1.1 to prove the existence and uniqueness of a solution for the following integral equation of Volterra type:

$$u(t) = g(t) + \int_0^t K(t,s,u(s))\,ds, \quad t \in [0,a], \tag{3.28}$$

where $a > 0$, $K : [0,a] \times [0,a] \times \mathbb{R} \to \mathbb{R}$ and $g : [0,a] \to \mathbb{R}$.

The suitable functional framework is as follows: $X = \mathscr{C}([0,a], \mathbb{R})$ be the set of all continuous functions $u : [0,a] \to \mathbb{R}$. It is well known that X equipped with Bielecki's norm

$$\|u\| = \sup_{t\in[0,a]} e^{-t} |u(t)|$$

is a Banach space. Thus, X endowed with the distance associated to Bielecki's norm

$$d(u,v) = \sup_{t\in[0,a]} e^{-t} |u(t) - v(t)|, \quad \text{for all } u, v \in X,$$

is a complete metric space. Now for $p \geq 1$, we define

$$\sigma(u,v) = (d(u,v))^p = \sup_{t\in[0,a]} e^{-pt} |u(t) - v(t)|^p, \quad \text{for all } u, v \in X. \tag{3.29}$$

Obviously, by Example 1.1.1, (X,σ) is a complete b-metric space with the constant $s = 2^{p-1}$.

3.5 Chapter 3 and around

In the chapter 3 of this memoir, we study also the existence and uniqueness of the fixed point for expansive mappings-types and we generalize some results existing in the paper [28].

As prospects for this chapter, it will be interesting to weaken some assumptions of Theorem 3.2.2 to obtain better results and nice application for the integral equations with delay.

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