

People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research



ABBES LAGHROUR UNIVERSITY - KHENCHELA

Option: Applied Mathematics

Theme:

**Effects of diffusion in controlling chaos in some
tri-trophic food chains**

Presented by:

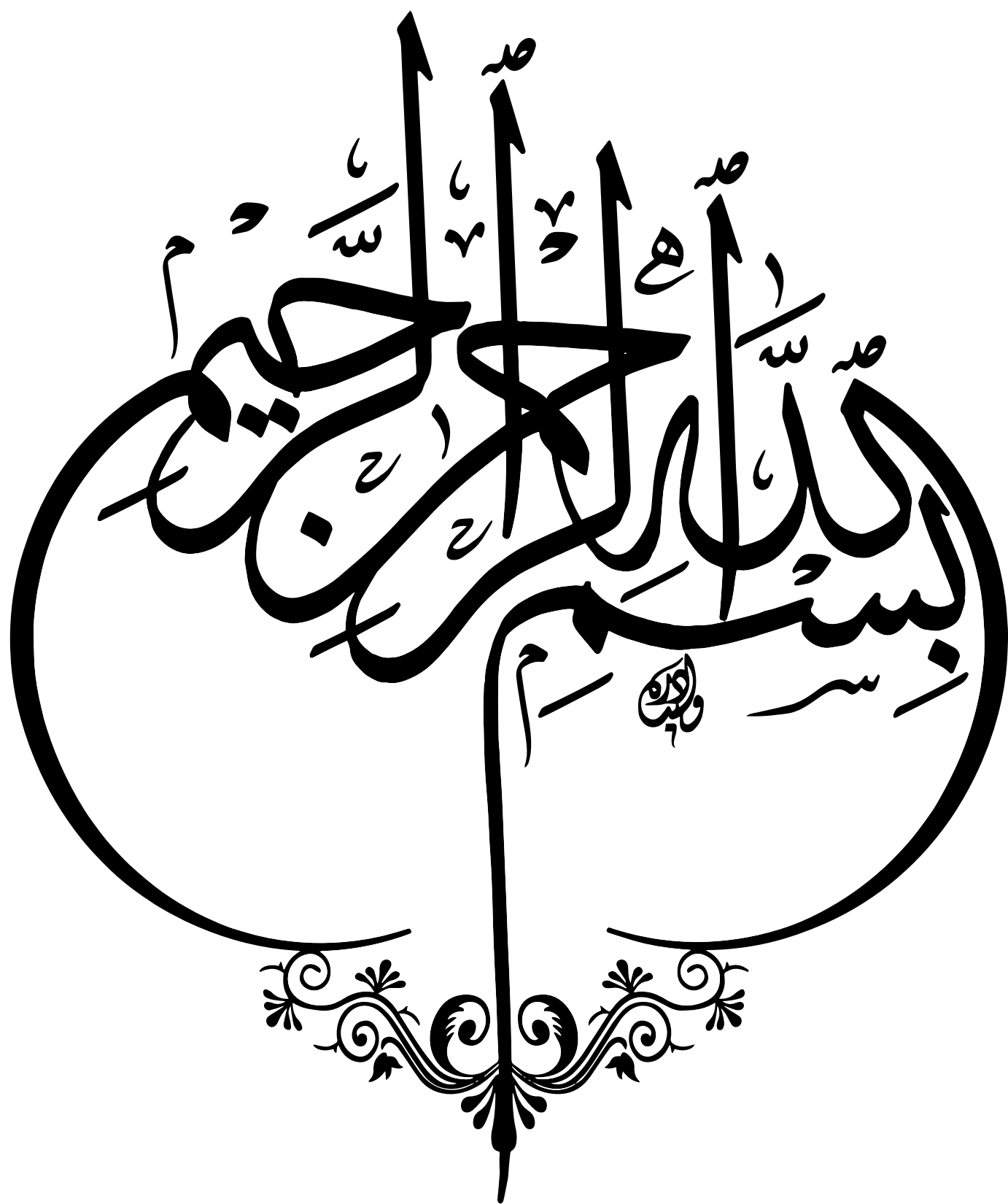
Mr. Abd Rraouf Abid

Supervisor:

Professor Emeritus Said Kouachi

**Thesis defended on, 2026, in front of
the jury composed of:**

Pr. SAOUDI Khaled	University Khenchela	President
Prof Emer. KOUACHI Saïd	University Khenchela	Supervisor
Dr. RAMOUL Hichem	University Khenchela	Examiner
Dr. Rabia Aouafi	University Khenchela	Examiner
Pr. Benhadid Ayache	University Batna 2	Examiner
Pr. Melkmi Khaled	University Batna 2	Examiner



Dedication

To my dear father Mansour Abid now gone, may God have mercy on your soul.

I know you have waited too long to see this day. You waited through the late nights, the doubts, the years of hard work. You believed in me when I struggled to believe in myself. And though you are no longer here to hold this finished thesis in your hands, I feel your presence in every page, every word, and every quiet moment of accomplishment.

I know you have waited too long to see this, but faith had something else in store. God had another plan one that I cannot fully understand, but one that I must learn to accept, even as my heart aches with your absence. Perhaps you are watching from a place far greater than any graduation hall. Perhaps you already know that this moment belongs to you as much as it belongs to me.

You taught me patience when I wanted to give up. You taught me integrity when no one was watching. You gave me your time, your wisdom, your silent sacrifices never asking for anything in return except that I try my best. This thesis is not only the result of my research; it is the result of your love, your encouragement, and your unwavering faith in me.

So I place this work at your feet, my dear father. It is yours. Every chapter, every correction, every small victory they all carry your name in my heart. May God have mercy on your soul, grant you peace in His eternal garden, and let you rest knowing that your child finally made it.

You waited too long for this, but you did not wait in vain. And though you left too soon, you will never be absent. Not from this thesis. Not from my life. Not from the person I will continue to become because of you.

With all my love, gratitude, and longing this is for you.

My deepest thanks go to my mother and my sister. Through every struggle and every victory, they have loved me unconditionally, encouraged me endlessly, and supported me without fail. In hardship and in happiness, their presence has been a true blessing.

And to my dear friends and colleagues thank you. Your constant encouragement, quiet understanding, and faithful companionship gave me the strength to keep going. Your friendship has been invaluable, and your belief in me has been a pillar I could always lean on throughout this unforgettable journey.

*I also dedicate this to the awesome and sweetest twins I ever seen my cousin
Adib & Moetez.*

Acknowledgement

All praise is due to Allah, to whom belongs all gratitude and thanks. We seek His help in times of ease and hardship and place our trust in Him in all matters. May peace and blessings be upon the best of creation, our beloved Prophet Muhammad, and upon his family and companions.

I would like first and foremost to express my profound gratitude to my supervisor, Professor **Said Kouachi**, for his exceptional guidance, continuous support, and insightful advice. His scientific rigor, generosity with his time, and constant encouragement have been invaluable throughout the preparation of this thesis. Working under his supervision has been a privilege and an inspiring experience that has deeply shaped my academic path.

I also gratefully acknowledge the support of Doctor **Nadir Teyar**, whose guidance and valuable advices have been of great help during the course of my doctoral studies.

I wish also to extend my sincere appreciation to Professor **Ramoul Hichem** who provided me with valuable assistance, advice, and encouragement. His generosity, open-mindedness, and readiness to help deserve my utmost respect and gratitude.

I would also like to offer my sincere thanks to the members of the jury Professor **Saoudi Khaled**, Doctor **Rabia Aouafi**, Professor **Benhadid Ayache**, and Professor **Melkmi Khaled** for their kindness in accepting my invitation to be part of this thesis defense.

And finally, to all my professors: thank you. I name each of you with gratitude. Every piece of advice, every word of guidance you so generously shared along my academic journey has shaped me profoundly.

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List of Symbols

$ \Omega $	Lebesgue measure of the domain Ω
$\text{tr}(\cdot)$	Trace of a matrix (sum of diagonal elements)
$\text{com } A$	Adjugate (classical adjoint) matrix of A
∇	Gradient operator
$\text{div}(\cdot)$	Divergence operator
$\mathcal{F}$	Fourier transform
k	Wave number
A	Jacobian matrix of reaction terms
D	Diffusion matrix
$L(t)$	Lyapunov functional
Σ	Invariant region
$Q(\cdot, \cdot)$	Quadratic form
$\tilde{k}_i^2$	Roots of the cubic polynomial φ_3
k_i^2	Roots of $\frac{\partial}{\partial k^2} \varphi_3$
$\hat{K}_i^2$	Roots of the polynomial φ
K_i^2	Roots of $\frac{\partial}{\partial k^2} \varphi$
i.e.	That is

Introduction

THE study of patterns in nature has fascinated humans for centuries. From ancient philosophers pondering the orderly structures in living organisms to Renaissance artists capturing the geometry of natural forms, the question of how complexity emerges from simplicity has persisted across cultures and disciplines. Yet, a rigorous scientific explanation for the origin of natural patterns was elusive until the mid-twentieth century. In 1952, the British mathematician and logician Alan Turing, best known for his foundational work in computer science, published a groundbreaking paper entitled *The Chemical Basis of Morphogenesis*. In this work, Turing proposed a quantitative mechanism explaining how uniform systems of interacting chemicals could spontaneously develop spatial patterns, a phenomenon now known as Turing instability. Turing's contribution was revolutionary in several ways. First, he challenged the prevailing notion that biological forms were entirely pre-determined by genetic or external instructions. Instead, he suggested that intrinsic chemical and physical interactions, governed by reaction kinetics and diffusion, could generate structure spontaneously. This idea introduced the notion of self-organization, in which order can emerge naturally from local interactions without centralized control. Second, Turing's approach merged mathematics with biology in an unprecedented way. While previous biological studies focused on observation and classification, Turing applied linear stability analysis, partial differential equations, and reaction-diffusion theory to explain phenomena as diverse as animal coat patterns and the segmentation of embryos. Though initially overlooked, Turing's theory gained prominence decades later, as experimental techniques in biology and chemistry began to reveal systems in which reaction-diffusion mechanisms could plausibly operate. Early skepticism centered on whether the conditions required for diffusion-driven instability were realizable in living organisms. However, the subsequent decades have shown that his predictions are not merely theoretical curiosities but can indeed describe real chemical and biological phenomena. Turing's work thus marks a turning point in the history of scientific thought, where abstract mathematical theory began to provide concrete explanations for the complexity observed in nature. One of the most remarkable aspects of Turing instability is its interdisciplinary relevance. Although initially conceived in the context of developmental biology, the underlying principles of reaction-diffusion systems have found applications in chemistry, ecology, physics, materials science, and even social sciences. In biology, Turing instability provides a framework for understanding morphogenesis, the process by which organisms develop their shape and structure. Pigmentation patterns in animals—such as the stripes of zebras, spots of leopards, and banded shells of mollusks—can be interpreted as outcomes of activator-inhibitor interactions, where morphogens diffuse through tissues and interact locally to generate spatial heterogeneity. Similarly, the periodic segmentation of vertebrate embryos and the arrangement of hair follicles or feather buds are thought to arise from Turing-like mechanisms, demonstrating how a simple set of chemical and physical rules can give rise to complex,

repeatable structures. In chemistry, experimental systems such as the Belousov-Zhabotinsky reaction and other oscillatory reactions exhibit spontaneous pattern formation that closely aligns with Turing's predictions. In these reactions, interacting chemical species diffuse at different rates, producing stable spatial structures such as spots, stripes, or labyrinthine patterns in otherwise uniform reaction media. These experiments not only confirm theoretical predictions but also allow scientists to systematically manipulate parameters like diffusion rates and reaction strengths, providing insights into the mechanisms underlying pattern formation. Ecology also benefits from the Turing framework. Spatial distribution of species in ecosystems, such as vegetation patches in arid landscapes or predator-prey population clusters, can be modeled using reaction-diffusion equations. In such systems, local interactions and differential dispersal rates lead to non-uniform population densities that resemble classical Turing patterns. This perspective has practical implications for ecosystem management, conservation biology, and understanding the resilience of ecological communities. In materials science and engineering, reaction-diffusion principles have inspired the design of micro- and nano-structured surfaces. Controlled chemical interactions and diffusion processes are used to generate surfaces with specific patterns, which can influence properties like wettability, adhesion, and optical characteristics. Similarly, in synthetic biology, engineered gene circuits employ activator-inhibitor dynamics to program bacterial colonies or tissue constructs to develop predetermined spatial patterns, demonstrating how Turing principles can be harnessed for applied science. Even in physics and social sciences, analogues of Turing instability appear. Nonlinear systems exhibiting pattern formation, such as convection rolls in fluid dynamics, chemical waves, or clustering behaviors in social networks, share common mathematical structures with classical Turing systems, underscoring the universality of diffusion-driven instabilities. Understanding Turing instability is not merely an academic exercise—it provides a fundamental framework for deciphering complexity in natural systems. One of its most important contributions is the demonstration that order can emerge spontaneously from uniformity, without requiring external templates or centralized control. This insight has deep implications across scientific disciplines: it challenges deterministic paradigms in biology, provides predictive power in chemistry and ecology, and informs engineering and design in synthetic systems. In practical terms, Turing instability allows scientists to predict and manipulate patterns. In developmental biology, it informs studies on morphogen distribution, tissue engineering, and regenerative medicine. In chemistry, it provides a guide for controlling spatial structures in reactive media. In ecology, it helps model spatial population dynamics and ecosystem patterns. Beyond applied science, the concept also enriches theoretical understanding of self-organization, nonlinear dynamics, and emergent behavior, which are central to modern complexity science. Moreover, Turing instability acts as a conceptual bridge between disciplines. It unites ideas from mathematics, physics, chemistry, and biology under a common theoretical framework, showing that the emergence of structure can often be explained using universal principles, rather than relying on context-specific rules. By studying Turing systems, scientists gain insight into how local interactions propagate to generate global order—a theme that resonates from molecular biology to social dynamics.

The organization of this thesis follows a logical progression from background material to original analytical contributions and their application. Chapter 1 surveys the historical development of Turing's instability theory for two-component systems, providing classical examples that illustrate diffusion-driven pattern formation. Chapter 2 assembles the necessary mathematical toolkit, covering Sobolev spaces, parabolic regularity theory, maximum principles, and stability results of dynamic systems.

In Chapter 3, we introduce three-component reaction-diffusion systems with cross-diffusion, justifying their importance in modeling complex multi-species interactions across scientific disciplines. This chapter also presents the linearization methodology around general steady states, forming the basis for subsequent instability analysis.

Chapter 4 delivers our principal theoretical results: complete spectral conditions for Turing instability in systems with full diffusion matrices, explicitly relating wave numbers to entries of both reaction Jacobian and diffusion matrices.

Chapter 5 applies this framework to a specific ecological model—a predator-prey system with three interacting species. We rigorously establish solution properties: local existence via contraction mapping arguments, invariant region construction based on ecological bounds, and global existence through energy estimates derived from tailored Lyapunov functionals, thereby connecting abstract spectral theory to concrete biological predictions.

Chapter 1

Foundation and Historical Background

THE study of spatial pattern formation in reaction–diffusion systems traces back to the seminal work of Alan Turing, who demonstrated that diffusion, typically viewed as a stabilizing mechanism, can in fact induce instabilities in systems that are otherwise stable in the absence of spatial effects. This phenomenon, now known as diffusion-driven instability, provides a theoretical explanation for the spontaneous emergence of ordered patterns in chemical and biological media. This chapter develops the mathematical framework necessary to analyze such instabilities. Starting from a general two-component reaction–diffusion system, the dynamics are linearized around a steady state, and the influence of diffusion on the stability of the linearized system is examined. Fourier analysis is employed to express the evolution of perturbations in terms of spatial modes, leading to the derivation of the corresponding dispersion relation. The stability of each mode is then assessed through the characteristic polynomial and standard criteria from linear stability theory. The purpose of this chapter is to establish the precise conditions required for the onset of diffusion-driven instability and to clarify how diffusion alters the eigenvalue spectrum of the system. These results form the basis for the analysis of concrete models and pattern-generating mechanisms in subsequent chapters.

Mathematically, the idea can be expressed as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = f(u, v), \\ \frac{\partial v}{\partial t} - d_2 \Delta v = g(u, v). \end{cases} \quad (\text{E})$$

let (u^*, v^*) denotes any steady state of $F(u, v) = (f(u, v), g(u, v))^t$, the reaction terms f and g can be written around (u^*, v^*) as follows

$$f(u, v) = f(u^*, v^*) + f_u(u^*, v^*)u + f_v(u^*, v^*)v + o(\|(u, v)\|), \quad (*)$$

$$g(u, v) = g(u^*, v^*) + g_u(u^*, v^*)u + g_v(u^*, v^*)v + o(\|(u, v)\|). \quad (**)$$

Substituting the expansions into system (E), we obtain:

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = f_u(u^*, v^*)u + f_v(u^*, v^*)v, \\ \frac{\partial v}{\partial t} - d_2 \Delta v = g_u(u^*, v^*)u + g_v(u^*, v^*)v. \end{cases}$$

At this stage, we apply the Fourier transform to both equations:

$$\begin{cases} \mathcal{F} \left(\frac{\partial u}{\partial t} - d_1 \Delta u \right) = \mathcal{F} (f_u (u^*, v^*) u + f_v (u^*, v^*) v), \\ \mathcal{F} \left(\frac{\partial v}{\partial t} - d_2 \Delta v \right) = \mathcal{F} (g_u (u^*, v^*) u + g_v (u^*, v^*) v). \end{cases}$$

Using the properties of the Fourier transform, we obtain:

$$\begin{cases} \frac{\partial}{\partial t} \mathcal{F} u + d_1 k^2 \mathcal{F} u = f_u (u^*, v^*) \mathcal{F} u + f_v (u^*, v^*) \mathcal{F} v, \\ \frac{\partial}{\partial t} \mathcal{F} v + d_2 k^2 \mathcal{F} v = g_u (u^*, v^*) \mathcal{F} u + g_v (u^*, v^*) \mathcal{F} v. \end{cases}$$

Let

$$\mathcal{F} u = \tilde{u}, \quad \mathcal{F} v = \tilde{v}.$$

This leads to

$$\frac{\partial}{\partial t} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \underbrace{(J - k^2 D)}_{A(k)} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix},$$

where,

- $J = \begin{pmatrix} f_u (u^*, v^*) & f_v (u^*, v^*) \\ g_u (u^*, v^*) & g_v (u^*, v^*) \end{pmatrix}$ is the Jacobian matrix,
- $D = \text{diag} (d_1, d_2)$ is the diffusion matrix.

The central idea is that, in the absence of diffusion (i.e., $d_1 = d_2 = 0$), the Jacobian matrix is stable; in other words, all roots have negative real parts. However, diffusion may cause $A(k)$ to have a positive eigenvalue.

The characteristic polynomial is given by,

$$\begin{aligned} P(\lambda) &= \det (J - \lambda I) \\ &= \lambda^2 - \text{tr} J \lambda + \det J, \end{aligned}$$

If $(\text{tr} J)^2 - 4 \det J > 0$, then the polynomial has two real roots:

$$\lambda_1 = \frac{\text{tr} J - \sqrt{(\text{tr} J)^2 - 4 \det J}}{2}, \quad \lambda_2 = \frac{\text{tr} J + \sqrt{(\text{tr} J)^2 - 4 \det J}}{2}.$$

To ensure that both λ_1 and λ_2 are negative, we assume that

$$(\text{tr} J)^2 - 4 \det J \geq 0, \quad \text{tr} J < 0, \quad \det J > 0.$$

We now consider the matrix $A(k)$. Its characteristic polynomial is:

$$\begin{aligned} \tilde{P}(\lambda) &= \det (A(k) - \lambda I) \\ &= \lambda^2 - (\text{tr} J - k^2 \text{tr} D) \lambda + \det J - (j_{11} d_2 + j_{22} d_1) k^2 + \det D k^4. \end{aligned}$$

The discriminant of $\tilde{P}$ is given by:

$$\Delta = \left((\text{tr} D)^2 - 4 \det D \right) (k^2)^2 + 2 (\det A - \det B) k^2 + (\text{tr} J)^2 - 4 \det J,$$

where,

$$A = \begin{pmatrix} j_{11} & d_1 \\ j_{11} & d_2 \end{pmatrix}, \quad B = \begin{pmatrix} j_{22} & d_1 \\ j_{22} & d_2 \end{pmatrix}.$$

We already assume that $(\text{tr}J)^2 - 4 \det J > 0$, it appears that $(\text{tr}D)^2 - 4 \det D > 0$, then if we select $\det A \geq \det B$ we get

$$\Delta(k^2) > 0 \quad \forall k^2 \geq 0.$$

At this point, by applying the Routh–Hurwitz stability criterion, we obtain:

$$\begin{aligned} \varphi(k^2) &= -(\text{tr}J - k^2 \text{tr}D) > 0, \\ \text{and} \\ \psi(k^2) &= \det J - (j_{11}d_2 + j_{22}d_1)k^2 + \det D k^4 > 0. \end{aligned}$$

It follows that both eigenvalues Λ_1 and Λ_2 of $\tilde{P}$ are negative. So if we take

$$\begin{aligned} \text{tr}D k^2 - \text{tr}J &\leq 0, \\ \text{or} \\ \det J - (j_{11}d_2 + j_{22}d_1)k^2 + \det D k^4 &\leq 0, \end{aligned}$$

This implies that one eigenvalue becomes positive. Since $\varphi(k^2) > 0$, it suffices to determine values of k^2 for which $\psi(k^2) \leq 0$.

Clearly ψ is a polynomial on k^2 , if

$$\Delta_\psi = (j_{11}d_2 + j_{22}d_1)^2 - 4 \det J \det D \geq 0,$$

then

$$\psi(k^2) \leq 0 \quad \forall k^2 \in [k_1^2, k_2^2],$$

where k_1^2 and k_2^2 are the roots of ψ .

Finally, we obtain the disiated result under the following assumption:

- f and g are continiously differentiable,
- $(\text{tr}J)^2 - 4 \det J \geq 0$,
- $\text{tr}J < 0$ and $\det J > 0$,
- $\det A \geq \det B$,
- $(j_{11}d_2 + j_{22}d_1)^2 - 4 \det J \det D \geq 0$.

1.1 Some examples

1.1.1 The Brusselator model

This is a standard form of the Brusselator model, where chemical u is produced from a constant source a and autocatalytically with v , while v is produced from u . The terms $d\Delta u$ and Δv represent Fickian diffusion in space. The central result of the paper is that for a fixed $d > 1$, a Turing instability occurs when b exceeds a critical threshold, transforming the homogeneous steady state into stable, stationary spatial patterns.

$$\begin{cases} \frac{\partial u}{\partial t} = a - (b+1)u + u^2v + d\Delta u, \\ \frac{\partial v}{\partial t} = bu - u^2v + \Delta v. \end{cases} \quad (1.1)$$

Muntari, Umar Faruk and Sengül, Taylan [23] get Turing's instability under the condition

$$\sqrt{d} < \frac{\sqrt{1+a^2}-1}{a}.$$

1.1.2 Ecological Competition with Allee

This model describes a spatially extended predator-prey system where prey u exhibits a strong Allee effect ($\delta u - 1$) and predators v benefit from cooperative hunting ($1 + \beta v$). The predator's growth is logistic but capped by a ratio-dependent carrying capacity ηu . The paper's goal is to analyze how these mechanisms, combined with differential diffusion (d_1, d_2), can destabilize homogeneous coexistence to generate stable Turing patterns.

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = u(1-u)(\delta u - 1) + (1 + \beta v)uv, \\ \frac{\partial v}{\partial t} - d_2 \Delta v = sv \left(1 - \frac{v}{\eta u}\right). \end{cases} \quad (1.2)$$

Danxia Song, Yongli Song, and Chao Li [22] established Turing's instability subject to the following assumptions

$$\begin{aligned} d_1 < d_2 \quad \text{and} \quad S_H^{(0)} < S_T^{(1)}, \\ S_H^{(0)} &= \frac{1 + \delta - \eta + \sqrt{Q_1}}{2(\delta + \beta\eta^2)^2} (\eta(\delta + \beta\eta\delta + \beta\eta) - \delta\sqrt{Q_1}), \\ S_T^{(1)} &= \frac{-2d_1d_2(a_{11} + 2\eta a_{12}) - \sqrt{Q_2}}{2d_1^2}, \\ Q_1 &= (1 + \delta - \eta)^2 - 4(\delta + \eta^2\beta), \quad Q_2 = 16d_1^2d_2^2\eta a_{12}(a_{11} + \eta a_{12}), \\ A &= (a_{ij})_{2 \times 2} \quad \text{is the Jacobian matrix.} \end{aligned}$$

1.1.3 Spatial SIS Epidemic Model

This model extends a standard SIS framework by incorporating cross-diffusion d_{12} and a nonlinear incidence rate $\beta S^n I$. The terms $a(S + I)$ and bs represent population-level dynamics like birth and death, while recovery and disease-induced mortality are captured by γ and α . The authors demonstrate that the cross-diffusion term is crucial in breaking the symmetry necessary to trigger a Turing instability, leading to stationary spatial patterns in disease prevalence.

$$\begin{cases} \frac{\partial S}{\partial t} = -\beta S^n I^m - bs + \gamma I + a(S + I) + d_{11} \frac{\partial^2 S}{\partial x^2} + d_{12} \frac{\partial^2 I}{\partial x^2} \\ \frac{\partial I}{\partial t} = \beta S^n I^m - (\alpha + \beta + \gamma) I + d_{22} \frac{\partial^2 I}{\partial x^2}. \end{cases} \quad (1.3)$$

Shaban Aly, Houari B. Khenous, and Fatma Hussien [3] demonstrate pattern formation under

$$\begin{cases} A_{11} + A_{22} - \left(\frac{\pi j}{\ell}\right)^2 (d_{11} + d_{22}) > 0, \\ \det A > 0, \\ d_{22} \geq \frac{A_{22}\ell^2}{\pi^2} \quad \text{and} \quad d_{11} > 0, \\ \frac{A_{22}\ell^2}{\pi^2} > d_{22} > \frac{4A_{22}\ell^2}{\pi^2}. \end{cases}$$

here, the matrix A is given by

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} (a-b)(1-nT) & -m(\alpha+\beta+\gamma)+\alpha+\gamma \\ (a-b)nT & (m-1)(\alpha+\beta+\gamma) \end{pmatrix}$$

which is the Jacobian matrix.

1.1.4 FitzHugh-Nagumo Type Model

This is a spatially extended FitzHugh-Nagumo model, a canonical activator-inhibitor system for neural and cardiac excitable media. The parameters control excitability, with c scaling the fast kinetics and a, b defining the slow recovery variable's dynamics. The authors' novel contribution lies in demonstrating that the standard Turing condition fails here, and a Turing instability leading to stationary patterns can only emerge when the diffusion process is placed on a random network topology rather than a regular lattice.

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = c \left(u - \frac{u^3}{3} - v \right), \\ \frac{\partial v}{\partial t} - d_2 \Delta v = c (au - bv). \end{cases} \quad (1.4)$$

Qianqian Zheng and Jianwei Shen [27] have showed that the critical condition for Turing's instability is given by

$$\frac{4d_1d_2(ac^2 - bc^2) - (bcd_1 - cd_2)^2}{4d_1d_2} = 0.$$

1.1.5 Gierer–Meinhardt Model

The Gierer–Meinhardt equations describe an activator u that autocatalytically promotes its own production via $\frac{u^2}{v}$, while also linearly producing the faster-diffusing inhibitor v . The system's steady state becomes diffusionally unstable under the condition $\sigma_2 > \sigma_1$, initiating a Turing bifurcation. This leads to the spontaneous formation of highly localized, stable activator peaks amidst a shallow inhibitor background, characteristic of the model's singularly perturbed nature.

$$\begin{cases} \frac{\partial u}{\partial t} - \sigma_1 \Delta u = \frac{u^2}{v} - \beta u, \\ \frac{\partial v}{\partial t} - \sigma_2 \Delta v = u^2 - v. \end{cases} \quad (1.5)$$

Yong Wang, Mengping Guo, and Weihua Jiang [24] proved Turing's instability such that the following hold

$$\sigma_1 < \sigma_2, \\ \left(3 + 2\sqrt{2}\right) < \frac{\sigma_2}{\sigma_1} \quad \text{and} \quad \beta \in \left(\frac{3 + 2\sqrt{2}}{\sigma_2} \sigma_1, 1\right).$$

1.1.6 Gray–Scott Model

This analysis introduces a relaxation time τ to the classic Gray–Scott model's activator kinetics, introducing memory effects. The reaction terms capture cubic autocatalysis uv^2 , where v consumes u , coupled with a constant feed F and removal K . The core result is that the derived amplitude equation near the Turing bifurcation point quantitatively predicts both stationary patterns and their transition to spatiotemporal chaos as F increases.

$$\begin{cases} (1 + \tau) \frac{\partial u}{\partial t} - D_u \Delta u = -uv^2 + F(1 - u), \\ \frac{\partial v}{\partial t} - D_v \Delta v = uv^2 + (F + K)v. \end{cases} \quad (1.6)$$

Wen-Xian Xie, Shu-Ping Cao, Li Cai, and Xiao-Xuan Zhang [26] derived the Turing's instability under the following conditions

- $0 < \tau < \frac{b_2^2 - K}{F + K},$
- $\frac{a_{11}^*}{\sigma_*} + a_{22}^* > 2\sqrt{\frac{\Delta_0^*}{\sigma_*}},$

the parameters are defined as:

- $a_{11}^* = \frac{-(F + v_*)^2}{p},$
- $a_{22}^* = 2u_*v_*,$
- $D_u^* = \frac{D_u}{p}$ and $D_v^* = \frac{D_v}{p},$
- $\sigma_* = \frac{D_u^*}{D_v^*},$
- $\Delta_0^* = \frac{(F + K)(v_*^2 - F)}{1 + \tau},$
- where (u_*, v_*) denotes the positive equilibria of system (1.6).

Chapter 2

Preliminaries

THIS chapter is devoted to all essential mathematical tools used throughout this thesis. First, we recall necessary concepts from the theory of ordinary differential equations, such as the stability of dynamical systems. Next, we present important formulas and inequalities. Finally, we define useful functional spaces and present results concerning both local and global existence for reaction-diffusion systems. In what follows Ω denotes an open bounded domain of $\mathbb{R}^n$.

2.1 Dynamic systems

2.1.1 Taylor expansion

Definition 2.1. Let Ω be an open subdomain of $\mathbb{R}^n$, assuming that f has m continuous partial derivatives, then in the neighborhood of any a in Ω the function f is written as follows

$$f(x) = \sum_{k=0}^m \frac{1}{k!} \left[\sum_{i_1=1}^n \dots \sum_{i_k=1}^n \frac{\partial^k f(a)}{\partial x_{i_1} \dots \partial x_{i_k}} (x_{i_1} - a_{i_1}) \dots (x_{i_k} - a_{i_k}) \right] + o(\|x - a\|^m),$$

where $o(\|x - a\|^m)$ is the remainder term of m order. In case where $\Omega \subset \mathbb{R}^2$ we have the first order linear approximation around $a = (a_1, a_2)$

$$f(x, y) = f(a_1, a_2) + f_x(a_1, a_2)x + f_y(a_1, a_2)y + o(\|x - a_1, y - a_2\|).$$

2.1.2 Fourier Transform

Definition 2.2. For any integrable function $f : \mathbb{R}^n \rightarrow \mathbb{C}$ i.e., $\left(\int_{\mathbb{R}^n} |f(x)| dx < \infty \right)$ the Fourier transform of f is

$$\mathcal{F}\{f\}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) e^{-i\xi x} dx.$$

2.1.3 Some properties of the Fourier transform

For any f and g in $L^1(\mathbb{R}^n)$ and any scalars a, b in $\mathbb{C}$ we have

Property	Function	Fourier transform
Linearity	$af + bg$	$a\mathcal{F}\{f\} + b\mathcal{F}\{g\}$
Convolution	$f * g$	$\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}$
Multiplication	$f \cdot g$	$\frac{1}{(2\pi)^n} \mathcal{F}\{f\} * \mathcal{F}\{g\}$
Scaling	$H_a f = f(ax)$	$ a ^{-n} H_{\frac{1}{a}} \mathcal{F}\{f\}$
Tensor product	$f \otimes g$	$\mathcal{F}\{f\} \otimes \mathcal{F}\{g\}$
Translation	$f(x - a)$	$e^{-i\xi a} \mathcal{F}\{f\}$
Transform of derivative	$D^\alpha f$	$(i\xi)^\alpha \mathcal{F}\{f\}$
Derivative of the Transform	$D_\xi^\alpha \{f\}$	$\mathcal{F}\{(-ix)^\alpha f(x)\}$

2.2 Stability of the dynamic systems

Consider the following system

$$\frac{d}{dt} \begin{pmatrix} y_1(t) \\ \vdots \\ y_k(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1k} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \dots & \vdots \\ a_{k1} & \dots & a_{kk} & \dots & a_{kn} \\ \vdots & \dots & \dots & \ddots & \vdots \\ a_{n1} & \dots & a_{nk} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1(t) \\ \vdots \\ y_k(t) \\ \vdots \\ y_n(t) \end{pmatrix},$$

which is equivalent to

$$\frac{d}{dt} Y(t) = AY(t),$$

let $\sigma(A) = \{\lambda_k\}$ be the set of the eigenvalues of A .

Definition 2.3. The system is said to be asymptotically stable if and only if:

$$\Re(\lambda_k) < 0 \quad \forall \lambda_k \in \sigma(A).$$

The system is stable in the sense of Lyapunov if and only if:

$$\Re(\lambda_k) \leq 0 \quad \forall \lambda_k \in \sigma(A).$$

The system is unstable if either

$$\left\{ \begin{array}{l} \exists \lambda_k \in \sigma(A) \text{ such that } \Re(\lambda_k) > 0, \\ \text{or} \\ \exists \lambda_k \in \sigma(A) \text{ with } \Re(\lambda_k) = 0. \end{array} \right.$$

2.2.1 Routh-Hurwitz stability criterion

Given the polynomial

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n,$$

where the coefficients a_i are real, we define n Hurwitz matrices by the coefficients a_i :

$$H_1 = a_1, \quad H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, \quad H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix},$$

and

$$H_n = \begin{pmatrix} a_1 & 1 & 0 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & 1 & \dots & 0 \\ a_5 & a_4 & a_3 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_n \end{pmatrix},$$

such that $a_j = 0$ if $j > n$, then we have the following theorem.

Theorem 2.4. *All the roots of P are negative or have a negative real part if and only if the determinant of all Hurwitz matrices are positive*

$$\det H_j > 0 \quad \forall j = 1 \dots n.$$

Corollary 2.5. *Routh-Hurwitz criteria for*

$$n = 2 : a_1 > 0 \text{ and } a_2 > 0,$$

$$n = 3 : a_1 > 0, a_3 > 0 \text{ and } a_1 a_2 - a_3 > 0.$$

2.2.2 Quadratic equations

Let

$$ax^2 + bx + c = 0,$$

where a, b and c are real, the following proposition assures the existence and the positivity of the roots.

Proposition 2.6. *All roots of the second degree algebraic equation exist and are positive if and only if*

$$b^2 - 4ac > 0 \quad a, c > 0 \quad \text{and} \quad b < 0.$$

2.3 Functional spaces

2.3.1 Lebesgue spaces

Definition 2.7. Let $p \in [1, +\infty)$, we call Lebesgue space the set of all p -th power absolutely integrable functions denoted by $L^p(\Omega)$ i.e.,

$$L^p(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ measurable} / \int_{\Omega} |f(x)|^p dx < \infty \right\}.$$

For $p = \infty$ we obtain Lebesgue space of essentially bounded functions defined by

$$L^\infty(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ measurable} / \sup_{x \in \Omega} |f(x)| < \infty \right\}.$$

The spaces $L^p(\Omega)$ and $L^\infty(\Omega)$ are equipped, respectively, with the norms

$$\|f\|_{L^p(\Omega)} = \left(\int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}} \quad \text{when } 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |f(x)| \quad \text{when } p = \infty.$$

Proposition 2.8. For $1 \leq p \leq \infty$ the spaces $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ are Banach spaces. For $p = 2$ the space $L^2(\Omega)$ is a Hilbert space.

2.3.2 Sobolev space

Definition 2.9. Let Ω be a subdomain of $\mathbb{R}^n$ and f in Lebesgue space then,

$$H^1(\Omega) = \left\{ f \in L^2(\Omega) \text{ such that } \frac{\partial}{\partial x_i} f \in L^2(\Omega) \text{ for all } 1 \leq i \leq n \right\},$$

with the norm,

$$\|f\|_{H^1(\Omega)} = \left(\int_{\Omega} f^2 dx + \sum_{i=1}^n \int_{\Omega} \left(\frac{\partial f}{\partial x_i} \right)^2 dx \right)^{\frac{1}{2}}.$$

The structure $(H^1(\Omega), \|\cdot\|_{H^1(\Omega)})$ defines a Hilbert space.

2.3.3 Spaces of Banach-valued functions

Let X denote any Banach space, we define $L^p(0, T, X)$ and $L^\infty(0, T, X)$ as

$$L^p(0, T, X) = \left\{ f : [0, T] \rightarrow X \text{ measurable} / \int_0^T \|f\|_X^p dt < \infty \right\},$$

$$L^\infty(0, T, X) = \left\{ f : [0, T] \rightarrow X \text{ measurable} / \sup_{t \in [0, T]} \|f\|_X < \infty \right\},$$

associated with their norms,

$$\|f\|_{L^p(0, T, X)} = \left(\int_0^T \|f\|_X^p dt \right)^{\frac{1}{p}} \quad \text{when } 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(0, T, X)} = \sup_{t \in [0, T]} \|f\|_X \quad \text{when } p = \infty.$$

and we have,

$$L^p(0, T, X) = L^p((0, T) \times \Omega) \text{ for } 1 \leq p \leq \infty.$$

We define as well the space $C^k(0, T, X)$ for $k \in \mathbb{N}$ by

$$C^k(0, T, X) = \left\{ f : [0, T] \rightarrow X \text{ continuous such that } \frac{\partial^i f}{\partial t^i} \in C(0, T, X) \text{ for all } 1 \leq i \leq k \right\},$$

where $\frac{\partial^i f}{\partial t^i}$ is the i^{th} derivative of f with respect to t , the usual norm is given by

$$\|f\|_{C^k(0,T,X)} = \sum_{i=0}^k \sup_{t \in [0,T]} \left\| \frac{\partial^i f}{\partial t^i} \right\|_X,$$

note that $(C^k(0,T,X), \|\cdot\|_{C^k(0,T,X)})$ is a Banach space since X is a Banach space.

2.4 Important inequalities and formulas

2.4.1 Hölder's inequality

Theorem 2.10. For any p -integrable function u over Ω and any q -integrable function v over Ω , if $\frac{1}{p} + \frac{1}{q} = 1$ then,

$$uv \in L^1(\Omega) \quad \text{and} \quad \|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

2.4.2 Grönwall's inequalities

General Grönwall Inequality

Theorem 2.11. Let $y : [0, T] \rightarrow \mathbb{R}$ be a non-negative, continuous function, and let

$$\alpha : [0, T] \rightarrow \mathbb{R}, \quad \beta : [0, T] \rightarrow \mathbb{R}$$

be continuous, non-negative functions. Assuming that:

$$y(t) \leq \alpha(t) + \int_0^t \beta(s) y(s) ds \quad \text{for any } t \in [0, T].$$

Then, for all $t \in [0, T]$, we have the estimation:

$$y(t) \leq \alpha(t) + \int_0^t \alpha(s) \beta(s) \exp\left(\int_s^t \beta(r) dr\right) ds.$$

Classical Grönwall Inequality

Lemma 2.12. Let $y : [0, T] \rightarrow \mathbb{R}$ be a non-negative, continuous function. If there exist two positive reals C and β that fulfill

$$y(t) \leq C + \beta \int_0^t y(s) ds \quad \text{for any } t \in [0, T].$$

Then the function $y(t)$ satisfies the bound:

$$y(t) \leq Ce^{\beta t} \quad \text{for any } t \in [0, T].$$

2.4.3 Green's Formula

Theorem 2.13. *Let Ω be an open bounded set of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Then for any u and v in $H^1(\Omega)$ we have the multidimensional formula of integration by parts:*

$$\int_{\Omega} \frac{\partial u}{\partial x_i} v \, dx = - \int_{\Omega} \frac{\partial v}{\partial x_i} u \, dx + \int_{\partial\Omega} u v \eta_i \, d\sigma, \quad 1 \leq i \leq n,$$

where η_i is the i^{th} consinus director of the normal η on $\partial\Omega$ directed towards the outside of Ω and we write $\eta_i = (\eta \cdot e_i)$ and $d\sigma$ the superficial measure on $\partial\Omega$.

Corollary 2.14. *For all functions u and v in $H^1(\Omega)$, we have the Green formula*

$$\int_{\Omega} \Delta u \, v \, dx = - \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v \, d\sigma.$$

2.5 Reaction-diffusion systems

2.5.1 Local existence

Let introduce the system

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = f(u, v, w), \\ \frac{\partial v}{\partial t} - d_2 \Delta v = g(u, v, w), \\ \frac{\partial w}{\partial t} - d_3 \Delta w = h(u, v, w), \end{cases} \quad \text{in } \{t > 0\} \times \Omega$$

with the boundary conditions,

$$\begin{aligned} \alpha_1 \frac{\partial u}{\partial \eta} + (1 - \alpha_1) u &= \beta_1, \\ \alpha_2 \frac{\partial v}{\partial \eta} + (1 - \alpha_2) v &= \beta_2, \\ \alpha_3 \frac{\partial w}{\partial \eta} + (1 - \alpha_3) w &= \beta_3, \end{aligned} \quad \text{on } \{t > 0\} \times \partial\Omega$$

and the initial data,

$$u(0, \cdot) = u_0, \quad v(0, \cdot) = v_0 \text{ and } w(0, \cdot) = w_0, \quad \text{in } \Omega,$$

where the functions f, g and h are continuously differentiable over $\mathbb{R}^3$, the diffusion coefficients d_1, d_2 and d_3 are positive, the parameters α_1, α_2 and α_3 are chosen in $[0, 1]$, the functions β_1, β_2 and β_3 are positive and belong to $C^2([0, T] \times \partial\Omega)$.

A classical solution of the system on $[0, T]$ we mean that, at least

$$\begin{cases} (u, v, w) \in C([0, T], L^1(\Omega)^3) \cap L^\infty([0, T - \tau] \times \Omega)^3, \quad \forall \tau \in (0, T - \tau), \\ \forall k, l = 1 \dots 3, \quad \forall p \in [1, \infty), \quad \partial_t u, \partial_{x_k} u, \partial_{x_k x_l} u \in L^p((\tau, T - \tau) \times \Omega)^3, \\ \text{all equations of the system along with their associated boundary conditions} \\ \text{and initial data are satisfied.} \end{cases}$$

Now we give a classical result of local existence.

Theorem 2.15. Assume $(u_0, v_0, w_0) \in L^\infty(\Omega)^3$. Then, there exist $T > 0$ and a unique classical solution of the system on $[0, T)$. If T^* denotes the greatest of these T 's, then

$$\sup_{t \in [0, T^*)} \left\{ \|u(t, \cdot)\|_{L^\infty(\Omega)}, \|v(t, \cdot)\|_{L^\infty(\Omega)}, \|w(t, \cdot)\|_{L^\infty(\Omega)} \right\} < +\infty \Rightarrow T^* = +\infty.$$

If, moreover, the non linearities f, g and h are quasi-positive, then

$$u_0, v_0 \text{ and } w_0 \geq 0 \Rightarrow \forall t \in [0, T^*) \ u(t), v(t) \text{ and } w(t) \geq 0.$$

Non-negativity of the solutions is preserved if (and only if) the non linearities f, g and h are quasi-positive which means that

$$\begin{cases} f(0, v, w) \geq 0 & \forall v, w \geq 0, \\ g(u, 0, w) \geq 0 & \forall u, w \geq 0, \\ h(u, v, 0) \geq 0 & \forall u, v \geq 0. \end{cases}$$

2.5.2 Global existence

Definition 2.16. The system admits a global classical solution if there exists a constant $M > 0$ such that for any local solution u, v and w we have

$$\|u(t, \cdot)\|_{L^\infty(\Omega)} + \|v(t, \cdot)\|_{L^\infty(\Omega)} + \|w(t, \cdot)\|_{L^\infty(\Omega)} < +\infty \quad \forall t \in [0, T),$$

then,

$$T = +\infty.$$

Remark 2.17 (Criterion for Global Existence). To investigate the global existence of solutions to the system, that is, to determine whether $T_{\max} = +\infty$, we rely on the contrapositive form of the characterization of the maximal existence time:

$$\begin{cases} \text{If there exists a continuous function } M : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ such that} \\ \|u(t)\|_\infty + \|v(t)\|_\infty + \|w(t)\|_\infty \leq M(t), \\ \text{then } T_{\max} = +\infty. \end{cases}$$

In other words, if the functions $u(t)$, $v(t)$, and $w(t)$ remain bounded for every $t \in [0, T]$, with $T < T_{\max}$, then it follows that $T_{\max} = +\infty$.

Consequently, to establish the global existence of classical solutions, it is sufficient to prove that these solutions remain uniformly bounded throughout their interval of existence.

2.5.3 Lyapunov functional

Definition 2.18. We call L a Lyapunov functional associated to the system any function

$$L : [0, +\infty) \rightarrow [0, +\infty),$$

such that

$$\frac{d}{dt} L(u(t, \cdot), v(t, \cdot), w(t, \cdot)) \leq 0,$$

for all $t > 0$ and any positive local solution $u(t, \cdot)$, $v(t, \cdot)$ and $w(t, \cdot)$ of the system.

2.5.4 Invariant region

Definition 2.19. A closed subset Σ of $\mathbb{R}^3$ is said to be an invariant region for the system if and only if

all initial data belong to Σ then all corresponding solutions remain in Σ ,

which is equivalent to,

$$\text{if } (u_0, v_0, w_0) \in \Sigma \Rightarrow (u(t, \cdot), v(t, \cdot), w(t, \cdot)) \in \Sigma.$$

2.5.5 Regularization effect

Theorem 2.20. If the reaction terms of the system satisfy

$$f(u, v, w)(t, \cdot), g(u, v, w)(t, \cdot), h(u, v, w)(t, \cdot) \in L^\infty(0, T, L^p(\Omega)),$$

for some $p > \frac{n}{2}$ where $n = \dim \Omega$ i.e.,

$$\sup \left\{ \|f(u, v, w)(t, \cdot)\|_{L^p(\Omega)}, \|g(u, v, w)(t, \cdot)\|_{L^p(\Omega)}, \|h(u, v, w)(t, \cdot)\|_{L^p(\Omega)} \right\} < \infty,$$

for all $t \in [0, T)$ then the solutions are global in time.

2.6 Linear algebra abstract results

2.6.1 Quadratic form

Definition 2.21. A homogenous polynomial Q with n variables $u_1 \dots u_n$ of second degree is called a quadratic form. If $A = (a_{i,j})_{n \times n}$ is a matrix then Q is written as follows

$$Q(u_1, \dots, u_i, \dots, u_n) = \sum_{i=1}^n \sum_{j=1}^n a_{i,j} u_i u_j,$$

if $U = (u_1, \dots, u_i, \dots, u_n)$, and if we set $\sum_{i=1}^n \sum_{j=1}^n a_{i,j} u_i u_j = A(U, U)$ then

$$A(U, U) = U^t A U = \langle A U, U \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product of the Euclidean space $\mathbb{R}^n$.

Definition 2.22. The quadratic form $A(U, U)$ is said to be positive definite if

$$A(U, U) > 0 \quad \forall U \in \mathbb{R}^n - \{0\}.$$

Theorem 2.23. The quadratic form $A(U, U)$ is positive definite if and only if the determinants of the principal minors of the matrix A are positive i.e., if

$$\det A_i > 0 \quad \forall i = 1, \dots, n,$$

where

$$A_1 = a_{11}, \quad A_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \dots, \quad A_n = A.$$

Chapter 3

Initialization

MULTI-component reaction-diffusion systems constitute a fundamental mathematical framework for describing spatial-temporal dynamics across diverse scientific disciplines. While two-component systems have been extensively studied since Turing's seminal work on morphogenesis [1], three-component systems exhibit richer dynamical behavior and can model more complex phenomena where binary interactions prove insufficient. The general form with cross-diffusion terms, as presented in equation (3.1), represents a significant generalization that captures intricate coupling mechanisms observed in nature and engineered systems.

Consider the reaction-diffusion system

$$\begin{cases} \frac{\partial u}{\partial t} = d_{11}\Delta u + d_{12}\Delta v + d_{13}\Delta w + f(u, v, w), \\ \frac{\partial v}{\partial t} = d_{21}\Delta u + d_{22}\Delta v + d_{23}\Delta w + g(u, v, w), \\ \frac{\partial w}{\partial t} = d_{31}\Delta u + d_{32}\Delta v + d_{33}\Delta w + h(u, v, w), \end{cases} \quad (3.1)$$

where f , g and h are nonlinear reaction terms and d_{ij} , $1 \leq i, j \leq 3$ represent diffusion coefficients of u , v and w respectively. The coefficients d_{ij} satisfy the conditions

$$\begin{cases} d_{11} > 0 \\ 4d_{11}d_{22} - (d_{12} + d_{21})^2 > 0 \\ d_{33} - \frac{(d_{13} + d_{31})^2}{d_{11}} - \frac{(2d_{11}(d_{23} + d_{32}) - (d_{12} + d_{21})(d_{13} + d_{31}))^2}{4d_{11}(4d_{11}d_{22} - (d_{12} + d_{21})^2)} > 0 \end{cases} \quad (P)$$

which reflects the parabolicity of the system and implies at the same time that the matrix D is positive definite.

3.1 Various applications in other fields of science

Three-component reaction–diffusion systems play a significant role in the mathematical modeling of a wide range of natural and engineered processes. Such systems arise naturally when the interaction of multiple substances or agents is coupled with spatial transport, and they provide a flexible framework for describing complex spatiotemporal dynamics. In chemistry, they are frequently employed to

capture reaction mechanisms involving activators and inhibitors, leading to pattern formation, oscillatory behavior, and wave propagation. In physics, reaction–diffusion models are used to describe phenomena such as plasma discharges, semiconductor devices, and nonlinear transport processes.

Beyond the physical sciences, these systems are equally relevant in ecological and biological contexts, where they are applied to model population interactions, competition between species, and the spread of biological or chemical signals in space. Similar mathematical formulations also appear in other scientific disciplines, each adapting the general structure of reaction–diffusion equations to reflect domain-specific mechanisms and assumptions. In the following, we outline several representative applications to illustrate the breadth and versatility of the three-component systems considered in this work.

3.1.1 Chemistry

Chemical reactions can be modeled using three-component reaction–diffusion systems by representing the concentrations of interacting chemical species as spatially and temporally evolving fields. In this framework, the reaction terms describe the local kinetics of transformation between the species, including production and depletion, while diffusion accounts for the transport of molecules across space. The presence of three coupled components allows the system to capture complex behaviors such as oscillations, pattern formation, or localized structures that arise from the interplay between nonlinear reactions and differential diffusion rates.

Let's consider

$$\sum_{i \in I} n_i R_i \xrightleftharpoons[k]{h} \sum_{j \in J} n_j R_j \text{ with } I \cup J = \{1, \dots, p\} \text{ and } I \cap J = \emptyset,$$

where n_i are the number of molecules R_i respectively for all $i = 1, \dots, p$, the parameters h and k represent the level of temperature for any instant t at the position x .

By application Fick (flux) law which describe conservation of mass gives

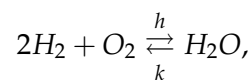
$$\begin{cases} \frac{\partial c_k}{\partial t} = \frac{d_k}{n_k} \Delta c_k + \frac{1}{n_k} \left(\ell \prod_{j \in J} c_j^{n_j} - h \prod_{i \in I} c_i^{n_i} \right) & k \in I, \\ \frac{\partial c_k}{\partial t} = \frac{d_k}{n_k} \Delta c_k + \frac{1}{n_k} \left(h \prod_{i \in I} c_i^{n_i} - \ell \prod_{j \in J} c_j^{n_j} \right) & k \in J, \end{cases}$$

where c_i denotes the concentrations of R_i respectively for all $i = 1, \dots, p$, and n_i are nonnegative describing the order of R_i respectively, constants h and ℓ are positive. Now for the special case where

$$p = 3, I = \{1, 2\}, J = \{3\}, \quad n_1 = n_3 = 2 \quad \text{and} \quad n_2 = 1$$

$$R_1 = \text{Hydrogen}, \quad R_2 = \text{Oxygen}, \quad R_3 = \text{water},$$

one gets the famous equation,



which can be written in terms of reaction-diffusion systems

$$\begin{cases} 2\frac{\partial}{\partial t} \langle H_2 \rangle - d_{H_2} \Delta \langle H_2 \rangle &= \ell \langle H_2 O \rangle^2 - h \langle H_2 \rangle^2 \langle O_2 \rangle, \\ \frac{\partial}{\partial t} \langle O_2 \rangle - d_{H_2} \Delta \langle O_2 \rangle &= \ell \langle H_2 O \rangle^2 - h \langle H_2 \rangle^2 \langle O_2 \rangle, \\ 2\frac{\partial}{\partial t} \langle H_2 O \rangle - d_{H_2} \Delta \langle H_2 O \rangle &= h \langle H_2 \rangle^2 \langle O_2 \rangle - \ell \langle H_2 O \rangle^2. \end{cases} \quad (3.2)$$

For more information see [11]

3.1.2 Ecology

Three-component reaction–diffusion systems provide a mathematically structured way to represent interacting ecological quantities that evolve in both space and time. In ecological contexts, the three components are often interpreted as distinct biological or environmental variables—such as multiple species or a species coupled with regulating factors—whose local interactions are governed by nonlinear reaction terms, while dispersal, movement, or transport across a habitat is captured by diffusion. This framework allows ecologists to study how spatial heterogeneity, interaction strength, and mobility jointly influence pattern formation, coexistence, and stability, without reducing the system to purely local dynamics or purely spatial processes.

Lotka–Volterra model In a reaction–diffusion formulation of the Lotka–Volterra type involving two prey species and a single predator, the spatial domain is assumed to be closed so that no population flux occurs across the habitat boundary. The model also allows for the presence of time delays arising from interspecific interactions. Under these assumptions, the spatiotemporal evolution of the prey densities u and v , together with the predator density w , is described by a system of coupled partial differential equations

$$\begin{cases} \partial_t u = E_1 u + a_1(x) u (1 - u - b_1 v - c_1 w - \beta_1 J_2 * v - \gamma_1 J_3 * w), \\ \partial_t v = E_2 v + a_2(x) v (1 - v - b_2 u - c_2 w - \beta_2 J_1 * u - \gamma_2 J_3 * w), \\ \partial_t w = E_3 w + a_3(x) w (1 - w + b_3 u + c_3 v + \beta_3 J_1 * u + \gamma_3 J_2 * v) \\ t > 0, x \in \Omega. \end{cases} \quad (3.3)$$

Where Ω is an open bounded subset of $\mathbb{R}^n$, the functions $a_1(x)$, $a_2(x)$ and $a_3(x)$ belong to $C^\alpha(\Omega)$ and are nonnegative, the constants b_i, c_i, β_i and γ_i are in $\mathbb{R}^+$ and satisfy $b_i + \beta_i > 0$, $c_i + \gamma_i > 0$ for $i = 1, 2, 3$, we denote by E_i the elliptic operator defined by

$$E_i u_i = \sum_{j,k=1}^n a_{jk}^{(i)}(x) \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \sum_{j=1}^n b_j^{(i)}(x) \frac{\partial u_i}{\partial x_j}.$$

The functions $J_i * u_i$ for $i = 1, 2, 3$ take two forms

either by the discrete delay time

$$J_i * u_i = u_i(t - r_i, x),$$

or by the continuous delay time

$$J_i * u_i = \int_{-r_i}^t J_i(t-s) u_i(s, x) dx.$$

The functions J_i are positive and piece wise continuous and satisfy the following properties

$$J_i(t) = 0 \text{ when } t > r_i,$$

and

$$\int_0^{r_i} J_i(t) dt = 1,$$

for $i = 1, 2, 3$. The model formulation follows the approach outlined in [17]

3.1.3 Genetic of populations

Consider a population in which all individuals possess the same chromosomal number as the fertilized ovum. Assume that, at a given genetic locus on a pair of chromosomes, two alternative variants (hereafter referred to as alleles) exist, denoted by a and A . Under this assumption, the population can be categorized into three distinct genetic groups, or genotypes. Two of these groups consist of homozygous individuals, each carrying identical alleles. These genotypes are represented as aa and AA , and their characteristics depend exclusively on the allele they contain. The third group comprises heterozygous individuals, which carry one copy of each allele and are denoted by aA . Assume further that the population is distributed along a one-dimensional habitat. Let $\sigma_1(x, t)$, $\sigma_2(x, t)$, and $\sigma_3(x, t)$ represent the linear population densities of genotypes aa , aA , and AA , respectively, at spatial position x and time $t > 0$. Mating within the population is assumed to occur randomly, with a constant birth rate μ , and movement across the habitat is characterized by a uniform flux. Additionally, mortality is assumed to depend solely on genetic composition. Let λ_1 , λ_2 , and λ_3 denote the death rates associated with genotypes aa , aA , and AA , respectively. In general, these mortality parameters are not identical, reflecting small differences in survival among genotypes; some genetic types may exhibit higher viability than others, as discussed by Fischer [9]. Under these assumptions, the dynamics of the population are governed by the following system:

$$\begin{cases} \frac{\partial \sigma_1}{\partial t} = \mu \frac{\partial^2 \sigma_1}{\partial x^2} - \lambda_1 \sigma_1 + \frac{R}{\sigma} \left(\sigma_1 + \frac{1}{2} \sigma_2 \right)^2 \\ \frac{\partial \sigma_2}{\partial t} = \mu \frac{\partial^2 \sigma_2}{\partial x^2} - \lambda_2 \sigma_2 + \frac{2R}{\sigma} \left(\sigma_1 + \frac{1}{2} \sigma_2 \right) \left(\sigma_3 + \frac{1}{2} \sigma_2 \right) \\ \frac{\partial \sigma_3}{\partial t} = \mu \frac{\partial^2 \sigma_3}{\partial x^2} - \lambda_3 \sigma_3 + \frac{R}{\sigma} \left(\sigma_3 + \frac{1}{2} \sigma_2 \right)^2 \end{cases} \quad (3.4)$$

where $\sigma = \sigma_1 + \sigma_2 + \sigma_3$. This modeling framework originates from the work of [4]

3.1.4 Medicine

Tumor growth can be formulated using a three-component reaction–diffusion framework by introducing coupled fields that represent key interacting constituents of the system. One component typically describes the spatial density of tumor cells, while the remaining components account for additional factors such as nutrients, signaling substances, or regulatory agents that influence proliferation and survival. The reaction terms model local processes including cell division, consumption, and interaction effects, whereas diffusion captures spatial redistribution within the tissue. This formulation enables the analysis of how local biological interactions, combined with spatial transport, give rise to heterogeneous growth patterns and evolving tumor morphologies over time.

Let

$$\begin{cases} \frac{\partial p}{\partial t} = \left[\operatorname{div} (d(x, t) \cdot \nabla p) + g(\eta) p \left(1 - \frac{r}{\Theta} \right) - f p \right] (t, x), \\ \frac{\partial q}{\partial t} = \left[\operatorname{div} (d(x, t) \cdot \nabla q) + f(\eta) p - h(\eta) q \right] (t, x), \\ \frac{\partial n}{\partial t} = [h(\eta) q] (t, x), \end{cases} \quad (3.5)$$

here $p(t, x)$, $q(t, x)$, $n(t, x)$ and $r(t, x)$ denote, respectively, the populations of proliferating cells, quiescent (hypoxic) cells, necrotic cells, and the overall cell population, the variable t corresponds to time, while x specifies a spatial position within a three-dimensional domain, the parameter Θ represents the maximum cell capacity of a voxel and is computed as the ratio between the voxel volume (taken here as 0.0751 mm^3) and the volume occupied by a single cell. The reaction terms are given by

$$\begin{aligned} f(\eta(t, x)) &= \exp(-c_f \cdot \eta(t, x)), \\ g(\eta(t, x)) &= c_{g0} \exp(-\exp(-c_g(\eta(t, x) - ip))), \\ h(\eta(t, x)) &= \exp(-c_h \eta(t, x)), \end{aligned}$$

in this formulation, c_{g0} , c_f and c_h are model-specific parameters, the coefficient c_g defines the steepness of the Gompertz function and is fixed at $c_g = 0.31 / \text{mmHg}$, which corresponds to reaching 99% of the asymptotic value c_{g0} under conditions of maximal nutrient supply, the parameter ip identifies the location at which the nutrient concentration increases most rapidly and thus coincides with the inflection point of the Gompertz curve. The system structure incorporates elements introduced by [20]

3.1.5 Physics

Many phenomena encountered in the physical sciences can be formulated within the framework of three-component reaction-diffusion systems by representing the underlying processes through coupled fields that evolve in space and time. In such formulations, the reaction terms encode local interactions or conversion mechanisms, while diffusion accounts for spatial transport and coupling between neighboring regions. The presence of three interacting components allows the model to capture competing influences and multiple time or length scales, which are often essential for describing the emergence of localized structures, oscillatory behavior, or complex spatiotemporal dynamics observed in physical systems.

The system presented below models the dynamics of a gas-discharge device incorporating a semiconductor electrode

$$\begin{cases} \frac{\partial u}{\partial t} - d_u \nabla^2 u = \lambda u - u^3 + k_1 + k_2 u - k_3 v - k_4 w, \\ \frac{\partial v}{\partial t} - \frac{d_v}{\tau} \nabla^2 v = \frac{u - v}{\tau}, \\ \frac{\partial w}{\partial t} - \frac{d_w}{\theta} \nabla^2 w = \frac{u - w}{\theta}. \end{cases} \quad (3.6)$$

In this model, u corresponds to the activating species, while v and w act as inhibitory components. The parameters d_u, d_v and d_w are strictly positive and represent the respective diffusion coefficients. The constants τ and θ both nonzero, are dimensionless quantities defined as the ratios between the characteristic time scales of the two inhibitors and that of the activator. The coefficients k_i for $i = 1, 2, 3, 4$ are kinetic constants governing the reaction terms. Owing to its nonlinear structure, this system may be interpreted as an extension of the classical Bonhoeffer–van der Pol kinetics to a three-variable framework through the inclusion of an additional inhibitory component w . These equations build upon earlier formulations developed in [8].

3.2 Linearization and Stability of the Jacobian Matrix

Let $U^* = (u^*, v^*, w^*)$ denote any arbitrary equilibrium of system (3.1). In the neighborhood of (u^*, v^*, w^*) , the non-linearities f, g and h can be approximated as follows:

$$\begin{cases} f(u, v, w) = f(U^*) + f_u(U^*)u + f_v(U^*)v + f_w(U^*)w + o(\|(u, v, w)\|), \\ g(u, v, w) = g(U^*) + g_u(U^*)u + g_v(U^*)v + g_w(U^*)w + o(\|(u, v, w)\|), \\ h(u, v, w) = h(U^*) + h_u(U^*)u + h_v(U^*)v + h_w(U^*)w + o(\|(u, v, w)\|). \end{cases}$$

So the right-hand side of system becomes

$$\begin{cases} \frac{\partial u}{\partial t} - (d_{11}\Delta u + d_{12}\Delta v + d_{13}\Delta w) = f_u(U^*)u + f_v(U^*)v + f_w(U^*)w, \\ \frac{\partial v}{\partial t} - (d_{21}\Delta u + d_{22}\Delta v + d_{23}\Delta w) = g_u(U^*)u + g_v(U^*)v + g_w(U^*)w, \\ \frac{\partial w}{\partial t} - (d_{31}\Delta u + d_{32}\Delta v + d_{33}\Delta w) = h_u(U^*)u + h_v(U^*)v + h_w(U^*)w. \end{cases}$$

Now we apply the Fourier transform $\mathcal{F}()$

$$\begin{cases} \mathcal{F}\left(\frac{\partial u}{\partial t} - (d_{11}\Delta u + d_{12}\Delta v + d_{13}\Delta w)\right) = \mathcal{F}(f_u(U^*)u + f_v(U^*)v + f_w(U^*)w), \\ \mathcal{F}\left(\frac{\partial v}{\partial t} - (d_{21}\Delta u + d_{22}\Delta v + d_{23}\Delta w)\right) = \mathcal{F}(g_u(U^*)u + g_v(U^*)v + g_w(U^*)w), \\ \mathcal{F}\left(\frac{\partial w}{\partial t} - (d_{31}\Delta u + d_{32}\Delta v + d_{33}\Delta w)\right) = \mathcal{F}(h_u(U^*)u + h_v(U^*)v + h_w(U^*)w). \end{cases}$$

$$\left\{ \begin{array}{l} \frac{\partial \mathcal{F}u}{\partial t} + (d_{11}\mathcal{F}(u) + d_{12}\mathcal{F}(v) + d_{13}\mathcal{F}(w))k^2 = f_u(U^*)\mathcal{F}(u) + f_v(U^*)\mathcal{F}(v) \\ \quad + f_w(U^*)\mathcal{F}(w), \\ \frac{\partial \mathcal{F}v}{\partial t} + (d_{21}\mathcal{F}(u) + d_{22}\mathcal{F}(v) + d_{23}\mathcal{F}(w))k^2 = g_u(U^*)\mathcal{F}(u) + g_v(U^*)\mathcal{F}(v) \\ \quad + g_w(U^*)\mathcal{F}(w), \\ \frac{\partial \mathcal{F}w}{\partial t} + (d_{31}\mathcal{F}(u) + d_{32}\mathcal{F}(v) + d_{33}\mathcal{F}(w))k^2 = h_u(U^*)\mathcal{F}(u) + h_v(U^*)\mathcal{F}(v) \\ \quad + h_w(U^*)\mathcal{F}(w). \end{array} \right.$$
$$\mathcal{F}(u) = \tilde{u}, \mathcal{F}(v) = \tilde{v}, \mathcal{F}(w) = \tilde{w},$$
$$\frac{\partial}{\partial t} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} = (A - k^2 D) \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix},$$

According to definition the steady state is stable that means all eigenvalues of the matrix A have negative real part. The characteristic polynomial of A is given by:

$$P(\lambda) = -\lambda^3 + (\operatorname{tr} A) \lambda^2 - (\operatorname{tr} \operatorname{com} A) \lambda + (\det A).$$

$$\begin{cases} a = -\operatorname{tr} A, \\ b = \operatorname{tr} \operatorname{com} A, \\ c = -\det A, \end{cases}$$
$$\lambda^3 + a\lambda^2 + b\lambda + c = 0. \quad (3.7)$$
$$a > 0, \quad c > 0, \quad \text{and} \quad ab - c > 0.$$

$$\text{tr } A < 0, \quad (\text{i})$$

$$\det A < 0, \quad (\text{ii})$$

$$\det A - \operatorname{tr} A \cdot \operatorname{tr} \operatorname{com} A > 0. \quad (\text{iii})$$

Chapter 4

Main Results

This chapter presents a detailed investigation of diffusion-driven instability commonly termed Turing instability in three component reaction-diffusion systems with fully coupled diffusion matrices. The central objective is to characterize precisely when spatially uniform steady states become unstable to spatially heterogeneous perturbations, leading to spontaneous pattern formation.

Unlike two-component systems where Turing conditions involve relatively simple relationships between diffusion coefficients, the three-component case presents significantly richer spectral structure. Our analysis accounts for all nine diffusion coefficients d_{ij} interacting with the complete reaction Jacobian matrix A . Through systematic linear stability analysis and Fourier decomposition, we derive explicit conditions on the wave number k that determine instability thresholds.

The analytical approach proceeds through several stages: first, linearizing the system about homogeneous equilibria; second, examining the dispersion relation obtained via spatial Fourier transforms; and third, identifying parameter regions where certain wave numbers experience positive growth rates. Particular attention is given to how cross-diffusion terms d_{ij} modify classical Turing conditions, potentially enabling pattern formation even when all diffusion coefficients are of comparable magnitude.

The results provide a comprehensive framework for predicting pattern formation in multi-species biological and chemical systems where three interacting components with general diffusive coupling are involved.

Before we give the expression of the characteristic polynomial of the matrix $A(k)$, we first identify some tools.

We set the submatrices

$$M_1 = \begin{pmatrix} a_{11} & a_{12} \\ d_{21} & d_{22} \end{pmatrix} \quad M_2 = \begin{pmatrix} a_{11} & a_{13} \\ d_{31} & d_{33} \end{pmatrix} \quad M_3 = \begin{pmatrix} a_{22} & a_{21} \\ d_{12} & d_{11} \end{pmatrix} \quad M_4 = \begin{pmatrix} a_{33} & a_{31} \\ d_{13} & d_{11} \end{pmatrix}$$

$$M_5 = \begin{pmatrix} a_{33} & a_{23} \\ d_{32} & d_{22} \end{pmatrix} \quad M_6 = \begin{pmatrix} a_{22} & a_{32} \\ d_{23} & d_{33} \end{pmatrix} \quad M_7 = \begin{pmatrix} a_{11} & d_{12} & d_{13} \\ a_{21} & d_{22} & d_{23} \\ a_{31} & d_{32} & d_{33} \end{pmatrix}$$

$$M_8 = \begin{pmatrix} d_{11} & a_{12} & a_{13} \\ d_{21} & a_{22} & a_{23} \\ d_{31} & a_{32} & a_{33} \end{pmatrix}.$$

We also define two constants σ_1 and σ_2 as follows:

$$\sigma_1 = -d_{11}(a_{23}d_{32} + a_{32}d_{23} - a_{22}d_{33} - a_{33}d_{22}) + d_{21}(a_{13}d_{32} + a_{32}d_{13} - a_{12}d_{33} - a_{33}d_{12}) \\ - d_{31}(a_{13}d_{22} + a_{22}d_{13} - a_{12}d_{23} - a_{23}d_{12}),$$

$$\sigma_2 = a_{11}(a_{23}d_{32} + a_{32}d_{23} - a_{22}d_{33} - a_{33}d_{22}) + a_{21}(a_{12}d_{33} + a_{33}d_{12} - a_{13}d_{32} - a_{32}d_{13}) \\ + a_{31}(a_{13}d_{22} + a_{22}d_{13} - a_{12}d_{23} - a_{23}d_{12}),$$

then we have the following lemma.

Lemma 4.1. *The characteristic polynomial of $A(k)$ is*

$$\tilde{P}(\lambda) = -\lambda^3 + [\text{tr}A - k^2\text{tr}D] \lambda^2 + \left[-\text{tr}(\text{com}D) k^4 + \left(\sum_{i=1}^6 \det M_i \right) k^2 - \text{tr}(\text{com}A) \right] \lambda \\ - (\det D) k^6 + (\sigma_1 + \det M_7) k^4 + (\sigma_2 - \det M_8) k^2 + \det A.$$

Proof. It suffices to apply the definition and use the above auxiliary tools. $\square$

We observe that $\tilde{P} = 0$ takes the form

$$\lambda^3 + \varphi_1(k^2) \lambda^2 + \varphi_2(k^2) \lambda + \varphi_3(k^2) = 0,$$

where

$$\begin{aligned} \varphi_1(k^2) &= \text{tr}(D) k^2 - \text{tr}(A), \\ \varphi_2(k^2) &= \text{tr}(\text{com}(D)) (k^2)^2 - \left(\sum_{i=1}^6 \det M_i \right) k^2 + \text{tr}(\text{com}(A)), \\ \varphi_3(k^2) &= (\det D) (k^2)^3 - (\sigma_1 + \det M_7) (k^2)^2 + (\det M_8 - \sigma_2) k^2 - \det A. \end{aligned}$$

We notice that the coefficients φ_1 , φ_2 , and φ_3 are polynomial functions in k^2 . Now, we apply the Routh-Hurwitz stability criterion, see F. R. Gantmacher [7],

$$\varphi_1(k^2) > 0 \quad \text{and} \quad \varphi_3(k^2) > 0 \quad \text{and} \quad (\varphi_1\varphi_2 - \varphi_3)(k^2) > 0.$$

This means that all eigenvalues of $A(k)$ have a negative real part. In order to achieve our goal "Diffusion Driven Instability", we violate at least one of the above conditions, so

$$\begin{aligned} &A(k) \text{ has at least one eigenvalue with positive real part} \\ \Leftrightarrow &\varphi_1(k^2) \leq 0 \quad \text{or} \quad \varphi_3(k^2) \leq 0 \quad \text{or} \quad (\varphi_1\varphi_2 - \varphi_3)(k^2) \leq 0. \end{aligned}$$

Remark 4.2. Since the functions we are dealing with, φ_1 , φ_3 , and $(\varphi_1\varphi_2 - \varphi_3)$, are even, we limit our study over $\mathbb{R}^+ = [0, +\infty)$.

At this point, we are now ready to establish the explicit conditions on k^2 .

4.1 First condition

We have $\varphi_1(k^2) = \text{tr}(D) k^2 - \text{tr}(A)$, and since $\text{tr}(D) > 0$, thanks to $d_{ii} > 0$, $i = 1, 2, 3$, and using the assumption $\text{tr}(A) < 0$ (i), we get that $\forall k^2 \geq 0$, $\varphi_1(k^2) > 0$. This means we get nothing as a result.

4.2 Second condition

The derivative of φ_3 with respect to k^2 is given by

$$\frac{\partial}{\partial k^2} \varphi_3(k^2) = 3 \det D (k^2)^2 - 2 (\sigma_1 + \det M_7) k^2 + \det M_8 - \sigma_2. \quad (4.1)$$

It appears that the derivative of φ_3 is a second degree polynomial in k^2 . Set:

$$\tilde{A} = 3 \det D, \quad \tilde{B} = -2 (\sigma_1 + \det M_7), \quad \tilde{C} = \det M_8 - \sigma_2.$$

The discriminant $\tilde{\Delta}$ of (4.1) is

$$\tilde{\Delta} = 4 \left[(\sigma_1 + \det M_7)^2 - 3 \det D (\det M_8 - \sigma_2) \right].$$

We denote by $\tilde{k}_i^2$ the roots of φ_3 , and by k_i^2 the roots of $\frac{\partial}{\partial k^2} \varphi_3$. Following the signs of $\det D$ and $\tilde{\Delta}$, we give two propositions.

Proposition 4.3. *If the following conditions hold:*

$$\det D > 0, \quad \tilde{\Delta} > 0, \quad \tilde{B} < 0 \text{ and } \tilde{C} > 0, \quad (4.2)$$

then $A(k)$ has at least one eigenvalue with positive real part furthermore,

$$\forall k^2 \in [\tilde{k}_1^2, \tilde{k}_2^2], \quad \varphi_3(k^2) \leq 0.$$

Proof. Since $\tilde{\Delta} > 0$, there exist two distinct roots, obviously with $k_1^2 < k_2^2$:

$$k_1^2 = \frac{-\tilde{B} - \sqrt{\tilde{\Delta}}}{2\tilde{A}}, \quad k_2^2 = \frac{-\tilde{B} + \sqrt{\tilde{\Delta}}}{2\tilde{A}}.$$

We must ensure that both roots are positive; otherwise, the analysis cannot proceed. So, the following conditions are required:

$$\tilde{B} < 0 \quad \text{and} \quad \tilde{C} > 0, \quad (4.3)$$

equivalently,

$$-2 (\sigma_1 + \det M_7) < 0, \quad \det M_8 - \sigma_2 > 0.$$

Under the hypotheses given by (4.2), the behavior of φ_3 is determined as follows.

- The value $\varphi_3(k_1^2)$ must be positive:

$$\begin{aligned} & (\det D) \left(\frac{-\tilde{B} - \sqrt{\tilde{\Delta}}}{2\tilde{A}} \right)^3 - (\sigma_1 + \det M_7) \left(\frac{-\tilde{B} - \sqrt{\tilde{\Delta}}}{2\tilde{A}} \right)^2 \\ & + (\det M_8 - \sigma_2) \left(\frac{-\tilde{B} - \sqrt{\tilde{\Delta}}}{2\tilde{A}} \right) - \det A > 0. \end{aligned}$$

- Regarding $\varphi_3(k_2^2)$, there are two cases:

- If $\varphi_3(k_2^2) > 0$, then

$$\forall k^2 \geq 0, \quad \varphi_3(k^2) \geq 0.$$

– If $\varphi_3(k_2^2) < 0$, then there exist $\tilde{k}_1^2 \in [k_1^2, k_2^2]$ and $\tilde{k}_2^2 > k_2^2$ such that

$$\varphi_3(\tilde{k}_1^2) = \varphi_3(\tilde{k}_2^2) = 0.$$

Thus,

$$\forall k^2 \in [\tilde{k}_1^2, \tilde{k}_2^2], \quad \varphi_3(k^2) \leq 0.$$

□

Proposition 4.4. *If the following assumptions hold:*

$$\det D < 0 \quad \text{and} \quad \tilde{\Delta} < 0, \tag{4.4}$$

then $A(k)$ has at least one eigenvalue with positive real part. Furthermore,

$$\forall k^2 \in [\tilde{k}_3^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

Moreover, if

$$\det D < 0, \quad \tilde{\Delta} > 0, \quad \varphi_3(k_3^2) > 0 \quad \text{and} \quad \varphi_3(k_4^2) > 0,$$

then $A(k)$ has at least one eigenvalue with positive real part, and

$$\forall k^2 \in [\tilde{k}_4^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

If

$$\det D < 0, \quad \tilde{\Delta} > 0, \quad \varphi_3(k_3^2) < 0 \quad \text{and} \quad \varphi_3(k_4^2) < 0,$$

then $A(k)$ has at least one eigenvalue with positive real part, and

$$\forall k^2 \in [\tilde{k}_5^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

Finally, if

$$\det D < 0, \quad \tilde{\Delta} > 0, \quad \varphi_3(k_3^2) < 0 \quad \text{and} \quad \varphi_3(k_4^2) > 0,$$

then $A(k)$ has at least one eigenvalue with positive real part, and

$$\forall k^2 \in [\tilde{k}_6^2, \tilde{k}_7^2] \cup [\tilde{k}_8^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

Proof. We begin with the trivial case: if $\tilde{\Delta} < 0$, then there are no real roots of (4.1). The behavior of φ_3 under the hypotheses given by (4.4), ensures the existence of some $\tilde{k}_3^2 > 0$ such that $\varphi_3(\tilde{k}_3^2) = 0$. Hence,

$$\forall k^2 \in [\tilde{k}_3^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

Now, consider the case $\tilde{\Delta} > 0$. Then, there exist two distinct real roots of (4.1), given by

$$k_3^2 = \frac{-\tilde{B} + \sqrt{\tilde{\Delta}}}{2\tilde{A}}, \quad k_4^2 = \frac{-\tilde{B} - \sqrt{\tilde{\Delta}}}{2\tilde{A}}, \quad \text{with} \quad k_3^2 < k_4^2.$$

Again, the roots must be positive and we require

$$\tilde{B} > 0 \quad \text{and} \quad \tilde{C} < 0. \tag{4.5}$$

Depending on the sign of $\varphi_3(k_3^2)$ and $\varphi_3(k_4^2)$, we have the following cases:

- If $\varphi_3(k_3^2) > 0$ and $\varphi_3(k_4^2) > 0$, then there exists $\tilde{k}_4^2 > k_4^2$ such that $\varphi_3(\tilde{k}_4^2) = 0$. Thus,

$$\forall k^2 \in [\tilde{k}_4^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

- If $\varphi_3(k_3^2) < 0$ and $\varphi_3(k_4^2) < 0$, then there exists $\tilde{k}_5^2 < k_3^2$ such that $\varphi_3(\tilde{k}_5^2) = 0$.
Therefore,

$$\forall k^2 \in [\tilde{k}_5^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

- If $\varphi_3(k_3^2) < 0$ and $\varphi_3(k_4^2) > 0$, then there exist $\tilde{k}_6^2 \in (0, k_3^2)$, $\tilde{k}_7^2 \in (k_3^2, k_4^2)$, and $\tilde{k}_8^2 > k_4^2$ such that

$$\varphi_3(\tilde{k}_6^2) = \varphi_3(\tilde{k}_7^2) = \varphi_3(\tilde{k}_8^2) = 0.$$

In this case, we get

$$\forall k^2 \in [\tilde{k}_6^2, \tilde{k}_7^2] \cup [\tilde{k}_8^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

□

In addition, we discuss the sub-cases where $\det D = 0$. In this case, the function φ_3 takes the form

$$\varphi_3(k^2) = -(\sigma_1 + \det M_7)(k^2)^2 + (\det M_8 - \sigma_2)k^2 - \det A.$$

Suppose that $\sigma_1 + \det M_7 \neq 0$. Then, φ_3 can be rewritten as

$$\varphi_3(k^2) = \tilde{A}_1 (k^2)^2 + \tilde{B}_1 k^2 + \tilde{C}_1, \quad (4.6)$$

where

$$\tilde{A}_1 = -(\sigma_1 + \det M_7), \quad \tilde{B}_1 = \det M_8 - \sigma_2, \quad \tilde{C}_1 = -\det A.$$

The discriminant of (4.6) is

$$\tilde{\Delta}_1 = \tilde{B}_1^2 - 4\tilde{A}_1\tilde{C}_1.$$

If $\tilde{\Delta}_1 > 0$ and $\tilde{A}_1 > 0$ (equivalently, $\sigma_1 + \det M_7 < 0$), then there exist two real roots

$$\tilde{k}_9^2 = \frac{-\tilde{B}_1 - \sqrt{\tilde{\Delta}_1}}{2\tilde{A}_1}, \quad \tilde{k}_{10}^2 = \frac{-\tilde{B}_1 + \sqrt{\tilde{\Delta}_1}}{2\tilde{A}_1}, \quad \text{with } \tilde{k}_9^2 < \tilde{k}_{10}^2.$$

To ensure that both roots are positive, we require

$$\tilde{B}_1 < 0 \quad \text{and} \quad \tilde{C}_1 > 0. \quad (4.7)$$

That is,

$$\det M_8 - \sigma_2 < 0 \quad \text{and} \quad -\det A > 0.$$

In this case, we have

$$\forall k^2 \in [\tilde{k}_9^2, \tilde{k}_{10}^2], \quad \varphi_3(k^2) \leq 0.$$

Furthermore, if $\sigma_1 + \det M_7 = 0$, then φ_3 reduces to

$$\varphi_3(k^2) = (\det M_8 - \sigma_2)k^2 - \det A.$$

If $\det M_8 - \sigma_2 < 0$, then there exists $\tilde{k}_{11}^2 > 0$ such that $\varphi_3(\tilde{k}_{11}^2) = 0$. Hence,

$$\forall k^2 \in [\tilde{k}_{11}^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

4.3 Third condition

The third and final term is given by:

$$\varphi(k^2) = (\varphi_1 \varphi_2 - \varphi_3)(k^2).$$

Explicitly,

$$\begin{aligned} \varphi(k^2) &= (\operatorname{tr} D \cdot \operatorname{tr}(\operatorname{com} D) - \det D)(k^2)^3 \\ &\quad + \left[\det M_7 + \sigma_1 - \operatorname{tr} D \cdot \left(\sum_{i=1}^6 \det M_i \right) - \operatorname{tr} A \cdot \operatorname{tr}(\operatorname{com} D) \right] (k^2)^2 \\ &\quad + \left[\operatorname{tr} D \cdot \operatorname{tr}(\operatorname{com} A) + \operatorname{tr} A \cdot \left(\sum_{i=1}^6 \det M_i \right) + \sigma_2 - \det M_8 \right] k^2 \\ &\quad + \det A - \operatorname{tr} A \cdot \operatorname{tr}(\operatorname{com} A). \end{aligned}$$

Following the same reasoning, the derivative of φ with respect to k^2 is:

$$\begin{aligned} \frac{\partial}{\partial k^2} \varphi(k^2) &= 3 (\operatorname{tr} D \cdot \operatorname{tr}(\operatorname{com} D) - \det D)(k^2)^2 + 2 \left[\det M_7 + \sigma_1 - \operatorname{tr} D \cdot \left(\sum_{i=1}^6 \det M_i \right) \right. \\ &\quad \left. - \operatorname{tr} A \cdot \operatorname{tr}(\operatorname{com} D) \right] k^2 + \operatorname{tr} D \cdot \operatorname{tr}(\operatorname{com} A) + \operatorname{tr} A \cdot \left(\sum_{i=1}^6 \det M_i \right) + \sigma_2 \\ &\quad - \det M_8. \end{aligned}$$

We set:

$$\begin{aligned} \hat{A} &= 3 (\operatorname{tr} D \cdot \operatorname{tr}(\operatorname{com} D) - \det D), \\ \hat{B} &= 2 \left(\det M_7 + \sigma_1 - \operatorname{tr} D \cdot \left(\sum_{i=1}^6 \det M_i \right) - \operatorname{tr} A \cdot \operatorname{tr}(\operatorname{com} D) \right), \\ \hat{C} &= \operatorname{tr} D \cdot \operatorname{tr}(\operatorname{com} A) + \operatorname{tr} A \cdot \left(\sum_{i=1}^6 \det M_i \right) + \sigma_2 - \det M_8. \end{aligned}$$

Thus,

$$\frac{\partial}{\partial k^2} \varphi(k^2) = \hat{A}(k^2)^2 + \hat{B}k^2 + \hat{C}. \quad (4.8)$$

The discriminant of (4.8) is:

$$\hat{\Delta} = \hat{B}^2 - 4\hat{A}\hat{C}.$$

We denote by $\hat{K}_i^2$ the roots of φ , and by K_i^2 the roots of $\frac{\partial}{\partial k^2} \varphi$.

Depending on the signs of $\hat{A}$ and $\hat{\Delta}$, we distinguish two propositions.

Proposition 4.5. *If the following conditions hold:*

$$\hat{A} > 0, \quad \hat{\Delta} > 0, \quad \hat{B} < 0, \quad \hat{C} > 0, \quad \varphi(K_1^2) > 0 \quad \text{and} \quad \varphi(K_2^2) < 0 \quad (4.9)$$

then $A(k)$ has at least one eigenvalue with positive real part. Furthermore,

$$\forall k^2 \in [\hat{K}_1^2, \hat{K}_2^2] \quad \varphi(k^2) \leq 0.$$

Proof. Since $\hat{\Delta} > 0$, the derivative $\frac{\partial}{\partial k^2} \varphi$ has two distinct real roots given by

$$K_1^2 = \frac{-\hat{B} - \sqrt{\hat{\Delta}}}{2\hat{A}}, \quad K_2^2 = \frac{-\hat{B} + \sqrt{\hat{\Delta}}}{2\hat{A}}, \quad \text{with } K_1^2 < K_2^2.$$

To ensure these roots are positive, we require

$$\hat{B} < 0 \quad \text{and} \quad \hat{C} > 0. \quad (4.10)$$

By the hypothesis (4.9), the behavior of φ at these critical points is

$$\varphi(K_1^2) > 0, \quad \varphi(K_2^2) < 0.$$

In this situation, by continuity, there must exist $\hat{K}_1^2 \in (K_1^2, K_2^2)$ and $\hat{K}_2^2 > K_2^2$ such that

$$\varphi(\hat{K}_1^2) = 0, \quad \varphi(\hat{K}_2^2) = 0.$$

Consequently,

$$\forall k^2 \in [\hat{K}_1^2, \hat{K}_2^2] \quad \varphi(k^2) \leq 0.$$

This proves the proposition. $\square$

Proposition 4.6. *Assume the polynomial φ defined above. Then:*

1. *If*

$$\hat{A} < 0, \quad \hat{\Delta} < 0,$$

then the matrix $A(k)$ has at least one eigenvalue with positive real part, and

$$\forall k^2 \in [\hat{K}_3^2, +\infty) \quad \varphi(k^2) \leq 0.$$

2. *If*

$$\hat{A} < 0, \quad \hat{\Delta} > 0, \quad \hat{B} > 0, \quad \hat{C} < 0, \quad \varphi(K_3^2) > 0, \quad \varphi(K_4^2) > 0,$$

then $A(k)$ has at least one eigenvalue with positive real part, and

$$\forall k^2 \in [\hat{K}_4^2, +\infty) \quad \varphi(k^2) \leq 0.$$

3. *If*

$$\hat{A} < 0, \quad \hat{\Delta} > 0, \quad \hat{B} > 0, \quad \hat{C} < 0, \quad \varphi(K_3^2) < 0, \quad \varphi(K_4^2) < 0,$$

then $A(k)$ has at least one eigenvalue with positive real part, and

$$\forall k^2 \in [\hat{K}_5^2, +\infty) \quad \varphi(k^2) \leq 0.$$

4. *If*

$$\hat{A} < 0, \quad \hat{\Delta} > 0, \quad \hat{B} > 0, \quad \hat{C} < 0, \quad \varphi(K_3^2) < 0, \quad \varphi(K_4^2) > 0,$$

then $A(k)$ has at least one eigenvalue with positive real part, and

$$\forall k^2 \in [\hat{K}_6^2, \hat{K}_7^2] \cup [\hat{K}_8^2, +\infty) \quad \varphi(k^2) \leq 0.$$

Proof. We first consider the case $\hat{\Delta} < 0$. Since the discriminant is negative, the quadratic polynomial φ has no real roots. Given the sign $\hat{A} < 0$, the parabola opens downward, so $\varphi(k^2)$ is negative for sufficiently large k^2 . Hence, there exists $\hat{K}_3^2 > 0$

such that $\varphi(\hat{K}_3^2) = 0$, and thus,

$$\forall k^2 \in [\hat{K}_3^2, +\infty) \quad \varphi(k^2) \leq 0.$$

Now, assume $\hat{A} < 0$ and $\hat{\Delta} > 0$. Then φ has two distinct real roots:

$$K_3^2 = \frac{-\hat{B} + \sqrt{\hat{\Delta}}}{2\hat{A}}, \quad K_4^2 = \frac{-\hat{B} - \sqrt{\hat{\Delta}}}{2\hat{A}}, \quad \text{with } K_3^2 < K_4^2.$$

To ensure that both roots are positive, we require

$$\hat{B} > 0 \quad \text{and} \quad \hat{C} < 0. \quad (4.11)$$

The sign of φ at these roots determines its intervals of negativity:

- If $\varphi(K_3^2) > 0$ and $\varphi(K_4^2) > 0$, then φ crosses zero again for some $\hat{K}_4^2 > K_4^2$.
Therefore,

$$\forall k^2 \in [\hat{K}_4^2, +\infty) \quad \varphi(k^2) \leq 0.$$

- If $\varphi(K_3^2) < 0$ and $\varphi(K_4^2) < 0$, then φ must cross zero for some $\hat{K}_5^2 \in (0, K_3^2)$.
Thus,

$$\forall k^2 \in [\hat{K}_5^2, +\infty) \quad \varphi(k^2) \leq 0.$$

- If $\varphi(K_3^2) < 0$ and $\varphi(K_4^2) > 0$, then there exist three positive zeros

$$\hat{K}_6^2 \in (0, K_3^2), \quad \hat{K}_7^2 \in (K_3^2, K_4^2), \quad \hat{K}_8^2 > K_4^2,$$

such that φ crosses zero at each point. Hence,

$$\forall k^2 \in [\hat{K}_6^2, \hat{K}_7^2] \cup [\hat{K}_8^2, +\infty) \quad \varphi(k^2) \leq 0.$$

Special cases.

Case 1: When

$$\text{tr } D \cdot \text{tr}(\text{com } D) - \det D = 0$$

is satisfied, the cubic term in $\varphi(k^2)$ disappears. Hence, $\varphi(k^2)$ reduces to a quadratic polynomial in k^2 given by

$$\begin{aligned} \varphi(k^2) = & \left[\det M_7 + \sigma_1 - \text{tr } D \cdot \left(\sum_{i=1}^6 \det M_i \right) - \text{tr } A \cdot \text{tr}(\text{com } D) \right] (k^2)^2 \\ & + \left[\text{tr } D \cdot \text{tr}(\text{com } A) + \text{tr } A \cdot \left(\sum_{i=1}^6 \det M_i \right) + \sigma_2 - \det M_8 \right] k^2 \\ & + \det A - \text{tr } A \cdot \text{tr}(\text{com } A). \end{aligned}$$

Then φ reduces to a quadratic of the form

$$\varphi(k^2) = \hat{A}_1(k^2)^2 + \hat{B}_1 k^2 + \hat{C}_1, \quad (4.12)$$

put

$$\hat{A}_1 = \det M_7 + \sigma_1 - \text{tr } D \sum_{i=1}^6 \det M_i - \text{tr } A \cdot \text{tr}(\text{com } D).$$

Assume $\hat{A}_1 \neq 0$. The discriminant of (4.12) is

$$\hat{\Delta}_1 = \hat{B}_1^2 - 4\hat{A}_1\hat{C}_1.$$

If $\hat{\Delta}_1 > 0$ and $\hat{A}_1 > 0$, then there are two real roots:

$$\hat{K}_9^2 = \frac{-\hat{B}_1 - \sqrt{\hat{\Delta}_1}}{2\hat{A}_1}, \quad \hat{K}_{10}^2 = \frac{-\hat{B}_1 + \sqrt{\hat{\Delta}_1}}{2\hat{A}_1}, \quad \text{with} \quad \hat{K}_9^2 < \hat{K}_{10}^2.$$

Both roots are positive if

$$\hat{B}_1 < 0, \quad \hat{C}_1 > 0.$$

Therefore,

$$\forall k^2 \in [\hat{K}_9^2, \hat{K}_{10}^2] \quad \varphi(k^2) \leq 0. \quad (4.13)$$

Case 2: $\hat{A}_1 = 0$. Then φ becomes:

$$\varphi(k^2) = \hat{B}_1 k^2 + \hat{C}_1,$$

$$\text{with} \quad \begin{cases} \hat{B}_1 &= \text{tr } D \text{ tr}(\text{com } A) + \text{tr } A \sum_{i=1}^6 \det M_i + \sigma_2 - \det M_8, \\ \hat{C}_1 &= \det A - \text{tr } A \text{ tr}(\text{com } A). \end{cases}$$

If $\hat{B}_1 < 0$, then there exists $\hat{K}_{11}^2 > 0$ such that $\varphi(\hat{K}_{11}^2) = 0$ and

$$\forall k^2 \in [\hat{K}_{11}^2, +\infty) \quad \varphi(k^2) \leq 0.$$

This completes the proof. □

Chapter 5

Application

This chapter bridges abstract mathematical theory with concrete ecological modeling by applying the preceding analytical results to a three-component predator-prey system with spatial dispersal. The model incorporates realistic biological features: prey competition or mutualism, predator functional responses, and diffusive movement that may include cross-diffusion terms representing avoidance or attraction behaviors.

The analysis follows a structured sequence:

1. **Model Specification:** We define the reaction terms f, g, h to reflect ecological principles.
2. **Local Analysis:** We establish local in time existence of classical solutions for biologically admissible initial data.
3. **Invariant Region Construction:** Ecological constraints (non-negative populations, bounded resources) naturally define an invariant region $\mathcal{R}$ in phase space. We verify that the reaction-diffusion dynamics respect these bounds.
4. **Pattern Formation Potential:** Applying the Turing instability criteria from Chapter 4, we compute the Jacobian at coexistence equilibria and identify diffusion coefficient ranges that permit spatial pattern formation potentially explaining patchy species distributions observed in field studies.
5. **Global Existence:** Through carefully designed polynomial Lyapunov functionals, we obtain uniform bounds that preclude finite-time blow-up, ensuring solutions persist for all time.

This implementation serves a dual purpose: it validates the abstract theory through concrete computation, and it provides ecologists with verifiable conditions for pattern formation and population persistence in multi-species communities with spatial structure.

5.1 Mathematical Model

We consider a reaction-diffusion system of three interacting components, which can be formulated as follows:

$$\frac{\partial u}{\partial t} - d_u \Delta u = f(u, v, w) = \alpha u - \beta uv + \delta v, \quad \text{in } (0, \infty) \times \Omega, \quad (5.1)$$

$$\frac{\partial v}{\partial t} - d_v \Delta v - d_{vw} \Delta w = g(u, v, w) = \gamma uv - \mu v + \eta w, \quad \text{in } (0, \infty) \times \Omega, \quad (5.2)$$

$$\frac{\partial w}{\partial t} - d_w \Delta w = h(u, v, w) = \tau w - \xi uw, \quad \text{in } (0, \infty) \times \Omega, \quad (5.3)$$

The system is equipped with homogeneous Neumann boundary conditions:

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \quad (5.4)$$

and initial conditions:

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad w(0, x) = w_0(x), \quad x \in \Omega. \quad (5.5)$$

Here, Ω denotes an open bounded domain of class $\mathcal{C}^1$ in $\mathbb{R}^n$ with boundary $\partial\Omega$, and $\frac{\partial}{\partial \nu}$ represents the outward normal derivative on $\partial\Omega$. All initial data are in $L^\infty(\Omega)$, according to (P) the diffusion coefficients satisfy the following conditions:

$$d_u > 0, \quad d_w > 0, \quad 4d_v d_w - d_{vw}^2 > 0, \quad \text{and} \quad d_w > d_v. \quad (5.6)$$

These constraints ensure the parabolicity of the system (5.1)–(5.3) and imply that the diffusion matrix

$$\Lambda = \begin{pmatrix} d_u & 0 & 0 \\ 0 & d_v & d_{vw} \\ 0 & 0 & d_w \end{pmatrix} \quad (5.7)$$

is positive definite. Introducing the parameter $\Psi = \frac{d_{vw}}{d_v - d_w}$, we impose the following structural assumptions:

$$\begin{cases} \delta v + \Psi(1 - \Psi)w \geq 0, & \forall v \geq 0, \forall w \geq 0, \\ (\mu\Psi + \Psi\tau + \eta)w - (\xi + \gamma)\Psi uw \geq 0, & \forall u \geq 0, \forall w \geq 0. \end{cases} \quad (5.8)$$

The initial data are assumed to belong to the admissible region:

$$\Sigma = \left\{ (u_0, v_0, w_0) \in \mathbb{R}^3 : \begin{cases} u_0 \geq 0, \\ v_0 \geq \frac{d_{vw}}{d_w - d_v} w_0, \\ w_0 \geq 0. \end{cases} \right\}. \quad (5.9)$$

This ecological system models the interactions of predator-prey dynamics, and the solutions $u(x, t)$, $v(x, t)$ and $w(x, t)$ represent the concentrations of three interacting biological species. Now, we give the biological interpretation of constants.

Constant	Signification
α	Growth rate of species u
β	Interaction strength between u and v
γ	Production rate of v via interaction with u
μ	Natural decay rate of v
δ	Effect of v on u
η	Coupling term from w to v
τ	Growth rate of species w
ξ	Interaction term between v and w
d_{vw}	Cross-diffusion coefficient

Equating the left-hand side of (5.1)-(5.3) to $(0,0,0)$, we get the equilibrium points:

- $E_0(0,0,0)$ exists always;
- $E_m\left(\frac{\mu}{\gamma}, \frac{\alpha\mu}{\beta\mu - \delta\gamma}, 0\right)$ exists under the conditions $\gamma \neq 0$ and $\beta\mu - \delta\gamma \neq 0$,
- $E_*\left(\frac{\tau}{\xi}, \frac{\alpha\tau}{\beta\tau - \delta\xi}, \frac{\alpha\tau(\mu\xi - \gamma\tau)}{\eta\xi(\beta\tau - \delta\xi)}\right)$ exists under the conditions $\xi \neq 0$, $\beta\tau - \delta\xi \neq 0$, and $\eta\xi(\beta\tau - \delta\xi) \neq 0$.

5.2 Turing instability

To obtain the Turing instability, we need find the values of k^2 , for which the matrix $A(k^2) = A - k^2D$ can be unstable; in other words, to define conditions on k^2 under which that the matrix $A(k)$ has at least one eigenvalue with positive real part.

Remark 5.1. The Routh-Hurwitz criterion implies that $\varphi_1(k^2) \leq 0$, or $\varphi_3(k^2) \leq 0$, or $(\varphi_1\varphi_2 - \varphi_3)(k^2) \leq 0$, for some values of k^2 , and because $\varphi_1(k^2) \leq 0$ is impossible to fulfil, the only option left is to deal with $\varphi_3(k^2) \leq 0$ or $(\varphi_1\varphi_2 - \varphi_3)(k^2) \leq 0$. Since we have the logical coordinator "or", it is much easier to handle $\varphi_3(k^2) \leq 0$ instead of $(\varphi_1\varphi_2 - \varphi_3)(k^2) \leq 0$. In other words, it suffices to focus on finding the values of k^2 such that $\varphi_3(k^2) \leq 0$.

The discriminant D of the matrix is equal to $d_u d_v d_w > 0$, we apply only Proposition 4.3.

The Jacobian matrix of linearization in the neighborhood of the first equilibrium point E_0 is given by

$$A_0 = \begin{pmatrix} \alpha & \delta & 0 \\ 0 & -\mu & \eta \\ 0 & 0 & \tau \end{pmatrix}.$$

Note that this matrix is stable under the conditions (i), (ii) and (iii), respectively, hence

$$\begin{aligned} \alpha - \mu + \tau &< 0, \\ -\alpha\mu\tau &< 0, \\ \alpha\mu\tau + (\alpha - \mu + \tau)(\alpha\tau - \mu\tau - \alpha\mu) &< 0. \end{aligned}$$

To obtain the desired result, as we have mentioned earlier, the matrix $A_0(k^2) = A_0 - k^2D$ must have an eigenvalue with positive real part.

By Proposition 4.3, if

$$\begin{aligned}\tilde{A}_0 &= 3d_u d_v d_w > 0, \\ \tilde{B}_0 &= 2\mu d_u d_w - 2\tau d_u d_v - 2\alpha d_v d_w < 0, \\ \tilde{C}_0 &= \alpha\tau d_v - \mu\tau d_u - \alpha\mu d_w > 0, \\ \tilde{\Delta}_0 &= \tilde{B}_0^2 - 4\tilde{A}_0\tilde{C}_0 > 0, \\ \varphi_3\left(\frac{-\tilde{B}_0 - \sqrt{\tilde{\Delta}_0}}{2\tilde{A}_0}\right) &> 0 \text{ and } \varphi_3\left(\frac{-\tilde{B}_0 + \sqrt{\tilde{\Delta}_0}}{2\tilde{A}_0}\right) < 0,\end{aligned}$$

it results in

$$\forall k^2 \in \left[\frac{-\tilde{B}_0 - \sqrt{\tilde{\Delta}_0}}{2\tilde{A}_0}, \frac{-\tilde{B}_0 + \sqrt{\tilde{\Delta}_0}}{2\tilde{A}_0} \right] \quad \varphi_3(k^2) \leq 0,$$

where

$$\begin{aligned}\varphi_3(k^2) = & (\det D)(k^2)^3 - (\sigma_1 + \det M_7)(k^2)^2 \\ & + (\det M_8 - \sigma_2)k^2 - \det A_0,\end{aligned}$$

and we get the Turing instability. Moving now to the second equilibrium point E_m , its associated matrix of linearization is given by

$$A_m = \begin{pmatrix} \alpha\left(1 - \frac{\beta\mu}{\beta\mu - \gamma\delta}\right) & \delta - \frac{\beta\mu}{\gamma} & 0 \\ \frac{\alpha\gamma\mu}{\beta\mu - \gamma\delta} & 0 & \eta \\ 0 & 0 & \tau - \frac{\xi\mu}{\gamma} \end{pmatrix}.$$

A_m is stable under the following conditions:

$$\begin{aligned}\tau - \frac{\xi\mu}{\gamma} - \frac{\alpha\gamma\delta}{\beta\mu - \gamma\delta} &< 0, \\ \frac{\alpha\mu}{\delta}(\xi\mu - \gamma\tau) &< 0, \\ \frac{\alpha\mu}{\delta}(\xi\mu - \gamma\tau) - \left(\tau - \frac{\xi\mu}{\gamma} - \frac{\alpha\gamma\delta}{\beta\mu - \gamma\delta}\right)\left(\alpha + \alpha\frac{\xi\mu - \gamma\tau}{\beta\mu - \gamma\delta}\right) &> 0.\end{aligned}$$

Following the same reasoning, we apply Proposition 4.3 under the following assumptions:

$$\begin{aligned}\tilde{A}_m &= 3d_u d_v d_w > 0, \\ \tilde{B}_m &= 2\frac{\xi\mu - \gamma\tau}{\gamma}d_u d_v + 2\frac{\alpha\gamma\delta}{\beta\mu - \gamma\delta}d_v d_w < 0, \\ \tilde{C}_m &= \alpha\mu d_w - \frac{\alpha(\gamma\tau - \xi\mu)}{\beta\mu - \gamma\delta}d_v > 0, \\ \tilde{\Delta}_m &= \tilde{B}_m^2 - 4\tilde{A}_m\tilde{C}_m > 0, \\ \varphi_3\left(\frac{-\tilde{B}_m - \sqrt{\tilde{\Delta}_m}}{2\tilde{A}_m}\right) &> 0 \quad \text{and} \quad \varphi_3\left(\frac{-\tilde{B}_m + \sqrt{\tilde{\Delta}_m}}{2\tilde{A}_m}\right) < 0.\end{aligned}$$

It results in

$$\forall k^2 \in \left[\frac{-\tilde{B}_m - \sqrt{\tilde{\Delta}_m}}{2\tilde{A}_m}, \frac{-\tilde{B}_m + \sqrt{\tilde{\Delta}_m}}{2\tilde{A}_m} \right] \quad \varphi_3(k^2) \leq 0,$$

where

$$\begin{aligned}\varphi_3(k^2) = & (\det D)(k^2)^3 - (\sigma_1 + \det M_7)(k^2)^2 \\ & + (\det M_8 - \sigma_2)k^2 - \det A_m,\end{aligned}$$

and we get the Turing instability. Finally, the matrix

$$A_* = \begin{pmatrix} \alpha \left(1 - \frac{\beta\tau}{\beta\tau - \delta\xi}\right) & \delta - \frac{\beta\tau}{\xi} & 0 \\ \frac{\alpha\gamma\tau}{\beta\tau - \delta\xi} & \frac{\gamma\tau}{\xi} - \mu & \eta \\ \frac{\alpha\tau(\gamma\tau - \mu\xi)}{\eta(\beta\tau - \delta\xi)} & 0 & 0 \end{pmatrix}$$

is the Jacobian matrix of linearization around the final equilibrium point E_* , with the following conditions of stability:

$$\frac{\gamma\tau}{\xi} - \mu - \frac{\alpha\delta\xi}{\beta\tau - \delta\xi} < 0,$$

$$\frac{\alpha\tau}{\xi}(\mu\xi - \gamma\tau) < 0,$$

$$\frac{\alpha\tau}{\xi}(\mu\xi - \gamma\tau) - \alpha \left(\frac{\gamma\tau}{\xi} - \mu - \frac{\alpha\delta\xi}{\beta\tau - \delta\xi} \right) \left(\frac{\gamma\tau}{\xi} - \delta \cdot \frac{\gamma\tau - \mu\xi}{\beta\tau - \delta\xi} \right) > 0.$$

We apply Proposition 4.3 under the following assumptions:

$$\tilde{A}_* = 3d_u d_v d_w > 0,$$

$$\tilde{B}_* = 2 \frac{\mu\xi - \gamma\tau}{\xi} d_u d_w + 2 \frac{\alpha\delta\xi}{\beta\tau - \delta\xi} d_v d_w < 0,$$

$$\tilde{C}_* = \left(\frac{\alpha\delta(\mu\xi - \gamma\tau)}{\beta\tau - \delta\xi} - \frac{\gamma\tau}{\xi} \right) d_w + \frac{\alpha\tau(\gamma\tau - \mu\xi)}{\xi} d_{vw} > 0,$$

$$\tilde{\Delta}_* = \tilde{B}_*^2 - 4\tilde{A}_*\tilde{C}_* > 0,$$

$$\varphi_3\left(\frac{-\tilde{B}_* - \sqrt{\tilde{\Delta}_*}}{2\tilde{A}_*}\right) > 0 \quad \text{and} \quad \varphi_3\left(\frac{-\tilde{B}_* + \sqrt{\tilde{\Delta}_*}}{2\tilde{A}_*}\right) < 0,$$

this leads to

$$\forall k^2 \in \left[\frac{-\tilde{B}_* - \sqrt{\tilde{\Delta}_*}}{2\tilde{A}_*}, \frac{-\tilde{B}_* + \sqrt{\tilde{\Delta}_*}}{2\tilde{A}_*} \right] \quad \varphi_3(k^2) \leq 0,$$

where

$$\begin{aligned} \varphi_3(k^2) = & (\det D)(k^2)^3 - (\sigma_1 + \det M_7)(k^2)^2 \\ & + (\det M_8 - \sigma_2)k^2 - \det A_*. \end{aligned}$$

and we get the Turing instability.

5.3 Local existence and Invariant region

The mathematical necessity of constructing invariant regions in reaction-diffusion analysis arises from two complementary theoretical requirements. First, the conventional analytical framework based on Lyapunov functionals encounters significant limitations when applied directly to these systems without predefined invariant structures. These classical stability techniques—essential for examining asymptotic behavior in simpler dynamical contexts—fundamentally depend on identifying state-space domains that remain positively invariant under the governing equations' flow. Thus, employing Lyapunov-based methodologies requires first delineating such invariant domains, which provide the necessary mathematical foundation for deriving crucial energy estimates and ensuring solution boundedness.

Concurrently, and equally critical in applications spanning biological, ecological, and chemical systems, is preserving solution positivity. Reaction-diffusion formulations that model population densities, chemical concentrations, or physiological variables inherently require non-negative state variables to maintain physical and biological validity. An invariant region confined to the phase space's non-negative orthant ensures that solutions originating from non-negative initial conditions remain within this domain, thereby upholding the model's structural coherence and practical applicability. In this dual capacity, the invariant region serves both as essential scaffolding for applying sophisticated analytical tools like Lyapunov functionals and as a vital safeguard for preserving positivity—a fundamental property indispensable for the model's consistency and real-world relevance.

5.3.1 Transformation to diagonal form

The primary motivation for diagonalizing the system arises from the inapplicability of standard abstract results for parabolic partial differential equations. Consequently, we cannot directly employ established theorems regarding the existence, uniqueness, and positivity of solutions. Before proceeding with our analysis, we present the following preparatory lemma.

Lemma 5.2. *The eigenvalues and corresponding eigenvectors of the transpose diffusion matrix Λ^t are given by:*

$$\begin{aligned} \mathbf{v}_1 &= (1, 0, 0)^t, & \lambda_1 &= d_u, \\ \mathbf{v}_2 &= \left(0, 1, \frac{d_{vw}}{d_v - d_w}\right)^t, & \lambda_2 &= d_v, \\ \mathbf{v}_3 &= (0, 0, 1)^t, & \lambda_3 &= d_w. \end{aligned}$$

Diagonalization Procedure The diagonalization of the system proceeds through the following linear combinations:

- Taking the equation for u directly yields:

$$\frac{\partial u}{\partial t} - d_u \Delta u = \alpha u - \beta uv + \delta v. \quad (5.10)$$

- Forming a linear combination of the equations for v and w with coefficients 1 and $\Psi = \frac{d_{vw}}{d_v - d_w}$, respectively, we obtain:

$$\frac{\partial}{\partial t} (v + \Psi w) - d_v \Delta (v + \Psi w) = (\gamma u - \mu)v + (\eta + \Psi\tau - \zeta\Psi u)w. \quad (5.11)$$

- The equation for w remains unchanged:

$$\frac{\partial w}{\partial t} - d_w \Delta w = \tau w - \xi uw. \quad (5.12)$$

Introducing the transformed variables

$$\begin{cases} U(t, x) = u(t, x), \\ V(t, x) = v(t, x) + \Psi w(t, x), \\ W(t, x) = w(t, x), \end{cases} \quad (5.13)$$

we obtain the following diagonalized system:

$$\frac{\partial U}{\partial t} - d_u \Delta U = F(U, V, W) = \beta \Psi U W - \beta U V + \alpha U + \delta V - \delta \Psi W, \quad (5.14)$$

$$\frac{\partial V}{\partial t} - d_v \Delta V = G(U, V, W) = \gamma U V - (\xi + \gamma) \Psi U W + (\mu \Psi + \tau \Psi + \eta) W - \mu V, \quad (5.15)$$

$$\frac{\partial W}{\partial t} - d_w \Delta W = H(U, V, W) = \tau W - \xi U W, \quad \text{in } (0, \infty) \times \Omega. \quad (5.16)$$

The transformed system inherits the homogeneous Neumann boundary conditions:

$$\frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = \frac{\partial W}{\partial \nu} = 0, \quad \text{on } \mathbb{R}^+ \times \partial \Omega, \quad (5.17)$$

with initial data:

$$U(0, x) = U_0(x), \quad V(0, x) = V_0(x), \quad W(0, x) = W_0(x), \quad x \in \Omega. \quad (5.18)$$

We now establish the invariance of the region Σ under the dynamics of the system.

Proposition 5.3 (Invariant Region). *Assume that the reaction terms f , g , and h satisfy condition (5.8), and that the vector field points inward on the boundary $\partial \Sigma$. Then, for any initial datum $(u_0, v_0, w_0) \in \Sigma$, the solution $(u(t, \cdot), v(t, \cdot), w(t, \cdot))$ of the initial-boundary value problem (5.1)–(5.5) remains in Σ for all $t \in [0, T^*)$, where T^* denotes the maximal existence time.*

Proof. We establish the invariance of Σ by verifying the quasi-positivity conditions for the diagonalized system and applying the invariant region theorem for parabolic systems.

Consider the diagonalized system with variables (U, V, W) and reaction terms (F, G, H) . For the nonnegative quadrant $\{U \geq 0, V \geq 0, W \geq 0\}$ to be invariant, it suffices to show:

- (i) $F(0, V, W) \geq 0$ for all $V \geq 0, W \geq 0$,
- (ii) $G(U, 0, W) \geq 0$ for all $U \geq 0, W \geq 0$,
- (iii) $H(U, V, 0) \geq 0$ for all $U \geq 0, V \geq 0$.

By direct computation, we obtain:

$$F(0, V, W) = \delta v + \Psi(1 - \Psi)w,$$

where $v = V - \Psi W$ and $w = W$. The structural assumption (5.8) guarantees that $F(0, V, W) \geq 0$ for all $V \geq 0, W \geq 0$. Since the initial data satisfies $U_0 = u_0 \geq 0$, the invariant region theorem developed by Smoller [21] ensures $U(t, x) \geq 0$ for all $t \in [0, T^*)$ and $x \in \Omega$.

Similarly,

$$G(U, 0, W) = (\mu\Psi + \tau\Psi + \eta)w - (\xi + \gamma)\Psi uw,$$

with $u = U$ and $w = W$. Assumption (5.8) implies $G(U, 0, W) \geq 0$ for all $U \geq 0$, $W \geq 0$. As $V_0 = v_0 + \Psi w_0 \geq 0$ by definition of Σ , we conclude $V(t, x) \geq 0$ for all $t \in [0, T^*)$ and $x \in \Omega$.

Finally, $H(U, V, 0) = 0$, which trivially satisfies condition (iii). Given $W_0 = w_0 \geq 0$, it follows that $W(t, x) \geq 0$ for all $t \in [0, T^*)$ and $x \in \Omega$. $\square$

Remark 5.4. Since the determinant of the transformation matrix in (5.13) is different from zero, it follows that the original system (5.1)–(5.3) is equivalent to the diagonalized system (5.14)–(5.16).

5.3.2 Local existence

Applying Theorem 2.15 to the diagonalized system (5.14)–(5.16) with the parameter choices $\alpha_i = 1$ and $\beta_i = 0$ for $i = 1, 2, 3$, we verify that the initial condition

$$X_0 = (U_0(x), V_0(x), W_0(x))^t \in L^\infty(\Omega)^3$$

satisfies the regularity requirement of the theorem. Consequently, there exists $T > 0$ and a unique classical solution to (5.14)–(5.16) defined on the interval $[0, T)$. Moreover, if the L^∞ norms of U , V , and W remain bounded up to the maximal existence time T^* , then $T^* = +\infty$ and the solution extends globally.

The quasi-positivity conditions in Theorem 2.15 translate to the requirements

$$\begin{cases} F(0, V, W) \geq 0 & \forall V, W \geq 0, \\ G(U, 0, W) \geq 0 & \forall U, W \geq 0, \\ H(U, V, 0) \geq 0 & \forall U, V \geq 0, \end{cases}$$

which, when satisfied, ensure that non-negative initial data yield non-negative solutions for all $t \in [0, T^*)$.

5.4 Global Existence

The centrality of establishing global existence in reaction-diffusion investigations stems from three interconnected imperatives: mathematical coherence, analytical completeness, and physical interpretation. First, a model's capacity to represent biological or physical processes requires that solutions persist throughout relevant temporal scales without succumbing to pathological singularities. Reaction-diffusion systems, which couple nonlinear interaction kinetics with spatial transport, can exhibit finite-time blow-up or catastrophic regularity loss, rendering models invalid beyond limited intervals. Demonstrating global existence ensures that solution trajectories remain well-defined and bounded, thus preserving the model's predictive utility over biologically or physically meaningful timeframes. Second, global regularity constitutes a prerequisite for any substantive investigation into long-term dynamical behavior. Without this foundation, analyses of stability properties, pattern formation mechanisms, attractor structures, and bifurcation phenomena remain inherently provisional, as their conclusions depend on the unverified assumption of solution persistence. Particularly in systems modeling

ecological dynamics, chemical networks, or physiological transport, the phenomena of primary interest—including spatial heterogeneity emergence, transitions between homogeneous and patterned states, or convergence to equilibrium—are inherently asymptotic. Consequently, the analytical program directed toward elucidating these features remains methodologically incomplete without rigorous verification that solutions exist on the unbounded temporal domains where these behaviors manifest.

Third, the techniques employed to secure global existence—including a priori estimate derivation, invariant region construction, comparison principle application, and energy method deployment—invariably yield significant structural insights into the system's intrinsic regulatory mechanisms. These analytical endeavors frequently reveal hidden conservation properties, dissipative structures, or compensatory balances between reaction kinetics and diffusive smoothing that govern the system's capacity for self-regulation. Thus, the proof of global existence transcends mere technical validation, serving as an investigative tool that uncovers the fundamental mathematical architecture constraining system evolution. The process of establishing global bounds on solution norms often illuminates critical parameter thresholds, identifies stabilizing nonlinearities, or exposes delicate balances determining whether the system sustains bounded, physically admissible dynamics or succumbs to explosive instabilities.

Together, these considerations position global existence at the confluence of mathematical consistency, analytical tractability, and physical interpretability—serving simultaneously as a criterion for model validation, a gateway to asymptotic analysis, and a source of deep structural understanding of reaction-diffusion dynamics in spatially extended systems.

The theory of parabolic partial differential equations provides that global existence for systems of type (5.14)-(5.16) follows from uniform L^p estimates of the reaction terms (see Henry [9],). Specifically, one must demonstrate that $\|F(U, V, W)\|_{L^p(\Omega)}$ and $\|G(U, V, W)\|_{L^p(\Omega)}$ remain bounded on $[0, T_{\max})$ for some $p > n/2$. To this end, we construct polynomial Lyapunov functionals to establish the required L^p -bounds for u and v , thereby ensuring global existence.

Applying the maximum principle to equation (5.16) under the conditions $\tau < 0$ and $\xi > 0$ yields the estimate

$$\|W(t, \cdot)\|_{L^\infty(\Omega)} \leq \|W_0\|_{L^\infty(\Omega)}.$$

With the boundedness of W established, attention turns to the remaining components U and V . These can be analyzed through the construction of polynomial Lyapunov functionals, which yield the necessary L^p estimates to complete the existence theory.

Theorem 5.5. *Let $U(t, \cdot)$ and $V(t, \cdot)$ be classical solutions of the diagonalized system (5.14)–(5.15). Define the Lyapunov functional*

$$L(t) = \int_{\Omega} \sum_{i=0}^p K^{i^2} \binom{p}{i} U^i(t, x) V^{p-i}(t, x) dx, \quad (5.19)$$

where p is a positive integer and $K \geq \frac{d_u + d_v}{2\sqrt{d_u d_v}}$ is a positive constant. Then $L(t)$ remains bounded on $[0, \bar{T})$, where $\bar{T}$ denotes the potential blow-up time.

Proof. Define the polynomial

$$\mathcal{L}(U, V) = \sum_{i=0}^p K^{i^2} \binom{p}{i} U^i V^{p-i}.$$

Its partial derivatives are

$$\begin{aligned} \partial_U \mathcal{L}(U, V) &= \sum_{i=1}^p i K^{i^2} \binom{p}{i} U^{i-1} V^{p-i}, \\ \partial_V \mathcal{L}(U, V) &= \sum_{i=0}^{p-1} (p-i) K^{i^2} \binom{p}{i} U^i V^{p-i-1}. \end{aligned}$$

Using the approach developed by Kouachi (see [11],[12] and [13]) differentiating $L(t)$ with respect to t and using the equations for U and V yields

$$\begin{aligned} \frac{d}{dt} L(t) &= \int_{\Omega} \left[\partial_U \mathcal{L}(U, V) (d_u \Delta U + F(U, V, W)) \right. \\ &\quad \left. + \partial_V \mathcal{L}(U, V) (d_v \Delta V + G(U, V, W)) \right] dx \\ &= I + J, \end{aligned}$$

where I contains the diffusion terms and J the reaction terms.

Estimation of I : After integration by parts, I becomes

$$I = - \int_{\Omega} (a |\nabla U|^2 + b \nabla U \cdot \nabla V + c |\nabla V|^2) dx,$$

with coefficients

$$\begin{aligned} a &= \sum_{i=2}^p d_u i(i-1) K^{i^2} \binom{p}{i} U^{i-2} V^{p-i}, \\ b &= \sum_{i=1}^{p-1} d_u i(p-i) K^{i^2} \binom{p}{i} U^{i-1} V^{p-i-1} \\ &\quad + \sum_{i=2}^p d_v (i-1)(p-i) K^{(i-1)^2} \binom{p}{i-1} U^{i-2} V^{p-i}, \\ c &= \sum_{i=1}^{p-1} d_v (p-i)(p-i+1) K^{(i-1)^2} \binom{p}{i-1} U^{i-1} V^{p-i-1}. \end{aligned}$$

Using the identity $i \binom{p}{i} = p \binom{p-1}{i-1}$, these simplify to

$$\begin{aligned} a &= \sum_{i=0}^{p-2} d_u p(p-1) K^{(i+2)^2} \binom{p-2}{i} U^i V^{p-i-2}, \\ b &= \sum_{i=0}^{p-2} (d_u + d_v) p(p-1) K^{(i+1)^2} \binom{p-2}{i} U^i V^{p-i-2}, \\ c &= \sum_{i=0}^{p-2} d_v p(p-1) K^{i^2} \binom{p-2}{i} U^i V^{p-i-2}. \end{aligned}$$

Thus,

$$I = -p(p-1) \sum_{i=0}^{p-2} \int_{\Omega} Q(\nabla U, \nabla V) dx,$$

where Q is the quadratic form

$$Q(\nabla U, \nabla V) = d_u K^{(i+2)^2} |\nabla U|^2 + (d_u + d_v) K^{(i+1)^2} \nabla U \cdot \nabla V + d_v K^{i^2} |\nabla V|^2.$$

Its discriminant is

$$\begin{aligned} \Delta_Q &= ((d_u + d_v) K^{(i+1)^2})^2 - 4d_u d_v K^{2(i+1)^2+2} \\ &= ((d_u + d_v)^2 - (2\sqrt{d_u d_v} K)^2) (K^{(i+1)^2})^2 \\ &= (d_u + d_v - 2\sqrt{d_u d_v} K)(d_u + d_v + 2\sqrt{d_u d_v} K) (K^{(i+1)^2})^2. \end{aligned}$$

For $I \leq 0$, it suffices that $\Delta_Q \leq 0$, which holds provided

$$K \geq \frac{d_u + d_v}{2\sqrt{d_u d_v}}.$$

Estimation of J : Let

$$r_1 = \beta \Psi, \quad r_2 = \delta \Psi, \quad q_1 = -(\xi + \gamma) \Psi, \quad q_2 = \mu \Psi + \tau \Psi + \eta.$$

Then

$$\begin{aligned} J &= \int_{\Omega} \sum_{i=1}^p i K^{i^2} \binom{p}{i} U^{i-1} V^{p-i} (r_1 U W - \beta U V + \alpha U + \delta V - r_2 W) dx \\ &\quad + \int_{\Omega} \sum_{i=1}^p (p-i+1) K^{(i-1)^2} \binom{p}{i-1} U^{i-1} V^{p-i} (\gamma U V + q_1 U W + q_2 W - \mu V) dx. \end{aligned}$$

Using $(p-i+1) \binom{p}{i-1} = i \binom{p}{i} = p \binom{p-1}{i-1}$ and rearranging,

$$J = p \sum_{i=1}^p K^{(i-1)^2} \int_{\Omega} [K^{2i-1} F(U, V, W) + G(U, V, W)] \binom{p-1}{i-1} U^{i-1} V^{p-i} dx.$$

By assumption, there exists $M_1 > 0$ such that for all $(U, V, W) \in \Sigma$,

$$K^{2i-1} F(U, V, W) + G(U, V, W) \leq M_1 (U + V + 1), \quad i = 1, \dots, p. \quad (5.20)$$

Hence,

$$J \leq p K^{(p-1)^2} M_1 \int_{\Omega} \sum_{i=1}^p (U + V + 1) \binom{p-1}{i-1} U^{i-1} V^{p-i} dx.$$

Observe that

$$\begin{aligned} \sum_{i=1}^p (U + V + 1) \binom{p-1}{i-1} U^{i-1} V^{p-i} &= \sum_{i=1}^p \binom{p-1}{i-1} U^i V^{p-i} + \sum_{i=1}^p \binom{p-1}{i-1} U^{i-1} V^{p-i+1} \\ &\quad + \sum_{i=1}^p \binom{p-1}{i-1} U^{i-1} V^{p-i} \\ &= (U + V)^p + (U + V)^{p-1}. \end{aligned}$$

Therefore,

$$J \leq pK^{(p-1)^2} M_1 \left(\int_{\Omega} (U + V)^p dx + \int_{\Omega} (U + V)^{p-1} dx \right).$$

From the definition of $L(t)$,

$$\inf_{0 \leq i \leq p} K^{i^2} \int_{\Omega} (U + V)^p dx \leq L(t) \leq \sup_{0 \leq i \leq p} K^{i^2} \int_{\Omega} (U + V)^p dx,$$

so

$$\int_{\Omega} (U + V)^p dx \leq \frac{1}{\kappa} L(t), \quad \text{where } \kappa = \inf_{0 \leq i \leq p} K^{i^2}.$$

By Hölder's inequality,

$$\int_{\Omega} (U + V)^{p-1} dx \leq |\Omega|^{1/p} \left(\int_{\Omega} (U + V)^p dx \right)^{\frac{p-1}{p}} \leq |\Omega|^{1/p} \left(\frac{1}{\kappa} L(t) \right)^{\frac{p-1}{p}}.$$

Thus,

$$J \leq \frac{pK^{(p-1)^2} M_1}{\kappa} L(t) + pK^{(p-1)^2} M_1 |\Omega|^{1/p} \left(\frac{1}{\kappa} \right)^{\frac{p-1}{p}} (L(t))^{\frac{p-1}{p}}.$$

Differential Inequality: Combining the estimates for I and J , we obtain:

$$\frac{d}{dt} L(t) \leq C_1 L(t) + C_2 (L(t))^{\frac{p-1}{p}},$$

where C_1 and C_2 are positive constants. Introducing the function $y(t) = (L(t))^{1/p}$, we derive the differential inequality:

$$\frac{dy}{dt} \leq C_1 y + C_2.$$

Applying Gronwall's inequality to this linear differential inequality yields:

$$y(t) \leq y(0)e^{C_1 t} + \frac{C_2}{C_1} (e^{C_1 t} - 1), \quad \text{for all } t \in [0, \bar{T}).$$

Since the right-hand side remains finite on the interval $[0, \bar{T})$, we conclude that $y(t)$ is bounded. Consequently, $L(t) = y(t)^p$ is also bounded on $[0, \bar{T})$, which completes the proof. $\square$

Proposition 5.6 (Global Existence). *Assume the reaction vector field (f, g) is continuously differentiable, points inward on $\partial\Sigma$, and satisfies (5.20), (5.21) with initial data in Σ that for some $m > 0$ and $p > \frac{mn}{2}$,*

$$|F(U, V, W)|^{p/m}, |G(U, V, W)|^{p/m} \leq M_2(U + V + 1)^p. \quad (5.21)$$

Then all solutions of the system are global in time.

Proof. Under the given assumptions (5.21), we have:

$$\begin{aligned}
\|\chi(U, V, W)\|_{L^{p/m}(\Omega)}^{p/m} &\leq M_2 \int_{\Omega} (U + V + 1)^p dx \\
&\leq M_2 \int_{\Omega} \sum_{i=0}^p \binom{p}{i} 1^i (U + V)^{p-i} dx \\
&\leq M_2 \int_{\Omega} \sum_{i=0}^p \binom{p}{i} (U + V)^p dx \\
&\leq M_2 \left(\sum_{i=0}^p \left(\sup_{0 \leq i \leq p} \binom{p}{i} \right) \right) \int_{\Omega} (U + V)^p dx \\
&\leq p M_2 \left(\sup_{0 \leq i \leq p} \binom{p}{i} \right) \int_{\Omega} \frac{\sup_{0 \leq i \leq p} \binom{p}{i}}{\inf_{0 \leq i \leq p} K^{i^2}} \sum_{i=0}^p K^{i^2} \binom{p}{i} U^i V^{p-i} dx \\
&\leq p M_2 \frac{\left(\sup_{0 \leq i \leq p} \binom{p}{i} \right)^2}{\inf_{0 \leq i \leq p} K^{i^2}} \int_{\Omega} \sum_{i=0}^p K^{i^2} \binom{p}{i} U^i V^{p-i} dx \\
&= p M_2 \frac{\left(\sup_{0 \leq i \leq p} \binom{p}{i} \right)^2}{\inf_{0 \leq i \leq p} K^{i^2}} L(t),
\end{aligned}$$

where $\chi = \sup \left(\|F(U, V, W)\|_{L^{p/m}(\Omega)}^{p/m}, \|G(U, V, W)\|_{L^{p/m}(\Omega)}^{p/m} \right)$.

Since the functional L is bounded on $[0, \bar{T})$ and we take $p > \frac{mn}{2}$, it follows directly that:

$$\|F(U, V, W)\|_{L^{p/m}(\Omega)}^{p/m}, \|G(U, V, W)\|_{L^{p/m}(\Omega)}^{p/m} \leq C,$$

for some constant $C > 0$. By the regularization effect of parabolic equations, we conclude that the solutions $U(t, \cdot)$ and $V(t, \cdot)$ exist globally in time. Given the equivalence of the two systems established earlier, the solutions $u(t, \cdot)$, $v(t, \cdot)$, and $w(t, \cdot)$ are also global in time. $\square$

Chapter 6

Final remarks

FINAL chapter consolidates several methodological extensions and unifying observations that complement the core results developed in preceding chapters. While our primary analysis focused on triangular diffusion matrices, the conceptual framework naturally extends to more general configurations. Here we present three interconnected analytical developments:

First, we examine Turing instability mechanisms in three-component reaction-diffusion systems, where differential diffusion rates interact with nonlinear kinetics to produce pattern-forming instabilities. The spectral conditions for such instabilities involve nontrivial relationships between all nine diffusion coefficients and reaction Jacobian entries.

Second, we construct invariant regions for systems with fully populated diffusion matrices, extending beyond the triangular structures studied earlier. This generalization requires careful spectral analysis of the complete diffusion matrix and appropriate reformulation of boundary conditions.

Third, we develop polynomial Lyapunov functionals applicable to these extended systems, providing a mechanism to obtain uniform bounds on solutions even when maximum principles fail. These functionals incorporate weighted combinations of solution components tailored to the system's spectral properties.

Each extension maintains the analytical rigor established throughout this work while addressing broader classes of parabolic systems encountered in applications. Together, they illustrate the adaptability of the methods developed in this thesis and suggest promising directions for future investigation.

Remark 6.1. The explicit expression of the characteristic polynomial of $A(k)$ is

$$\tilde{P}(\lambda) = -\lambda^3 + p\lambda^2 + q\lambda + r,$$

where

$$p = a_{11} + a_{22} + a_{33} - k^2 (d_{11} + d_{22} + d_{33}),$$

$$\begin{aligned} q = & \left[(d_{12}d_{21} - d_{11}d_{22}) + (d_{13}d_{31} - d_{11}d_{33}) + (d_{23}d_{32} - d_{22}d_{33}) \right] (k^2)^2 \\ & + \left[a_{11}(d_{22} + d_{33}) + d_{11}(a_{22} + a_{33}) + a_{22}d_{33} + a_{33}d_{22} \right. \\ & \left. - (a_{12}d_{21} + a_{21}d_{12} + a_{13}d_{31} + a_{31}d_{13} + a_{23}d_{32} + a_{32}d_{23}) \right] k^2 \\ & + (a_{12}a_{21} - a_{11}a_{22}) + (a_{13}a_{31} - a_{11}a_{33}) + (a_{23}a_{32} - a_{22}a_{33}), \end{aligned}$$

$$\begin{aligned}
r = & - \left[d_{11}(d_{22}d_{33} - d_{23}d_{32}) - d_{21}(d_{12}d_{33} - d_{13}d_{32}) \right. \\
& + d_{31}(d_{12}d_{23} - d_{13}d_{22}) \Big] k^6 \\
& + \left[a_{11}(d_{22}d_{33} - d_{23}d_{32}) - a_{21}(d_{12}d_{33} - d_{13}d_{32}) \right. \\
& + a_{31}(d_{12}d_{23} - d_{13}d_{22}) \\
& - d_{11}(a_{23}d_{32} + a_{32}d_{23} - a_{22}d_{33} - a_{33}d_{22}) \\
& + d_{21}(a_{13}d_{32} + a_{32}d_{13} - a_{12}d_{33} - a_{33}d_{12}) \\
& - d_{31}(a_{13}d_{22} + a_{22}d_{13} - a_{12}d_{23} - a_{23}d_{12}) \Big] k^4 \\
& + \left[-d_{11}(a_{22}a_{33} - a_{23}a_{32}) + d_{21}(a_{12}a_{33} - a_{13}a_{32}) \right. \\
& - d_{31}(a_{12}a_{23} - a_{13}a_{22}) \\
& + a_{11}(a_{23}d_{32} + a_{32}d_{23} - a_{22}d_{33} - a_{33}d_{22}) \\
& + a_{21}(a_{12}d_{33} + a_{33}d_{12} - a_{13}d_{32} - a_{32}d_{13}) \\
& + a_{31}(a_{13}d_{22} + a_{22}d_{13} - a_{12}d_{23} - a_{23}d_{12}) \Big] k^2 \\
& + a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22}).
\end{aligned}$$

Corollary 6.2. For $k = 0$, we get the characteristic polynomial of the matrix A given by (3.7).

Remark 6.3. We strongly recommend that roots k_i^2 of $\frac{\partial}{\partial k^2} \varphi_3$ respectively K_i^2 of $\frac{\partial}{\partial k^2} \varphi$ to be positives, otherwise we cannot proceed, we mention that the conditions (4.3), (4.5), (4.7), (4.10), (4.11), and (4.13) are necessary and sufficient.

Remark 6.4. The conditions $\text{sign}(k_i^2) = -\text{sign}(k_{i+1}^2)$ respectively $\text{sign}(K_i^2) = -\text{sign}(K_{i+1}^2)$ for all $i \geq 1$ assure the existence of $\tilde{k}_i^2$ respectively $\hat{K}_i^2$ for all $i \geq 1$.

Remark 6.5. The following cases don't assure the existence of an eigenvalue with positive real part or leads us to a contradiction.

Case. The special case when,

$$\varphi_3 = -(\sigma_1 + \det M_7)(k^2)^2 + (\det M_8 - \sigma_2)k^2 - \det A.$$

If $\tilde{\Delta}_1 < 0$ and $(\sigma_1 + \det M_7) < 0$ then $\varphi_3(k^2) > 0 \forall k^2 \geq 0$. If $(\tilde{\Delta}_1 < 0$ or $\tilde{\Delta}_1 > 0)$ and $(\sigma_1 + \det M_7) > 0$ it leads to a contradiction with hypotheses (ii).

Case. Concerning φ if $\hat{A} > 0$ and $\hat{\Delta} < 0$ it results $\varphi(k^2) > 0 \forall k^2 \geq 0$, so we get nothing.

Case. The special case when

$$\begin{aligned} \varphi(k^2) = & \overbrace{\left(\det M_7 + \sigma_1 - \operatorname{tr} D \cdot \sum_{i=1}^6 \det M_i - \operatorname{tr} A \cdot \operatorname{tr} \operatorname{com} D \right)}^{\hat{A}_1} (k^2)^2 \\ & + \left(\operatorname{tr} D \cdot \operatorname{tr} \operatorname{com} A + \operatorname{tr} A \cdot \sum_{i=1}^6 \det M_i + \sigma_2 - \det M_8 \right) k^2 \\ & + \det A - \operatorname{tr} A \cdot \operatorname{tr} \operatorname{com} A. \end{aligned}$$

if $\hat{A}_1 < 0$ and $(\hat{\Delta}_1 > 0 \text{ or } \hat{\Delta}_1 < 0)$ it leads to a contradiction with hypotheses (iii). if $\hat{\Delta}_1 < 0$ and $\hat{A}_1 > 0$ then $\varphi(k^2) > 0 \forall k^2 \geq 0$.

Remark 6.6 (Simplified Growth Conditions via L^∞ Estimates). Using the L^∞ bound obtained from the maximum principle,

$$\|w(t, \cdot)\|_{L^\infty(\Omega)} = \|W(t, \cdot)\|_{L^\infty(\Omega)} \leq \|W_0\|_{L^\infty(\Omega)},$$

we may simplify the growth conditions by removing explicit dependence on w from the right-hand sides. Define

$$C_w = \|w_0\|_{L^\infty(\Omega)} = \|W_0\|_{L^\infty(\Omega)}.$$

Then for any $(u, v, w) \in \Sigma$, we have $|w| \leq C_w$, and consequently

$$u + v + \Psi w + 1 \leq u + v + |\Psi|C_w + 1 \leq (1 + |\Psi|C_w)(u + v + 1).$$

Hence, condition (5.20) in the original variables becomes

$$K^{2i-1}[\alpha u - \beta uv + \delta v] + [(\gamma u - \mu)v + (\tau\Psi + \eta - \xi\Psi u)w] \leq \tilde{M}_1(u + v + 1), \quad (5.22)$$

where $\tilde{M}_1 = M_1(1 + |\Psi|C_w)$. Similarly, conditions (5.21) transform to

$$|\alpha u - \beta uv + \delta v|^{p/m} \leq \tilde{M}_2(u + v + 1)^p, \quad (5.23)$$

$$|(\gamma u - \mu)v + (\tau\Psi + \eta - \xi\Psi u)w|^{p/m} \leq \tilde{M}_2(u + v + 1)^p, \quad (5.24)$$

with $\tilde{M}_2 = M_2(1 + |\Psi|C_w)^p$.

These formulations exhibit the growth requirements purely in terms of u and v , leveraging the uniform boundedness of w . This reduction is particularly useful when applying Lyapunov functionals that depend only on $U = u$ and $V = v + \Psi w$, as the boundedness of w allows one to control the non-linearities exclusively through u and v .

Remark 6.7 (Generalization to Full Diffusion Matrices). The analytical framework developed in this thesis can be extended to reaction-diffusion systems with a full diffusion matrix of the form

$$\begin{aligned}\frac{\partial \phi_1}{\partial t} - \delta_{11}\Delta\phi_1 - \delta_{12}\Delta\phi_2 - \delta_{13}\Delta\phi_3 &= \mathcal{F}_1(\phi_1, \phi_2, \phi_3), \\ \frac{\partial \phi_2}{\partial t} - \delta_{21}\Delta\phi_1 - \delta_{22}\Delta\phi_2 - \delta_{23}\Delta\phi_3 &= \mathcal{F}_2(\phi_1, \phi_2, \phi_3), \\ \frac{\partial \phi_3}{\partial t} - \delta_{31}\Delta\phi_1 - \delta_{32}\Delta\phi_2 - \delta_{33}\Delta\phi_3 &= \mathcal{F}_3(\phi_1, \phi_2, \phi_3),\end{aligned}$$

subject to the boundary conditions

$$\lambda\phi_i + (1 - \lambda)\frac{\partial\phi_i}{\partial\eta} = \beta_i, \quad i = 1, 2, 3, \quad \text{on } \mathbb{R}^+ \times \partial\Omega,$$

with initial data

$$\phi_i(0, x) = \phi_{i0}(x), \quad x \in \Omega, \quad i = 1, 2, 3.$$

Here, δ_{ij} denote diffusion coefficients, $0 \leq \lambda \leq 1$ is a boundary parameter, and $\beta_i \in \mathbb{R}$ are boundary constants. Building upon the methodology established by Kouachi and Belgassam [14], an invariant region $\mathcal{R}$ can be constructed under the following conditions:

$$\mathcal{R} = \begin{cases} \mathcal{R}_1 & \text{if } \nu_2\beta_2 \leq \delta_{21}\beta_1 + \delta_{23}\beta_3 \leq \nu_1\beta_2 \text{ and } \delta_{32}\beta_1 \leq \delta_{12}\beta_3, \\ \mathcal{R}_2 & \text{if } \frac{1}{\nu_2}(\delta_{21}\beta_1 + \delta_{23}\beta_3) \leq \beta_2 \leq \frac{1}{\nu_1}(\delta_{21}\beta_1 + \delta_{23}\beta_3) \text{ and } \delta_{32}\beta_1 \leq \delta_{12}\beta_3, \end{cases}$$

where the subregions are defined by

$$\begin{aligned}\mathcal{R}_1 &= \left\{ (\phi_1^0, \phi_2^0, \phi_3^0) \in \mathbb{R}^3 : \begin{array}{l} \nu_2\phi_2^0 \leq \delta_{21}\phi_1^0 + \delta_{23}\phi_3^0 \leq \nu_1\phi_2^0, \\ \delta_{32}\phi_1^0 \leq \delta_{12}\phi_3^0 \end{array} \right\}, \\ \mathcal{R}_2 &= \left\{ (\phi_1^0, \phi_2^0, \phi_3^0) \in \mathbb{R}^3 : \begin{array}{l} \frac{1}{\nu_2}(\delta_{21}\phi_1^0 + \delta_{23}\phi_3^0) \leq \phi_2^0 \leq \frac{1}{\nu_1}(\delta_{21}\phi_1^0 + \delta_{23}\phi_3^0), \\ \delta_{32}\phi_1^0 \leq \delta_{12}\phi_3^0 \end{array} \right\},\end{aligned}$$

with $\nu_1 = \underline{\delta} - \zeta_1 > 0 > \nu_2 = \underline{\delta} - \zeta_2$, where ζ_1, ζ_2 are spectral parameters of the diffusion matrix.

The reaction terms must satisfy the directional conditions

$$\begin{aligned}\delta_{21}\mathcal{F}_1 + \delta_{23}\mathcal{F}_3 &\leq \nu_1\mathcal{F}_2, \\ \delta_{22}\mathcal{F}_2 &\leq \delta_{21}\mathcal{F}_1 + \delta_{23}\mathcal{F}_3, \\ \delta_{32}\mathcal{F}_1 &\leq \delta_{12}\mathcal{F}_3,\end{aligned}$$

on the corresponding boundaries of $\mathcal{R}$, ensuring the vector field points inward. Additionally, there exist positive constants E, D, C_1 such that

$$(E\mathcal{F}_1 + D\mathcal{F}_2 + \mathcal{F}_3)(\phi_1, \phi_2, \phi_3) \leq C_1(\phi_1 + \phi_2 + \phi_3 + 1), \quad \forall (\phi_1, \phi_2, \phi_3) \in \mathcal{R}.$$

These conditions, combined with the invariant region properties, establish a foundation for proving global existence and boundedness of solutions in the fully coupled diffusion setting.

Remark 6.8 (Generalized Lyapunov Functional Approach). In scenarios where the maximum principle cannot be directly applied—such as when the diffusion matrix is not cooperative or when the reaction terms lack appropriate monotonicity properties—alternative methods based on Lyapunov functionals become essential. Following the approach developed by Kouachi [12], consider the system with diagonal diffusion:

$$\begin{aligned}\frac{\partial \psi_1}{\partial t} - \gamma_1 \Delta \psi_1 &= \mathcal{G}_1(\psi_1, \psi_2, \psi_3), \\ \frac{\partial \psi_2}{\partial t} - \gamma_2 \Delta \psi_2 &= \mathcal{G}_2(\psi_1, \psi_2, \psi_3), \\ \frac{\partial \psi_3}{\partial t} - \gamma_3 \Delta \psi_3 &= \mathcal{G}_3(\psi_1, \psi_2, \psi_3),\end{aligned}$$

subject to the boundary conditions

$$\lambda \psi_i + (1 - \lambda) \frac{\partial \psi_i}{\partial \eta} = \varrho_i, \quad i = 1, 2, 3, \quad \text{on } \mathbb{R}^+ \times \partial\Omega,$$

with initial data

$$\psi_i(0, x) = \psi_{i0}(x), \quad x \in \Omega, \quad i = 1, 2, 3.$$

Assume the reaction terms satisfy a polynomial growth condition: there exist positive constants M, N and C_2 such that

$$M\mathcal{G}_1(\psi_1, \psi_2, \psi_3) + N\mathcal{G}_2(\psi_1, \psi_2, \psi_3) + \mathcal{G}_3(\psi_1, \psi_2, \psi_3) \leq C_2(\psi_1 + \psi_2 + \psi_3 + 1), \quad (5.25)$$

for all $\psi_1, \psi_2, \psi_3 \geq 0$. This condition generalizes (5.20) and ensures controlled non-linear growth.

Define parameters

$$\alpha = \sqrt{\frac{1 + \gamma^2}{\beta\kappa}}, \quad \beta_* = \sqrt{\frac{\gamma + \beta}{\gamma\kappa}}, \quad \kappa_* = \sqrt{\frac{\gamma + \beta}{\gamma\beta}},$$

and select positive constants ξ and ς such that

$$(\varsigma^2 - \alpha^2)(\xi^2 - \beta_*^2) - (\kappa_* - \alpha\beta_*)^2 > 0 \quad \text{and} \quad \xi > \kappa_*.$$

For a positive integer n , let

$$\xi_q = \xi^{q^2} \quad \text{and} \quad \varsigma_p = \varsigma^{p^2}, \quad \text{for } q = 1, \dots, p, \quad p = 1, \dots, n.$$

The key result involves the Lyapunov functional

$$\mathcal{L}(t) = \int_{\Omega} \mathcal{H}_n(\psi_1(t, x), \psi_2(t, x), \psi_3(t, x)) \, dx,$$

where

$$\mathcal{H}_n(\psi_1, \psi_2, \psi_3) = \sum_{p=0}^n \sum_{q=0}^p \binom{p}{q} \binom{q}{p} \zeta_q \zeta_p \psi_1^q \psi_2^{p-q} \psi_3^{n-p}.$$

Under condition (5.25) with sufficiently large M, N , the functional $\mathcal{L}(t)$ remains uniformly bounded for all $t \geq 0$. This boundedness yields a priori estimates that preclude finite-time blow-up, thereby ensuring global existence of classical solutions. The construction generalizes earlier results for two-component systems and provides a systematic method for handling polynomial nonlinearities in coupled parabolic systems.

Remark 6.9 (Monotone Lyapunov Functional Case). In the special case where the Lyapunov functional $L(t)$ is non-increasing—that is, when $\frac{d}{dt}L(t) \leq 0$ the a priori estimates can be strengthened. Under the same polynomial growth assumption

$$|F(U, V, W)|^{p/m}, |G(U, V, W)|^{p/m} \leq M_2 (U + V + 1)^p,$$

the boundedness of the reaction terms follows directly from the initial value of the Lyapunov functional. Specifically, one obtains

$$\|\chi((U, V, W))\|_{L^{p/m}(\Omega)}^{p/m} \leq p M_2 \frac{\left(\sup_{0 \leq i \leq p} \binom{p}{i}\right)^2}{\inf_{0 \leq i \leq p} K^{i^2}} L(0),$$

where $\chi = \max\{\|F\|_{L^{p/m}(\Omega)}^{p/m}, \|G\|_{L^{p/m}(\Omega)}^{p/m}\}$ and

$$L(0) = \int_{\Omega} \sum_{i=0}^p K^{i^2} \binom{p}{i} U_0(x)^i V_0(x)^{p-i} dx \in L^{\infty}(\Omega),$$

with $U_0(x)$ and $V_0(x)$ denoting the initial data.

Since $L(t) \leq L(0)$ for all $t \geq 0$, the right-hand side of the estimate is independent of time. This uniform bound immediately yields global $L^{p/m}$ -estimates for the nonlinearities F and G , which, together with the condition $p/m > n/2$, guarantees global existence of classical solutions via standard parabolic regularity theory. The monotonicity of $L(t)$ therefore simplifies the analysis and leads to more explicit bounds.

Conclusion

Concluding Synthesis

This thesis has developed a comprehensive analytical framework for three-component reaction-diffusion systems with coupled diffusion mechanisms. Through the systematic construction of invariant regions, polynomial Lyapunov functionals, and careful spectral analysis, we have established conditions ensuring global existence, boundedness, and pattern-forming capabilities of solutions.

Principal Achievements

The work advances the theory of parabolic systems in several directions:

1. **Generalized Invariant Regions:** We extended invariant region theory from triangular to fully populated diffusion matrices, accommodating Robin, Neumann, and Dirichlet boundary conditions within a unified formulation.
2. **Constructive Lyapunov Theory:** The development of explicit polynomial functionals provides a practical mechanism for deriving a priori estimates, even when maximum principles or comparison techniques are unavailable.
3. **Global Existence Framework:** By combining invariant regions with L^p estimates and blow-up alternatives, we obtained verifiable criteria for global solution existence without requiring monotonicity or cooperativity assumptions.
4. **Turing Instability Analysis:** The spectral conditions for diffusion-driven instability were rigorously analyzed, clarifying how differential diffusion rates interact with reaction kinetics to produce spatial patterning in three-component systems.

Methodological Contributions

The analytical approach demonstrates how spectral decomposition, diagonalization techniques, and energy methods can be synergistically combined to handle strongly coupled parabolic systems. The transformation to diagonal variables, while standard for linear problems, proves particularly powerful when coupled with invariant region arguments for nonlinear systems.

Broader Implications and Future Perspectives

The methods developed here are not restricted to specific reaction kinetics and can be adapted to various models in mathematical biology, including predator-prey interactions, chemical reaction networks, and morphogenetic patterning systems. Future work could explore:

- Extension to systems with cross-diffusion or time-dependent coefficients
- Numerical validation of the theoretical bounds and instability thresholds
- Application to specific biological models requiring three-component descriptions
- Connection to stochastic versions of the systems considered

In summary, this thesis provides both theoretical advances and practical tools for analyzing complex reaction-diffusion systems, contributing to the ongoing dialogue between partial differential equations theory and applied mathematical modeling.

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Abstract

This dissertation addresses the mathematical analysis of three-component reaction–diffusion systems, with a focus on their necessity in modelling complex phenomena across ecology, epidemiology, and developmental biology. Such systems provide a more realistic framework than classical two-component models, capturing intricate multi-species interactions, cross-diffusion processes, and spatially extended dynamics.

The first part of the work establishes explicit algebraic conditions for the onset of diffusion-driven instability (Turing patterns) in tri-trophic models. By analysing the linearised spectrum of the reaction–diffusion operator, we derive parameter-space criteria that determine when spatially homogeneous equilibria become unstable due to differential diffusion rates, leading to spontaneous pattern formation.

The second part investigates the global existence of solutions for systems with full (including cross-diffusion) matrices and polynomial growth nonlinearities. Using the method of invariant regions combined with Lyapunov-type functionals, we prove that, under appropriate structural assumptions on the reaction kinetics and diffusion coefficients, solutions remain uniformly bounded for all time and satisfy suitable a priori estimates. The results are valid for a wide class of boundary conditions and do not require the diffusion matrix to be triangular or diagonal.

The theoretical framework is illustrated with applications to ecological and biological models, highlighting the interplay between cross-diffusion, stability thresholds, and long-time behaviour. The thesis thus provides both a refined instability analysis for three-component systems and a robust existence theory that extends previous results in the literature.

Keywords: Reaction–diffusion systems, cross-diffusion, Turing instability, predator–prey, global existence, invariant region, Lyapunov functional.

Résumé

Cette thèse porte sur l'analyse mathématique des systèmes de réaction–diffusion à trois composantes, en mettant l'accent sur leur nécessité pour modéliser des phénomènes complexes en écologie, épidémiologie et biologie du développement. Ces systèmes offrent un cadre plus réaliste que les modèles classiques à deux composantes, permettant de décrire des interactions multi-espèces complexes, des processus de diffusion croisée et des dynamiques spatialement étendues.

La première partie établit des conditions algébriques explicites pour l'apparition d'une instabilité induite par la diffusion (structures de Turing) dans des modèles tri-trophiques. En analysant le spectre linéarisé de l'opérateur de réaction–diffusion, nous obtenons des critères sur l'espace des paramètres qui déterminent quand des équilibres spatialement homogènes deviennent instables sous l'effet de taux de diffusion différentiels, conduisant à la formation spontanée de motifs.

La seconde partie étudie l'existence globale de solutions pour des systèmes à matrices complètes (incluant la diffusion croisée) et à non-linéarités à croissance polynomiale. En utilisant la méthode des régions invariantes combinée à des fonctionnelles de type Lyapunov, nous démontrons que, sous des hypothèses structurelles adéquates sur les cinétiques de réaction et les coefficients de diffusion, les solutions restent uniformément bornées pour tout temps et vérifient des estimations a priori appropriées. Les résultats sont valables pour une large classe de conditions aux limites et n'exigent pas que la matrice de diffusion soit triangulaire ou diagonale.

Le cadre théorique est illustré par des applications à des modèles écologiques et biologiques, mettant en évidence l'interaction entre diffusion croisée, seuils de stabilité et comportement asymptotique. La thèse fournit ainsi à la fois une analyse fine de l'instabilité pour les systèmes à trois composantes et une théorie d'existence robuste qui étend les résultats antérieurs de la littérature.

Mots-clés : Systèmes de réaction–diffusion, diffusion croisée, instabilité de Turing, proie–prédateur, existence globale, région invariante, fonctionnelle de Lyapunov.

ملخص

تتناول هذه الأطروحة التحليل الرياضي لأنظمة التفاعل الانتشار ثلاثية المكونات، مع التركيز على ضرورتها في نمذجة الظواهر المعقدة في البيئة وعلم الأوبئة وعلم الأحياء التطوري. توفر هذه الأنظمة إطاراً أكثر واقعية من النماذج الكلاسيكية ثنائية المكونات، حيث تلتقط تفاعلات متعددة الأنواع المعقدة وعمليات الانتشار المتقاطع والديناميكيات المكانية الممتدة.

يُحدد الجزء الأول شروطاً جبرية صريحة لنشوء عدم الاستقرار الناتج عن الانتشار (أنماط تورينج) في النماذج ثلاثية التغذية. من خلال تحليل الطيف الخطي لمؤثر التفاعل الانتشار، نشق معايير في فضاء المعاملات تحدد متى تصبح التوازنات المتجانسة مكانياً غير مستقرة بسبب معدلات انتشار تفاضلية، مما يؤدي إلى تشكل تلقائي لأنماط مكانية.

يدرس الجزء الثاني الوجود الشامل للحلول لأنظمة ذات مصفوفات انتشار كاملة (تشمل الانتشار المتقاطع) ولا خطيات ذات نمو متعدد الحدود. باستخدام طريقة المناطق الثابتة المقرونة بمعاملات من نوع ليابونوف، نثبت أنه تحت افتراضات بنيوية مناسبة على حركات التفاعل ومعاملات الانتشار تبقى الحلول موحدة الحدود لكل الأزمنة وتحقق تقديرات أولية مناسبة. النتائج صالحة لمجموعة واسعة من الشروط الحدودية ولا تشترط أن تكون مصفوفة الانتشار مثلثية أو قطرية.

يوضح الإطار النظري بتطبيقات على نماذج بينية وبيولوجية، مسلطاً الضوء على التداخل بين الانتشار المتقاطع وعتبات الاستقرار والسلوك الزمني الطويل. تقدم الأطروحة بذلك تحليلاً دقيقاً لعدم الاستقرار في الأنظمة ثلاثية المكونات ونظرية وجود قوية تمتد على النتائج السابقة في الأدبيات.

الكلمات المفتاحية: أنظمة التفاعل-الانتشار، الانتشار المتقاطع، عدم استقرار تورينج، مفترس-فريسة، الوجود الشامل، المنطقة الثابتة، دالة ليابونوف