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Theme

**On the Robustness of Positive Linear Systems in Infinite  
Dimension**

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# ABSTRACT

In the present research, it has been demonstrated that an explicit robust stability bound is generalized from the finite dimensional case to the infinite dimensional case. The systems that were investigated are known as Positive Linear Time-Delay Differential Systems (PLTDDS (s)), and they are subject to time-varying structural perturbations in Banach lattices. The Semigroup Method, the Principle of Comparison, and the Perron-Frobenius Theorem are all utilized in the current study. Furthermore, the coincidence of complex, real, and positive stability radii for a variety of particular cases is also demonstrated, and an explicit formula for the radius is derived. This is in addition to the radius that is described in detail.

**Keywords:** Time varying delay differential equations, robustness on stability, delay positive systems in Banach lattices, time varying perturbation, exponential stability, stability radius.

# RESUME

Dans cette recherche, il a été démontré qu'une borne de stabilité robuste et explicite est généralisée du cas de dimension finie au cas de dimension infinie. Les systèmes étudiés sont appelés systèmes différentiels linéaire, positif et à retard dans le temps (PLTDDS (s)) et sont sujets à des perturbations structurelles variables dans le temps dans les treilles de Banach. Le principe de comparaison, la méthode des semi-groupes et le théorème de Perron-Frobenius sont utilisés dans cette étude. De plus, la coïncidence des rayons de stabilité complexes, réels et positifs pour des cas particuliers est démontrée, et une formule explicite pour le rayon est dérivée.

**Mots-clés :** Équations différentielles à retard variable dans le temps, robustesse en stabilité, systèmes différentiels positifs à retard dans les treilles de Banach, perturbation variable dans le temps, stabilité exponentielle, rayon de stabilité.

## ملخص

في هذا البحث، تم إثبات أن حد الاستقرار القوي الصريح يُعمم من الحالة محدودة الأبعاد إلى الحالة غير محدودة الأبعاد. تُعرف الأنظمة التي تمت دراستها باسم أنظمة التفاضل الخطي الموجب للتأخير الزمني وهي عرضة لاضطرابات هيكلية متغيرة مع الزمن في شبكات باناخ. تم استخدام مبدأ المقارنة، وطريقة شبه المجموعات، ونظرية بيرون-فروبينيوس في الدراسة الحالية. علاوة على ذلك، تم أيضاً إثبات تطابق أنصاف أقطار الاستقرار المركبة والحقيقية والموجبة لبعض الحالات الخاصة، وتم إيجاد صيغة صريحة لنصف قطر الاستقرار.

**الكلمات المفتاحية:** معادلات التفاضل المتغيرة مع الزمن، المتانة على الاستقرار، أنظمة التأخير الموجبة في شبكات باناخ، الاضطراب المتغير مع الزمن، الاستقرار الأسّي، نصف قطر الاستقرار.

# ABBREVIATIONS

|          |                                                 |
|----------|-------------------------------------------------|
| (TDDS)   | Time-Delay Differential System.                 |
| (PLTDDS) | Positive linear Time-Delay Differential System. |
| (DDE)    | Delay Differential equation.                    |
| (PDE)    | Partial Differential equation.                  |
| (ODE)    | Ordinary Differential equation.                 |

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# GENERAL INTRODUCTION

Many real-world biological and physical phenomena are not solely determined by their instantaneous conditions; rather, they bear the enduring influence of their past states. This crucial characteristic, recognized as a time delay, transcends purely theoretical considerations. It constitutes a significant and frequently appreciated determinant contributing to system instability, often culminating in sub-optimal control and degraded system performance. When a system exhibits this profound reliance on historical information, it is precisely characterized as a Time-Delay Differential System (TDDS). These systems offer a more sophisticated and vertical representation of dynamic processes, acknowledging that effects are not always instantaneous but may manifest following a discernible temporal delay [34].

The foundational theories underpinning (TDDS) were rigorously established in the 1950s and 1960s. This period witnessed seminal investigations centered upon demonstrating the existence and uniqueness of solutions to these complex differential equations, alongside conducting rigorous analyses of the stability of equilibrium (trivial) solutions. These early contributions constituted the foundational framework for all subsequent analysis, design, and control efforts concerning time-delay systems. The persistent relevance and ubiquitous presence of (TDDS) have since engendered considerable interest among a substantial number of researchers across diverse fields, thereby establishing it as a dynamic, rapidly expanding, and exceptionally significant domain of scientific inquiry [34, 37].

The broad applicability of systems that rely on past states was initially recognized and emphasized by insightful researchers such as Bellman and Danskin. They were instrumental in articulating the profound implications and indispensable nature of incorporating time delays across a diverse array of disciplines, particularly within fields such as biology and economics. With advancements in technological capabilities, especially in the capacity for intervention and modification in the design and control of real-world systems, the purview of (TDDS) applications continues to expand at an accelerating rate [22].

To begin with in the biological domain, time delays are fundamentally integral to the comprehension of a myriad of phenomena. This includes the intricate com-

plexities inherent in population growth dynamics, wherein natality rates may be contingent upon antecedent environmental conditions, or the complex processes of cellular metabolism, where enzymatic reactions can exhibit intrinsic temporal delays. Furthermore, the accurate comprehension of the complex propagation of infectious diseases frequently necessitates accounting for incubation periods and the delayed manifestation of symptomatic responses.

In chemistry, time delays assume a critical role in precisely modeling the reaction rates of chemical compounds, particularly in multi-step reaction mechanisms where the formation of intermediate products is characterized by a temporal delay. Moreover, their significance extends to comprehending the efficient transport of fluids through conduits or within biological systems, where fluid dynamic behaviors can be significantly influenced by prior states of pressure or viscosity.

Economists extensively leverage (TDDS) to elucidate the dynamic interplay within financial markets, where investor responses frequently exhibit a temporal delay relative to economic news, and to model the developmental courses of economies, wherein the ramifications of policy interventions may not be immediately discernible.

Furthermore, within the domain of engineering, these systems are indispensable for the precise and stable control of robotic systems, facilitating fluid and accurate movements notwithstanding intrinsic motor or sensor latencies. They are pivotal for the efficient and dependable operation of power systems, given that fluctuations in load and adjustments in generation capacity are characterized by inherent time constants.

Finally, they are of paramount importance in the sophisticated design of communication networks, where signal propagation delays exert a substantial influence on network performance and overall stability. Readers seeking more comprehensive insights into specific applications and detailed theoretical expositions are encouraged to consult specialized scholarly references and advanced textbooks dedicated to this subject [29, 45].

The robust control of Time-Delay Differential Systems (TDDS(s)) has constituted an exceptionally active and prolific field of research over the past five decades. This enduring academic interest arises from the intrinsic complications associated with time delays, which frequently result in the onset of instability and a deterioration of operational efficacy within dynamic systems [32]. Consequently, this domain has branched into numerous specialized areas of research, each addressing distinct challenges posed by the delayed nature of these systems [34].

Among these specialized branches, stability analysis stands as a foundational pillar. Its primary objective is to delineate the conditions under which a (TDDS) remains bounded or converges to an equilibrium state, a critical endeavor given that time delays are a well-known source of oscillatory behavior and instability.

Building upon these analytical insights, stabilization design focuses on developing methodologies to imbue inherently unstable (TDDS) with stable characteristics, typically through the synthesis of appropriate feedback control laws. This often involves intricate techniques to compensate for the delayed information flow [34]. Further advancements have led to the development of sophisticated paradigms such as  $H_\infty$  control [27]. This approach demonstrates particular utility for (TDDS(s)), owing to its intrinsic robustness against uncertainties and external disturbances. These factors are frequently encountered in practical applications, wherein temporal delays may be imprecise or subject to variability.  $H_\infty$  control seeks to minimize the worst-case effect of these disturbances on system performance, ensuring stable operation even under adverse conditions [32]. Concurrently, stochastic control addresses (TDDS) where system dynamics are influenced by random noise, uncertain parameters, or stochastically varying delays. This branch endeavors to ensure stability and achieve desired performance from a probabilistic perspective, by designing controllers that operate optimally despite inherent system randomness. Notwithstanding the diversity and specialization across these branches, stability unequivocally remains the fundamental prerequisite for any meaningful control design or performance guarantee in (TDDS). It is commonly known that an unstable system, by definition, exhibits unbounded behavior, rendering effective control or performance optimization unattainable. Therefore, significant developments in the study of (TDDS) have consistently initiated with a rigorous examination of stability, providing the essential analytical framework upon which all subsequent control strategies and design methodologies are constructed. This underscores the paramount importance of understanding and ensuring stability as the bedrock of robust control for time-delay systems [34].

Concerning the stability analysis, it is believed by many researchers that perturbations in delay systems involve small changes that impact their behaviour, stability and lead to degradation of performance, this shows the extreme importance of study of robustness in stability, that is, the ability of the system to maintain its stability properties despite the presence of external disturbances. A measure for the stability robustness of a stable system is the stability radius which was introduced in 1986 by Hinrichsen and Pritchard for linear systems, it is defined as the smallest real or complex disturbance, in norm, that destabilizes the system [25, 26]. Depending on the type of disturbance, whether complex, real or positive, there are complex, real or positive stability radii respectively, which are in general different. Over the years, this field has witnessed a great development, as it has expanded to include a large number of types of systems disturbances [4, 15, 16, 17, 23, 24, 35, 36, 37, 39, 40, 41].

Some researchers proposed utilising the concept of Banach lattices which is a mathematical framework that enables an analysis of different types of functions

and spaces. Accordingly, in the context of stability analysis, the use of Banach lattices allows one to study the behaviour of a system under different types of perturbations which is particularly important in applications where certainties or external disturbances play a significant role. Therefore, this paper aims to investigate the robust stability of positive linear time delay systems under time varying perturbations using the stability radius approach as a tool of analysis.

It is crucial to mention that the generalization to infinite dimensions is extremely complicated, thus, considerable amount of knowledge about the subject was exploited in order to achieve two goals. The first is to obtain a generalization of a result which has been found by P. H. A. Ngoc and C.T. Tinh in [39] concerning the robust stability of positive delay differential systems under time varying perturbations, from the finite dimension space to the infinite dimension, using semigroup method, the Perron-Frobenius theorem and the principle of comparison. The second is to determinate an upper bound of the complex stability radius. Moreover, we obtain coincidence in the radius of stability, complex, real or positive, in some special cases. It should be noted that the work of S. Murakami and P. H. A. Ngoc in infinite dimension space in [35] was the motivation of the the current research.

The organization of this thesis is as follows, in chapter one, and in order to make this work more specific, we give the most important concepts and theories related to the Banach lattice framework, as well as some results about positive operators, semigroup of bounded linear operators, Metzler operators and their properties. Then, in chapter two, we present the system under study and some results about the asymptotic behavior of the solutions. In chapter three, we apply time varying structured perturbations to this system and we study the robustness on stability. In chapter four, we present the results found about an upper bound of the complex stability radius and the coincidence of complex, real and positive stability radii in some special cases.

# Chapter 1

## PRELIMINARIES

The purpose of this piece of preliminary work is to provide a concise summary of the ideas and findings pertaining to Banach lattices, positive operators, Semi-group of bounded linear operators and Metzler operators. In order to make this work self-contained, we have divided this section into four subsections. Each and every one of these outcomes is essential for later usage.

### 1.1 BANACH LATTICES

To begin, we will provide some fundamental information on Banach lattices; for more information, the reader may consult [2, 17, 31, 33, 44, 48] for more details.

#### 1.1.1 Historical Development and Mathematical Significance of Banach Lattices

The study of Banach lattices represents a profound synthesis of two fundamental branches of mathematics: Functional Analysis and Order Theory [2, 44]. This convergence is often considered to offer a solid framework for examining spaces that feature both a metric structure, which facilitates the measurement of distances and convergence, and an order structure, which permits the comparison of elements. The historical evolution of these disciplines suggests a logical advancement, moving from the analysis of function spaces to the abstraction of their shared characteristics, and culminating in the formal notion of Banach lattices.

In the early stages of functional analysis, a notable observation was made in 1948: "every known instance of a Banach space was also a vector lattice". This observation was not only an interesting fact but a strong indicator that the order

structure was as basic as the metric and algebraic structures for numerous analytical issues. This implicit acknowledgment of the interaction between metric and order features in classical function spaces established the foundation for the formalization of Banach lattices. The notion of a Banach lattice arose not merely as a Banach space equipped with an order, but as a space where the order and norm are closely aligned [12]. The compatibility is argued to be essential, enabling the robust tools of functional analysis to engage meaningfully with order-theoretic concepts, thereby suggesting a more extensive range of theorems and wider applications than either theory could do independently.

In the 1930s, the theory of vector lattices, which is also referred to as Riesz spaces, emerged independently [12]. Frigyes Riesz introduced the foundational notion of a vector lattice in his 1928 address at the International Mathematical Congress [43]. Concurrently, Leonid Kantorovich defined K-spaces in 1935, which are vector lattices possessing Dedekind completeness and made significant contributions to ordered vector spaces and functional analysis, later receiving the Nobel Prize in Economics for his work [1]. Hans Freudenthal also played a crucial role, proving the seminal Freudenthal Spectral Theorem in 1936. Garrett Birkhoff further advanced lattice theory with his influential 1940 monograph *Lattice Theory*.

The formal integration of these ordered structures with the completeness property of Banach spaces led to the modern concept of Banach lattices, with major advancements occurring around the 1970s [12]. Key mathematicians during this period included H.H. Schaefer, whose 1974 book: *Banach Lattices and Positive Operators*, was highly influential, P. Meyer-Nieberg, J. Lindenstrauss, L. Tzafriri, and W. Johnson. The field's evolution from distinct "schools" (e.g., Russian, Japanese, German, Dutch) to a unified positivity community underscores the growing recognition of this powerful mathematical synthesis [30].

Banach lattices are used when you need to analyze not just the "size" of elements in a space (using the norm) but also their relative position with respect to an order relation. This is particularly useful when dealing with spaces of functions or operators where such order relationships are important. The theory of Banach lattices sits at the intersection of functional analysis and order theory. To understand these structures, one must first grasp how order relations can be combined with the algebraic structure of vector spaces.

### 1.1.2 Vector Lattices Over the Real Field

**Definition 1.1.** [44] (*Order*)

*Let  $X$  be an arbitrary set,  $X$  is ordered if it is endowed with a binary relation, usually denoted by  $\leq$ , which is supposed to be **reflexive**, **anti-symmetric** and **transitive**, that is,*

1.  $x \leq x$ , for all  $x \in X$ ;
2.  $x \leq y$  and  $y \leq x$  implies  $x = y$  for all  $x, y \in X$ ;
3.  $x \leq y$  and  $y \leq z$  implies  $x \leq z$  for all  $x, y, z \in X$ .

The pair  $(X, \leq)$  is called an **ordered set**. If  $x, y \in X$ , we write  $y \geq x$  to mean  $x \leq y$ , and  $x < y$  to express that  $x \leq y$  and  $x \neq y$ .

**Definition 1.2.** [44] (**Majorized, Minorized Subset, Maximal, Minimal Element**)

Let  $Y$  a subset of  $X$ .

1. The subset  $Y$  is called **majorized** if there exists  $x_0 \in X$  such that  $y \leq x_0$  for all  $y \in Y$ ,  $x_0$  is called a **majorant** or **upper bound** of  $Y$  in  $X$ .
2. An element  $x_0 \in Y$  is said to be **maximal** if there is no  $y \in Y$  such that  $y \neq x_0$  for which  $x_0 \leq y$ .
3. The subset  $Y$  is called **minorized** if there exists  $x_0 \in X$   $x_0 \leq y$  for all  $y \in Y$ ;  $x_0$  is called **minorant** or **lower bound** of  $Y$  in  $X$ .
4. An element  $x_0 \in Y$  is said to be **minimal** if there is no  $y \in Y$  such that  $y \neq x_0$  for which  $y \leq x_0$ .
5. By the order interval  $[x, y]$ , where  $x, y \in X$ , we understand the set

$$[x, y] := \{z \in X \text{ satisfying } x \leq z \leq y\};$$

6. A subset  $Y \subset X$  contained in some order interval of  $X$ , is called **order bounded**.
7. The set  $Y$  is called **totally ordered** if for each pair  $(x, y) \in Y$ , at least one of the relations  $x \leq y$  and  $y \leq x$  holds.

**Definition 1.3.** [44] (**Supremum and Infimum**)

If  $Y$  is a majorized subset of  $X$  and if there exists a majorant of  $Y$  that minorizes all majorants of  $Y$  (in  $X$ ), such an element is unique, called the **supremum** or **least upper bound** of  $Y$ , and denoted by  $\sup Y$ ; similarly for  $\inf Y$  (the **infimum** or **greatest lower bound** of  $Y$ ). And for a two point set  $\{x, y\}$  we write

$x \wedge y$  or  $\inf\{x, y\}$  to denote its infimum.

$x \vee y$  or  $\sup\{x, y\}$  to denote its supremum.

**Remark 1.1.** *The supremum and the infimum of a set need not exist.*

**Definition 1.4.** [44] (**Lattice**)

An ordered set  $(L, \leq)$  is called a **lattice** if for each pair  $(x, y) \in L \times L$ , the elements  $x \wedge y := \inf\{x, y\}$  and  $x \vee y := \sup\{x, y\}$  exists in  $L$ .

**Remark 1.2.** [48]

Let  $(L, \leq)$  lattice, for any  $x, y, w \in L$ . If  $x \leq w$  and  $y \leq w$  implies  $x \vee y \leq w$ . Hence, since both  $x \wedge y$  and  $x \wedge z$  are majorized by  $x \wedge (y \vee z)$ , it follows that

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z).$$

**Definition 1.5.** [48] (**Distributive Lattice**)

The lattice  $(L, \leq)$  is called **distributive** if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \text{ for all } x, y, z \in L$$

**Theorem 1.1.** [48]

The lattice  $(L, \leq)$  is distributive if and only if

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \text{ for all } x, y, z \in L$$

**Remark 1.3.** [44]

If  $(L, \leq)$  is a lattice, the mapping  $(x, y) \mapsto x \vee y$  and  $(x, y) \mapsto x \wedge y$  are usually called the **lattice operations**. As laws of composition they are idempotent, associative, commutative, and satisfy  $x \wedge (x \vee y) = x$  as well as  $x \vee (x \wedge y) = x$ ; conversely, if a non-void set  $L$  is endowed with two laws of composition having these properties, then  $x \leq y \Leftrightarrow x \vee y = y$  defines an ordering under which  $L$  is a lattice in the sense of definition 1.2.

### 1.1.3 Normed Vector Lattices

**Definition 1.6.** [44] (**Ordered Vector Space**)

A vector space  $X$  over  $\mathbb{R}$  equipped with an order relation  $\leq$  is called **ordered vector space** if it is compatible with its vector structure in the sense that

$$\begin{cases} x \leq y \Rightarrow x + z \leq y + z \text{ for all } x, y, z \in X; \\ x \leq y \Rightarrow \alpha x \leq \alpha y \text{ for all } x, y \in X, \alpha \in \mathbb{R}_+. \end{cases} \quad (1.1)$$

**Definition 1.7.** [44] (**Vector Lattice**)

A **vector lattice** is an ordered vector space  $X$  such that

$x \wedge y := \inf\{x, y\}$  and  $x \vee y := \sup\{x, y\}$  exists in  $X$ , for all  $x, y \in X$ .

Vector lattices are also called **Riesz spaces** or **linear lattices**.

Operations  $\vee$  and  $\wedge$  in vector lattices have many useful properties that are similar to how max and min work for numbers, but in a more general setting

**Theorem 1.2.** [2] (**Properties of Supremum and Infimum in Vector Lattice**)

Let  $X$  a vector lattice, For any  $x, y, z \in X$  the following assertions hold true.

1.  $(x \vee y) = -[(-x) \wedge (-y)]$  and  $(x \wedge y) = -[(-x) \vee (-y)]$ ;
2.  $x + y = (x \vee y) + (x \wedge y)$ ;
3.  $(x + y) \vee (x + z) = x + (y \vee z)$  and  $(x + y) \wedge (x + z) = x + (y \wedge z)$ ;
4.  $(\alpha x) \vee (\alpha y) = \alpha(x \vee y)$  and  $(\alpha x) \wedge (\alpha y) = \alpha(x \wedge y)$ , for all  $\alpha \geq 0$ .

**Corollary 1.1.** [2]

If  $A$  is a nonempty subset of a vector lattice  $X$  for which  $\sup A$  exists, then

1. The infimum of the set  $-A := \{-a : a \in A\}$  exists and

$$\inf(-A) = -\sup A.$$

2. For every  $x \in X$ , the supremum of the set  $x + A := \{x + a : a \in A\}$  exists and

$$\sup(x + A) = x + \sup A.$$

3. For every  $\alpha \geq 0$ , the supremum of the set  $\alpha A := \{\alpha a : a \in A\}$  exists and

$$\sup(\alpha A) = \alpha \sup A.$$

**Lemma 1.1.** [2] (**Identities between Positive Elements**)

If  $x, x_1, x_2, \dots, x_n$  are positive elements in a vector lattice  $X$ , then

$$x \wedge (x_1 + x_2 + \dots + x_n) \leq x \wedge x_1 + x \wedge x_1 + x \wedge x_2 + \dots + x \wedge x_n.$$

**Theorem 1.3.** [2] (**Decomposition Property of Vector Lattices**)

If

$$|x| \leq |y_1 + y_2 + \dots + y_n|$$

holds in a vector lattice  $X$ , then there exist  $x_1, x_2, \dots, x_n$  satisfying

$$x = x_1 + x_2 + \dots + x_n \text{ and } |x_i| \leq |y_i| \text{ for every } i = 1, \dots, n.$$

Moreover, if  $x$  is positive, then the  $x_i$ , for every  $i = 1, \dots, n$ , also can be chosen to be positive.

A key part of an ordered vector space is its positive cone.

**Definition 1.8.** [44] (**Positive Cone**)

If  $(X, \leq)$  is an ordered vector space, the subset of all elements  $x$  in  $X$  such that  $x \geq 0$  is called the **positive cone** of  $X$ , and noted by  $X^+$ .

$$X^+ := \{x \in X / x \geq 0\}.$$

The set  $X^+$  fulfills the following geometric properties

$$\begin{cases} X^+ + X^+ \subseteq X^+; \\ \mathbb{R}_+ X^+ \subseteq X^+; \\ X^+ \cap (-X^+) = \{0\}. \end{cases} \quad (1.2)$$

In particular,  $X^+$  is a convex cone in  $X$ .

Conversely, The order itself can be defined using this cone (i.e) every subset  $C$  of  $X$  satisfying (1.2) determines an order relation on  $X$  by

$$x \leq y : \Leftrightarrow y - x \in C \text{ such that (1.1) holds.}$$

Moreover, if  $X$  is a vector lattice, the set  $X^+$  is generating, that is, it satisfies

$$X = X^+ - X^+.$$

**Definition 1.9.** [44] (**Positive (Negative) Part, Absolute Value**)

For an element  $x$  of a vector lattice  $X$ , the positive part, the negative part and the absolute value of  $x$ , are defined respectively by

$$x^+ := x \vee 0, \quad x^- := -x \vee 0, \quad |x| = x \vee -x.$$

$|x|$  is called also the **modulus** of  $x$ .

The elements  $x^+$ ,  $x^-$ , and  $|x|$  satisfy the following important identities.

**Proposition 1.1.** [2] (**Identities between  $x^+$ ,  $x^-$ , and  $|x|$** )

Let  $X$  a vector lattice, and  $x$  an arbitrary element in  $X$ , then

$$1. \quad x = x^+ - x^- ;$$

2.  $|x| = x^+ + x^-$  ;

3.  $x^+ \wedge x^- = 0$  ;

Moreover, the decomposition in (1) satisfies the following minimality and uniqueness properties.

(a) If  $x = y - z$  with  $y, z \in X^+$ , then  $y \geq x^+$  and  $z \geq x^-$ .

(b) If  $x = y - z$  with  $y \vee z = 0$ , then  $y = x^+$  and  $z = x^-$ .

**Theorem 1.4.** [2] (**Lattice Identities in Vector Lattices**)

For arbitrary elements  $x, y$  of a vector lattice  $X$ , one has the relations:

1.  $x = (x - y)^+ + x \vee y$  ;

2.  $x \vee y = \frac{1}{2}(x + y + |x - y|)$  and  $x \wedge y = \frac{1}{2}(x + y - |x - y|)$  ;

3.  $|x - y| = x \vee y - x \wedge y$ ;

4.  $|x| \vee |y| = \frac{1}{2}(|x + y| + |x - y|)$  ;

5.  $|x| \vee |y| = \frac{1}{2}(|x + y| - |x - y|)$ ;

6.  $|x + y| \vee |x - y| = ||x| - |y||$ ;

7.  $|x + y| \wedge |x - y| = |x| + |y|$ .

**Definition 1.10.** [44] (**Orthogonality**)

1. Let  $X$  a vector lattice, two elements  $x, y \in X$  are called **orthogonal** (**disjoint**, or **lattice disjoint**) if  $|x| \wedge |y| = 0$ ; this is denoted by  $x \perp y$ .

2. Two subsets  $A, B$  of  $X$  are called **orthogonal** (**lattice disjoint**) if  $x \perp y$  for all pair  $(x, y) \in A \times B$ .

3. For  $A \subset X$ ,  $A^\perp$  denotes the set of all  $x \in X$  disjoint from each  $y \in A$ .

**Corollary 1.2.** [44]

In every vector lattice  $X$ , for arbitrary elements  $x, y \in X$ .

1. The relation  $x \leq y \Leftrightarrow (x^+ \leq y^+ \text{ and } y^- \leq x^-)$  ;

2.  $x \perp y \Leftrightarrow |x| \vee |y| = |x| + |y|$ .

Moreover,  $x \perp y \Rightarrow ((x + y)^+ = x^+ + y^+ \text{ and } |x + y| = |x| + |y|)$ .

**Theorem 1.5.** [2] (*Infinite Distributive Law*)

Let  $A$  is a nonempty subset of a vector lattice  $X$ . If  $\sup A$  exists, then for every  $x \in X$  the supremum of the set  $x \wedge A := \{x \wedge a : a \in A\}$  exists and

$$\sup (x \wedge A) = x \wedge \sup A.$$

Similarly, if  $\inf A$  exists, then for every  $x \in X$  the infimum of the set  $x \vee A := \{x \vee a : a \in A\}$  exists and

$$\inf (x \vee A) = x \vee \inf A.$$

**Theorem 1.6.** [2] (*Triangle, Birkhoff's Inequalities*)

For arbitrary elements  $x, y$ , and  $z$  in a vector lattice, we have the following inequalities.

1.  $||x| - |y|| \leq |x + y| \leq |x| + |y|$  (Triangle inequality).
2.  $|x \vee z - y \vee z| \leq |x - y|$  and  $|x \wedge z - y \wedge z| \leq |x - y|$  (Birkhoff's inequalities).

**Definition 1.11.** [48] (*Archimedean Riesz space*)

The Riesz space (or the lattice)  $X$  is said to be **Archimedean** if

$$\inf \{n^{-1}x : n = 1, 2, \dots\} = 0$$

holds for every  $x \in X^+$ .

**Theorem 1.7.** [48] (*Properties of Archimedean Riesz space*)

Let  $X$  be a Riesz space (or a lattice)

1. The space  $X$  is Archimedean if and only if it is true for every  $x \in X^+$  that

$$\inf \{\epsilon_n x : n = 1, 2, \dots\} = 0,$$

holds for any sequence  $\{\epsilon_n : n = 1, 2, \dots\} = 0$  of non-negative real numbers satisfying  $\epsilon_n \rightarrow 0$ .

2. The space  $X$  is Archimedean if and only if, given  $x$  and  $y$  in  $X^+$  such that  $0 \leq ny \leq x$  for  $n = 1, 2, \dots$ , it follows that  $y = 0$ . In other words, for any  $y \geq 0, y \neq 0$ , the sequence  $\{ny : n = 1, 2, \dots\}$  is not bounded above.
3. If  $X$  is Archimedean, then every Riesz subspace of  $X$  is also Archimedean.

The structures discussed so far are primarily algebraic and order-theoretic. To bring in the tools of functional analysis, we need to introduce a norm that interacts appropriately with the order structure.

**Definition 1.12.** [44] (*Normed Vector Lattice*)

1. A norm  $\|\cdot\|$  on a vector lattice (or on a Riesz space)  $X$  is called a **lattice norm** (or **Riesz norm**) if it satisfies the following compatibility property (lattice norm property), that is,

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|, \text{ for every } x, y \in X, \quad (1.3)$$

2. A **normed vector lattice** (or a **normed Riesz space**)  $X$  is vector lattice with a lattice norm (or a Riesz space equipped with a lattice norm).
3. The lattice norm property implies  $\| |x| \| = \|x\|$  for every  $x \in X$ .

**Proposition 1.2.** [44] (*Properties of Normed Vector Lattices*)

Let  $X$  normed vector lattice.

1. The mappings  $x \mapsto x^+$ ,  $x \mapsto x^-$ ,  $x \mapsto |x|$  and  $(x, y) \mapsto x \vee y$  and  $(x, y) \mapsto x \wedge y$  are uniformly continuous from  $X$  (respectively, from  $X \times X$ ) into  $X$ .
2. The positive cone  $X^+$  is closed.

### 1.1.4 Banach Lattices

**Definition 1.13.** [44] (*Banach Lattice*)

If  $X$  is a normed vector lattice and norm complete with respect to  $\|\cdot\|$ , then  $X$  is called a **real Banach lattice**, or only **Banach lattice**.

A Banach Lattice is a normed Riesz space that is complete with respect to its lattice norm. In other words, it is a Banach space that is also a Riesz space, where the norm and the order are compatible via the lattice norm property.

Now, we list some further properties of Banach lattices.

**Proposition 1.3.** [8] (*Properties of Banach Lattices*)

For a Banach lattice  $X$  the following hold.

1. The lattice operations are continuous.
2. The positive cone  $X^+$  is closed.

3. The order intervals are closed and bounded.

To summarize, a Banach lattice  $(X, \leq, \|\cdot\|)$  possesses:

1. A real vector space structure.
2. A partial order  $\leq$  making it an ordered vector space.
3. A lattice structure derived from the order (existence of  $x \vee y$  and  $x \wedge y$ ).
4. A complete norm  $\|\cdot\|$  (making it a Banach space).
5. Compatibility between the norm and the order:

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|, \text{ for every } x, y \in X.$$

Banach lattices thus harmoniously integrate algebraic, order-theoretic, and topological structures. This rich structure makes them powerful tools in many areas of analysis, including operator theory, measure theory, probability, and even mathematical economics. The systematic study of these spaces was proposed by F. Riesz in 1928 and developed significantly alongside the general theory of Banach spaces, pioneered by Stefan Banach. Key figures in the development of Riesz space and Banach lattice theory include L.V. Kantorovich, H. Freudenthal, G. Birkhoff, H. Nakano, T. Ogasawara, K. Yosida, M.H. Stone, W.A.J. Luxemburg, A.C. Zaanen, H.H. Schaefer, P. Meyer-Nieberg, and C.D. Aliprantis, O. Burkinshaw, whose books form the foundation of the subject.

### 1.1.5 Complexification

Ordered vector spaces and vector lattices over the complex field have been considered by various authors (cf. Schaefer [1966 a]); Lotz [1968] seems to be the first who recognized the possibility of defining complex vector lattices. Further recent references include Ito [1967], Luxemburg-Zaanen [1971a], and Mittelmeyer-Wolff [1974]. In the latter paper, the concept of a vector lattice (over the real or complex field) is axiomatized. Since it is often necessary to consider complex vector spaces, we extend the notion of Banach lattice to the complex case.

**Definition 1.14.** [44] (*Relatively Uniformly Complete Vector Lattice*)

Let  $X$  be a vector lattice. Then  $X$  is said to be **relatively uniformly complete** if, for every sequence  $(\lambda_n)_{n \in \mathbb{N}} \in \mathbb{R}$  satisfying  $\sum_{n=1}^{\infty} |\lambda_n| < +\infty$  and for every  $x \in X$  and for every sequence  $(x_n)_{n \in \mathbb{N}} \in X$  it holds that

$$0 \leq x_n \leq \lambda_n x \Rightarrow \sup_{n \in \mathbb{N}} \left( \sum_{i=1}^n x_i \right) \in X.$$

Now, let  $X$  be a relatively uniformly complete vector lattice. The complexification  $X_{\mathbb{C}}$  of  $X$  is the additive group  $X \times X$  with scalar multiplication defined by  $(\alpha, \beta)(x, y) := (\alpha x - \beta y, \beta x + \alpha y)$  for  $(\alpha, \beta) = \alpha + i\beta \in \mathbb{C}$ ; identifying each  $x \in X$  with  $(x, 0) \in X \times X$ , each  $z \in X \times X$  is uniquely represented as  $z = x + iy$ , where  $x, y \in X$ . Thus  $X_{\mathbb{C}}$  is frequently written as  $X + iX$ . With the understanding that  $X_{\mathbb{C}}$  is the direct sum over  $\mathbb{R}$  of the real vector subspace  $X$  and  $iX$ ; in particular,  $x + iy \mapsto x$  and  $x + iy \mapsto y$  are  $\mathbb{R}$ -linear projections of  $X_{\mathbb{C}}$  onto  $X$ . The modulus of  $z = x + iy \in X_{\mathbb{C}}$  is defined by

$$|z| = \sup_{0 \leq \varphi \leq 2\pi} |(\cos \varphi)x + (\sin \varphi)y| \in X. \quad (1.4)$$

Moreover, the modulus function  $z \mapsto |z|$  of  $X_{\mathbb{C}}$  into  $X$  is homogeneous, that is, it satisfies the following relations

$$\begin{cases} |z| = 0 \text{ if and only if } z = 0, \\ |\alpha z| = |\alpha| |z| \text{ for all } z \in X_{\mathbb{C}}, \alpha \in \mathbb{C}, \\ |z_1 + z_2| \leq |z_1| + |z_2| \text{ for all } z_1, z_2 \in X_{\mathbb{C}}. \end{cases} \quad (1.5)$$

**Definition 1.15.** [44] (**Complex Vector Lattice**)

A complex vector lattice  $(X_{\mathbb{C}}, |\cdot|)$  is defined as the complexification of a relatively uniformly complete vector lattice endowed with the modulus function  $z \mapsto |z|$  given by (1.4).

**Proposition 1.4.** [44]

Let  $z, z_1, z_2 \in X_{\mathbb{C}}$  satisfy  $|z| \leq |z_1| + |z_2|$ . There exist elements  $u_1, u_2 \in X_{\mathbb{C}}$  such that

$$z = u_1 + u_2 \text{ and } |u_1| \leq |z_1|, |u_2| \leq |z_2|$$

**Definition 1.16.** [44] (**Normed Complex Vector Lattice**)

If  $X_{\mathbb{C}}$  is a complex vector lattice and  $x \mapsto \|x\|$  is a lattice norm on  $X$  satisfying the lattice norm property (1.3), then the modulus function (1.4) suggests the extension to  $X_{\mathbb{C}}$  of the given lattice norm by

$$\|z\| := \| |z| \|, \text{ for every } z \in X_{\mathbb{C}}. \quad (1.6)$$

**Definition 1.17.** [44] (**Complex Banach Lattice**)

The complexification  $X_{\mathbb{C}}$  of a Banach lattice  $X$ , endowed with the modulus function (1.4) and the norm (1.6) is called a **complex Banach lattice**.

### 1.1.6 Key Examples of Banach Lattices

**Example 1.1.** [2, 8, 33] ( $L_p$  **Spaces**)

The most prototypical examples of Banach lattices are spaces of measurable or continuous functions. In these spaces, the partial order is defined pointwise. Let  $(\Omega, \Sigma, \mu)$  be a measure space and let  $1 \leq p \leq \infty$ . The space  $L_p(\mu)$  is a Banach lattice under the standard ordering:

$$f \leq g \iff f(t) \leq g(t) \text{ for } \mu\text{-almost all } t \in \Omega.$$

The lattice operations are defined pointwise almost everywhere:

$$\begin{aligned} (f \vee g)(t) &= \max(f(t), g(t)), \\ (f \wedge g)(t) &= \min(f(t), g(t)). \end{aligned}$$

The norm depends on  $p$ :

- For  $1 \leq p < \infty$ ,  $\|f\|_p = \left( \int_{\Omega} |f(t)|^p d\mu(t) \right)^{1/p}$ .
- For  $p = \infty$ ,  $\|f\|_{\infty} = \text{ess sup}_{t \in \Omega} |f(t)|$ .

For  $p < \infty$ , the norm is order continuous, meaning that if  $f_n \downarrow 0$  in order, then  $\|f_n\| \rightarrow 0$ .

**Example 1.2.** [2, 33] ( $C(K)$  **Spaces**)

Let  $K$  be a compact Hausdorff space. The space  $C(K)$  of real-valued continuous functions on  $K$  is a Banach lattice under the supremum norm:

$$\|f\|_{\infty} = \sup_{t \in K} |f(t)|.$$

The order is strict pointwise ordering:  $f \leq g$  if  $f(t) \leq g(t)$  for all  $t \in K$ . Unlike  $L_p$  spaces (for  $p < \infty$ ), the norm in  $C(K)$  is generally **not** order continuous, but the space always contains an order unit (the constant function **1**).

**Definition 1.18.** [2] (**Interior of a Set**)

Let  $(X, \|\cdot\|)$  be a Banach space (or more generally, a topological space). The **interior** of a subset  $F \subseteq X$ , denoted by  $\text{Int}F$ , consists of all points  $x \in F$  for which there exists an open neighborhood contained entirely within  $F$ .

Formally,  $x \in \text{Int}F$  if there exists an  $\varepsilon > 0$  such that the open ball centered at  $x$ :

$$B(x, \varepsilon) = \{y \in X : \|x - y\| < \varepsilon\} \subseteq F.$$

If  $\text{Int}F \neq \emptyset$ , the set  $F$  is said to have a non-empty interior.

**Definition 1.19.** [2] **Interior Point of the Positive Cone**

Let  $X$  be a Banach lattice. An element  $x \in X^+$  is an **interior point of the positive cone** if  $x \in \text{Int}X^+$ . This means there exists an  $\varepsilon > 0$  such that for any  $y \in X$  with  $\|x - y\| < \varepsilon$ , we have  $x \geq 0$ .

The most significant property regarding the interior of the positive cone is its relationship with *order units*.

**Definition 1.20.** [2] **Order Unit**

An element  $x \in X^+$  is called an **order unit** if for every  $y \in X$ , there exists a real number  $\lambda > 0$  such that  $|y| \leq \lambda x$ .

**Theorem 1.8 (Equivalence Theorem).**

Let  $X$  be a Banach lattice. An element  $x \in X^+$  is an interior point of  $X^+$  if and only if  $x$  is an order unit of  $E$ .

$$x \in \text{Int}E_+ \iff x \text{ is an order unit.}$$

Next, we summarize some concepts and theorems about positive operators, Metzler operators, spectrum, resolvent and stability. We obtained this information from the following references [2, 5, 6, 9, 17, 33, 42, 44].

## 1.2 POSITIVE OPERATORS

### 1.2.1 Positive Operators: Foundation, Evolution and Significance Across Disciplines

Positive operators represent a fundamental concept spanning linear algebra, operator theory, and functional analysis [2]. Their importance seems to be underscored by their critical role in various branches of mathematics and physics, including quantum mechanics, differential equations, signal processing, optimization, and machine learning.

The set of positive bounded operators on a Banach space forms a mathematical structure known as a cone in the algebra of all bounded operators [2, 8]. The

widespread presence and use of positive operators in several scientific and engineering fields demonstrate their fundamental significance. This is not merely a specialized mathematical concept but a unifying principle that formalizes the notion of "non-negativity" or "order preservation" in various contexts. These contexts range from the expectation values of energy in quantum systems to the characterization of convex functions in optimization. Their inherent ability to define order and structure within abstract spaces is central to their broad utility across these varied domains.

The concept of positive operators first emerged in the early nineteenth century, closely tied to the study of integral operators and matrices with non-negative entries. However, a systematic investigation into positive operators, particularly their spectral properties, did not commence until the early 1960s. This later development was contingent upon the prior establishment of the theory of Riesz spaces (vector lattices), which provided the necessary theoretical groundwork. Frigyes Riesz made seminal contributions to functional analysis in the early 20th century, including pioneering work on linear operators and functionals. His efforts laid the foundational stones for modern functional analysis. Notably, Riesz's work in 1910 is often cited as the genesis of operator theory. His eponymous Riesz Representation Theorem established a profound correspondence between positive linear functionals and measures, further solidifying the analytical framework. Helmut H. Schaefer's 1974 monograph, "Banach Lattices and Positive Operators," stands as a landmark work entirely dedicated to the theory of positive operators and their spectral analysis, solidifying his reputation as a leading international scholar in this specialized field [49].

### 1.2.2 Space of Linear Operators

**Definition 1.21.** [2] (*Linear Operator*)

Let  $X$  and  $Y$  be real vector spaces. The map  $T : X \rightarrow Y$  is called **Linear operator** if and only if

$$\begin{aligned} T(x + y) &= T(x) + T(y) \text{ for all } x, y \in X. \\ T(\alpha x) &= \alpha T(x) \text{ for all } x \in X \text{ and } \alpha \in \mathbb{R}. \end{aligned}$$

As usual, the value  $T(x)$  will also be designated by  $Tx$ .

The set of all linear operators  $T : X \rightarrow Y$  is denoted by  $L(X, Y)$ .

This set itself be turned into vector space with these operations

$$\begin{aligned} (T + S)(x) &= T(x) + S(x) \text{ for all } T, S \in L(X, Y) \text{ and } x \in X. \\ (\alpha T)(x) &= \alpha(T(x)) \text{ for all } T \in L(X, Y), \alpha \in \mathbb{R} \text{ and } x \in X. \end{aligned}$$

Now, we can introduce an order on  $L(X, Y)$

**Definition 1.22. (*Ordering the Space  $L(X, Y)$* )**

For two linear operators  $T, S \in L(X, Y)$ , we define  $S \preceq T$  if the operator  $T - S$  is a positive ( $T - S \succeq 0$ ). That is,

$$S \preceq T \text{ if and only if } (T - S)(x) \succeq 0 \text{ for all } x \in X^+.$$

it is called the **canonical ordering** of  $L(X, Y)$ .

With this order,  $L(X, Y)$  becomes an ordered vector space.

**Remark 1.4. [44]**

If  $X$  and  $Y$  be real vector lattices,  $L(X, Y)$  is **not a vector lattices**, in general, under the canonical ordering.

### 1.2.3 Bounded Linear Operators

If  $X$  and  $Y$  are normed vector spaces (with norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$  respectively), we are often most interested in bounded linear operators.

**Definition 1.23. (*Bounded Linear Operator*)**

A linear operators  $T : X \rightarrow Y$  is **bounded** if there exists a constant  $M \geq 0$  such that

$$\|T(x)\|_Y \leq M\|x\|_X \text{ for all } x \in X.$$

The set of all bounded linear operators from  $X$  to  $Y$  is denoted by  $\mathcal{L}(X, Y)$ .

$\mathcal{L}(X, Y)$  is a vector subspace of  $L(X, Y)$ .

**Definition 1.24. (*Operator Norm*)**

For a bounded linear operator  $T \in \mathcal{L}(X, Y)$ , we define its operator norm as

$$\|T\| = \sup\{\|T(x)\|_Y : x \in X, \|x\|_X \leq 1\}$$

**Theorem 1.9. [48]**

If  $X$  and  $Y$  are normed spaces and  $T : X \rightarrow Y$  a linear operator, then the following conditions of  $T$  are equivalent

1.  $T$  is continuous at one point  $x_0 \in X$ .
2.  $T$  is continuous on  $X$ .
3.  $T$  is norm bounded.

In particular, therefore, a linear functional on  $X$  is norm bounded if and only if it is continuous.

**Remark 1.5.** [44]

1.  $\mathcal{L}(X, Y)$  equipped with the operator norm, is a normed space.
2. If  $Y$  is a Banach space, then  $\mathcal{L}(X, Y)$  is also a Banach space.
3. If  $X$  and  $Y$  are Banach lattices,  $\mathcal{L}(X, Y)$  is **not a vector lattice**, in general, under the canonical ordering, see [44, example 2. p 231].

### 1.2.4 Lattice Homomorphism

**Definition 1.25.** [48] (**Lattice Homomorphism**)

Let  $X$  and  $Y$  be ordered vector spaces and let  $T : X \rightarrow Y$  be a linear operator,  $T$  is called a **lattice homomorphism** (**homomorphism of vector lattices**) If

$$T(x \vee y) = (Tx) \vee (Ty) \text{ for all } x, y \in X .$$

**Theorem 1.10.** [48]

Let  $X$  and  $Y$  be vector lattices and let  $T : X \rightarrow Y$  be a linear operator. Then the following conditions of  $T$  are equivalent

1.  $T$  lattice homomorphism.
2.  $T(x \wedge y) = (Tx) \wedge (Ty)$ , for all  $x, y \in X$  .
3.  $x \wedge y = 0$  implies  $(Tx) \wedge (Ty) = 0$ .
4.  $|Tx| = T(|x|)$ , for all  $x \in X$  .
5.  $Tx^+ = (Tx)^+$ , for all  $x \in X$  .

### 1.2.5 Positive Linear Operators

**Definition 1.26.** [44] (**Positive, Strictly Positive Operator**)

Let  $X$  and  $Y$  be two ordered vector spaces, and let  $X^+, Y^+$  be their positive cones respectively.

1. An operator  $T \in L(X, Y)$  is called **positive**, if  $Tx \geq 0$  for all  $x \geq 0$  and denoted by  $T \succeq 0$ . Clearly  $T \in L(X, Y)$  is positive if and only if  $T(X^+) \subset Y^+$  ( and also if and only if  $x \leq y$  implies  $Tx \leq Ty$ ).

2. An operator  $T \in L(X, Y)$  is called **strictly positive**, if  $Tx > 0$  for all  $x > 0$  and denoted by  $T \gg 0$ .

**Theorem 1.11.** [44]

Let  $X$  and  $Y$  are normed vector lattices and suppose to be norm complete. Every positive linear operator  $T : X \rightarrow Y$  is continuous.

We denote by  $\mathcal{L}^+(X, Y)$  the set of all positive linear operators from a Banach lattice  $X$  into a Banach lattice  $Y$ .

**Definition 1.27.** [44] (**Positive, Negative Part of Linear Operator**)

The concepts of positive part, negative Part of an element  $x \in X$  are extended to linear operators in Banach lattices by considering the action of the operator on positive elements and using the lattice operations.

1. The **positive part** of  $T$  noted by  $T^+$  is defined such that

$$T^+x = (Tx)^+ = (Tx) \vee 0 \text{ for all } x \geq 0.$$

2. The **negative part** of  $T$  noted by  $T^-$  is defined such that

$$T^-x = (Tx)^- = (-Tx) \vee 0 \text{ for all } x \geq 0.$$

**Lemma 1.2.** [8] (**Characterization of Positive Operator**)

The following assertions for a linear operator  $T : X \rightarrow Y$  between the Banach lattices  $X$  and  $Y$  are equivalent.

1.  $T$  is positive.
2. For all  $x \in X$ , we have  $(Tx)^+ \leq Tx^+$  and  $(Tx)^- \leq Tx^-$ .
3.  $|Tx| \leq T|x|$  for all  $x \in X$ .

**Proposition 1.5.** [8]

Let  $X$  and  $Y$  two Banach lattices and  $T \in \mathcal{L}^+(X, Y)$ , then the following properties hold.

1.  $\|T\| = \sup\{\|Tx\| : x \in X^+, \|x\| \leq 1\}$ .
2. If  $S \in \mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}})$  such that  $0 \preceq S \preceq T$  i.e.  $0 \preceq Sx \preceq Tx$  for all  $x \in X^+$ , then  $\|S\| \leq \|T\|$ .

**Theorem 1.12.** [2] (**Extension to a Positive Operator (Kantorovich theorem)**)

Suppose that  $X$  and  $Y$  are two vector lattices with  $Y$  Archimedean. Assume also that  $T : X^+ \rightarrow Y^+$  is an additive mapping, that is,  $T(x + y) = T(x) + T(y)$  holds for all  $x, y \in X^+$ . Then  $T$  has a **unique extension to a positive operator** from  $X$  to  $Y$ . Moreover, the extension (denoted by  $T$  again) is given by

$$T(x) = T(x^+) + T(x^-), \text{ for all } x \in X.$$

**Definition 1.28.** [44] (**Absolutely Majorized Operator**)

Let  $X$  a Banach lattice and  $Y$  a normed vector lattice. A linear operator  $S : X \rightarrow Y$  is called **absolutely majorized** if there exists a positive linear operator  $T : X \rightarrow Y$  such that  $|Sx| \leq Tx$ , for all  $x \in X^+$ .

**Corollary 1.3.** [44]

Every absolutely majorized operator  $S : X \rightarrow Y$  is continuous.

## 1.2.6 Modulus of Linear Operator

**Definition 1.29.** [2] (**Modulus of Linear Operator**)

For linear operator  $T : X \rightarrow Y$  between two vector lattices, we say that its **modulus**  $|T|$  **exists** or that  $T$  **possesses a modulus** whenever

$$|T| := T \vee (-T)$$

exists in the sense that  $|T|$  is the supremum of the set  $\{-T, T\}$  in  $L(X, Y)$ .

**Theorem 1.13.** [2]

For linear operator  $T : X \rightarrow Y$  between two vector lattices such that

$$\sup \{|Ty| : |y| \leq x\}$$

exists in  $Y$  for each  $x \in X^+$ . Then the modulus of  $T$  exists and

$$|T|(x) = \sup \{|Ty| : |y| \leq x\}$$

holds for all  $x \in X^+$ .

**Lemma 1.3.** [2]

If the modulus of an operator  $T : X \rightarrow Y$  exists, then

$$|T(x)| \leq |T|(|x|)$$

holds for all  $x \in X$ .

## 1.2.7 Regular Operator

**Definition 1.30.** [44] (*Regular Operator*)

Let  $X$  and  $Y$  be ordered vector spaces. A linear operator  $T : X \rightarrow Y$  is called **regular**, if  $T = T_1 - T_2$  for suitable positive linear operators  $T_i : X \rightarrow Y$  ( $i = 1, 2$ ). The set of all regular operators is noted by  $L_r(X, Y)$ .

Under the canonical ordering of  $L(X, Y)$ ,  $L_r(X, Y)$  is an ordered vector subspace of  $L(X, Y)$ .

**Proposition 1.6.** [44]

Let  $X$  and  $Y$  be vector lattices. For  $T \in L(X, Y)$  consider this assertions

1.  $|T| := T \vee (-T)$  exist in the canonical ordering of  $L(X, Y)$ .
2.  $T$  is regular.
3.  $T$  maps order bounded sets onto order bounded sets.

Then  $(1) \Rightarrow (2) \Rightarrow (3)$  and, if  $Y$  is order complete,  $(3) \Rightarrow (1)$ .

**Proposition 1.7.** [44]

If  $X$  and  $Y$  are vector lattices and  $Y$  is order complete, then  $L_r(X, Y)$  is an order complete vector lattice under its canonical ordering.

## 1.2.8 Complexification of Linear Operators

Let  $X$  and  $Y$  be real vector spaces, and let  $X_{\mathbb{C}}, Y_{\mathbb{C}}$  be their complexifications.

**Definition 1.31.** [44] (*Complexification of Linear Operator*)

Every  $\mathbb{R}$ -linear operator  $T \in L(X, Y)$  has a unique  $\mathbb{C}$ -linear extension  $T_{\mathbb{C}} \in L(X_{\mathbb{C}}, Y_{\mathbb{C}})$  given by

$$T_{\mathbb{C}}z := Tx + iTy, \text{ for every } z = x + iy \in X_{\mathbb{C}}.$$

$T_{\mathbb{C}}$  is a complex-linear operator. This means it respects addition and multiplication by complex scalars:

$$\begin{aligned} T_{\mathbb{C}}(z_1 + z_2) &= T_{\mathbb{C}}(z_1) + T_{\mathbb{C}}(z_2) \text{ for } z_1, z_2 \in X_{\mathbb{C}}. \\ T_{\mathbb{C}}(\alpha z) &= \alpha T_{\mathbb{C}}(z) \text{ for } z \in X_{\mathbb{C}} \text{ and } \alpha \in \mathbb{C}. \end{aligned}$$

$T_{\mathbb{C}}$  is usually again noted by  $T$ .

**Definition 1.32.** [44] (*Real and Positive Operator in  $L(X_{\mathbb{C}}, Y_{\mathbb{C}})$* )

1. An operator  $T \in L(X_{\mathbb{C}}, Y_{\mathbb{C}})$  is called **real** if  $TX \subset Y$ , and we introduce the notation

$$L^{\mathbb{R}}(X_{\mathbb{C}}, Y_{\mathbb{C}}) = \{T \in L(X_{\mathbb{C}}, Y_{\mathbb{C}}) / T \text{ is real}\}.$$

2. An operator  $T \in L(X_{\mathbb{C}}, Y_{\mathbb{C}})$  is called **positive** ( $T \succeq 0$ ) if  $T$  is real and  $TX^+ \subset Y^+$ , and we introduce the notation

$$L^+(X_{\mathbb{C}}, Y_{\mathbb{C}}) = \{T \in L^{\mathbb{R}}(X_{\mathbb{C}}, Y_{\mathbb{C}}) / T \succeq 0\}.$$

**Definition 1.33.** (*Real and Positive Operator in  $\mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}})$* )

1. An operator  $T \in \mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}})$  is called **real** if  $TX \subset Y$ , and we introduce the notation

$$\mathcal{L}^{\mathbb{R}}(X_{\mathbb{C}}, Y_{\mathbb{C}}) = \{T \in \mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}}) / T \text{ is real}\}.$$

2. An operator  $T \in \mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}})$  is called **positive** ( $T \succeq 0$ ) if  $T$  is real and  $TX^+ \subset Y^+$ , and we introduce the notation

$$\mathcal{L}^+(X_{\mathbb{C}}, Y_{\mathbb{C}}) = \{T \in \mathcal{L}^{\mathbb{R}}(X_{\mathbb{C}}, Y_{\mathbb{C}}) / T \succeq 0\}.$$

we emphasize the simple but important fact

$$\|T\| = \sup_{x \in X^+, \|x\|=1} \|Tx\|. \quad (1.7)$$

**Proposition 1.8.** [44]

Let  $X_{\mathbb{C}}, Y_{\mathbb{C}}$  be complex vector lattices and let  $T \in L(X_{\mathbb{C}}, Y_{\mathbb{C}})$ . Consider the assertions

1.  $T$  is regular.
2.  $T$  maps absolutely majorized subsets of  $X_{\mathbb{C}}$  onto like subsets of  $Y_{\mathbb{C}}$ .

Then (1)  $\Rightarrow$  (2) and if  $Y$  is order complete, (2)  $\Rightarrow$  (1).

**Definition 1.34.** [44] (*Modulus of Operator*)

Let  $X_{\mathbb{C}}, Y_{\mathbb{C}}$  be complex vector lattices and suppose  $Y$  to be order complete. For each regular  $T : X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$ , the positive mapping  $|T| : X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$  defined by

$$|T|x := \sup_{|z| \leq x} |Tz|, \text{ for all } x \in X^+.$$

is called **modulus of  $T$** .

**Remark 1.6.** [44]

1. The modulus  $|T|$  of  $T \in L_r(X_{\mathbb{C}}, Y_{\mathbb{C}})$  satisfies  $|Tz| \leq |T||z|$  for all  $z \in X_{\mathbb{C}}$ , and if  $|Tz| \leq S|z|$  for all  $z \in X_{\mathbb{C}}$ , then  $|T| \leq S$ .
2. Let  $T \in \mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ . If  $\sup_{|z| \leq x} |Tz| \in Y$  for every  $x \in X^+$ , then it holds by linear extension that  $|T| \in \mathcal{L}^+(X_{\mathbb{C}}, Y_{\mathbb{C}})$  and we introduce the denotation

$$\mathcal{L}^{|\cdot|}(X_{\mathbb{C}}, Y_{\mathbb{C}}) = \{T \in \mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}}) \mid |T| \in \mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}})\}.$$

**Theorem 1.14.** [44]

Let  $X_{\mathbb{C}}, Y_{\mathbb{C}}$  be complex vector lattices and suppose  $Y$  to be order complete. For  $T \in L_r(X_{\mathbb{C}}, Y_{\mathbb{C}})$ , the modulus  $|T|$  satisfies

$$|T| := \sup_{0 \leq \theta \leq 2\pi} |(\cos \theta) T_1 + (\sin \theta) T_2|.$$

Where  $T = T_1 + iT_2$  is the canonical decomposition of  $T$ . Hence  $L_r(X_{\mathbb{C}}, Y_{\mathbb{C}})$  endowed with the modulus function  $T \mapsto |T|$ , is isomorphic with the complexification of the vector lattice  $L_r(X, Y)$ .

**Remark 1.7.**

We have  $|Tx| \leq |T||x|$  for every  $T \in \mathcal{L}^{|\cdot|}(X_{\mathbb{C}}, Y_{\mathbb{C}})$  and every  $x \in X_{\mathbb{C}}$  as well as  $|Tx| \leq T|x|$  for every  $T \in \mathcal{L}^+(X_{\mathbb{C}}, Y_{\mathbb{C}})$  and every  $x \in X_{\mathbb{C}}$ . The lattice norm property (1.3) implies that

$$\|T\| \leq \| |T| \| \text{ for every } T \in \mathcal{L}^{|\cdot|}(X_{\mathbb{C}}, Y_{\mathbb{C}}).$$

**Definition 1.35.** [8] (**Lattice Homomorphism**)

Let  $X_{\mathbb{C}}$  and  $Y_{\mathbb{C}}$  be two complex Banach lattices. A linear operator  $T : X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$  is called a **lattice homomorphism** if  $|Tx| = T|x|$  for all  $x \in X$ .

**Definition 1.36.** [47] (**Resolvent set, Resolvent Operator, Spectrum, Spectral Radius, Spectral Bound**)

1. Let  $X_{\mathbb{C}}$  a complex Banach lattice. For a closed linear operator  $T$  on  $X_{\mathbb{C}}$ , the **resolvent set** of  $T$  is defined by

$$\rho(T) = \{\lambda \in \mathbb{C}; \lambda I - T : X_{\mathbb{C}} \rightarrow X_{\mathbb{C}} \text{ is invertible}\}.$$

$\rho(T)$  is an open subset of the complex plan  $\mathbb{C}$ .

2. We call  $(\lambda I - T)^{-1}$  the **resolvent** of  $T$  and denoted by

$$R(\lambda, T) := (\lambda I - T)^{-1} \text{ for } \lambda \in \rho(T).$$

3. The complement of  $\rho(T)$  in  $X_{\mathbb{C}}$  is called the **spectrum** of  $T$  and denoted by  $\sigma(T)$ . In general, it may be possible that either  $\rho(T)$  or  $\sigma(T)$  is empty.
4. The **spectral radius** of  $T$  is defined by

$$r(T) := \sup \{|\lambda|, \lambda \in \sigma(T)\}.$$

5. The **spectral bound** of  $T$  is defined by

$$s(T) := \sup \{\operatorname{Re} \lambda, \lambda \in \sigma(T)\}.$$

**Definition 1.37.** [17] (*Hurwitz Stable, Strictly Hurwitz Stable*)

1. A closed operator  $T$  on  $X$  is called **hurwitz stable** if  $\sigma(T) \subset \mathbb{C}_-$ , such that  $\mathbb{C}_- := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$ ,
2. A closed operator  $T$  on  $X$  is called **strictly hurwitz stable** if  $s(T) < 0$ .

## 1.3 SEMI-GROUP OF BOUNDED LINEAR OPERATORS

### 1.3.1 Semi-Group Theory as a Framework for Evolution Equations

It is commonly known the semi-group approach is used to solve differential systems because it provides a powerful and unified way to understand a wide variety of them, from simple ordinary differential equations (ODE(s)) to complex partial differential equations (PDE(s)) and delay differential equations (DDE(s)). In essence, this approach generalizes the familiar exponential function solution of basic differential equations to infinite-dimensional spaces.

Differential equation as a set of rules that describe how a system changes over time. For a simple linear (ODE) like  $y'(t) = ay(t)$ , with well-determined initial conditions  $y(0)$  the solution is straightforward:  $y(t) = e^{at}y(0)$ . Here, the term  $e^{at}y(0)$  tells exactly how the initial state  $y(0)$  evolves. The semi-group approach extends this idea to much more complex systems. It reformulates a differential system, which could be a (PDE) describing heat flow or a (DDE) modeling population dynamics, as an abstract (ODE) on a function space (a Banach space):  $\frac{du(t)}{dt} = Au(t)$  with  $u(0) = u_0$ . Here,  $u(t)$  represents the state of the system at

time  $t$  (which could be a function describing a temperature distribution, for example), and  $A$  is an operator that encapsulates the system's dynamics (like the Laplacian operator in the heat equation). The solution to this abstract equation is given by a semi-group of operators, denoted as  $S(t)$ . The state of the system at time  $t$  is then:  $u(t) = S(t)u_0$ . The family of operators  $(S(t))_{t \geq 0}$  acts as a generalized version of the exponential function, evolving the initial state  $u_0$  forward in time.

The use of semi-group theory offers several significant advantages in the study of differential systems. It provides a framework to analyze a vast range of evolution equations. This allows for a deeper understanding of the common mathematical structures that govern these different physical phenomena. Also it guaranteed existence and uniqueness of solutions and its dependence continuously on the initial conditions (i.e., the problem is well-posed), where the Hille-Yosida theorem is a cornerstone of semi-group theory that provides a complete answer to this question, it gives necessary and sufficient conditions for an operator  $A$  to be the "infinitesimal generator" of a semi-group. As well as it enables us to analysis of long-term behavior and stability where the qualitative properties of the solution  $u(t)$  are directly linked to the properties of the semi-group  $(S(t))_{t \geq 0}$ , which in turn are determined by the spectrum of the generator  $A$ .

### 1.3.2 Uniformly Continuous Semi-groups

**Definition 1.38.** [42] (*Semi-group*)

Let  $X$  be a Banach space. A one parameter family  $(S(t))_{t \geq 0}$  of bounded linear operators from  $X$  into  $X$  is a **semi-group** of bounded linear operators on  $X$  if

1.  $S(0) = I$  ( $I$  is the identity operator on  $X$ ).
2.  $S(t+s) = S(t)S(s)$  for all  $t, s \geq 0$  (the semigroup property)..

**Definition 1.39.** [42] (*Uniformly Continuous Semi-group*)

A semigroup of bounded linear operators,  $(S(t))_{t \geq 0}$ , is **uniformly continuous** if

$$\lim_{t \rightarrow 0} \|S(t) - I\| = 0.$$

The linear operator  $T$  defined by

$$D(T) = \{x \in X : \lim_{t \rightarrow 0} \frac{S(t)x - x}{t} \text{ exists}\},$$

and

$$Tx = \lim_{t \rightarrow 0} \frac{S(t)x - x}{t} = \frac{d^+ S(t)x}{dt} \Big|_{t=0} \text{ for } x \in D(T)$$

is the infinitesimal generator of the semi-group  $(S(t))_{t \geq 0}$ ,  $D(T)$  is the domain of the generator  $T$ .

**Theorem 1.15.** [42]

A linear operator  $T$  is the infinitesimal generator of a uniformly continuous semi-group  $(S(t))_{t \geq 0}$  if and only if  $T$  is a bounded linear operator.

**Theorem 1.16.** [42]

Let  $(T(t))_{t \geq 0}$  and  $(S(t))_{t \geq 0}$  be uniformly continuous semi-groups of bounded linear operators. If

$$\lim_{t \rightarrow 0} \frac{T(t)x - x}{t} = \lim_{t \rightarrow 0} \frac{S(t)x - x}{t},$$

then  $T(t) = S(t)$  for all  $t \geq 0$ .

**Corollary 1.4.** [42]

Let  $(S(t))_{t \geq 0}$  be a uniformly continuous semi-group of bounded linear operators. Then

1. There exists a constant  $\omega \geq 0$  such that  $\|S(t)\| \leq e^{\omega t}$ .
2. There exists a unique bounded linear operator  $T$  such that  $S(t) = e^{tT}$ .
3. The operator  $T$  in part (2) is the infinitesimal generator of  $(S(t))_{t \geq 0}$ .
4.  $t \rightarrow S(t)$  is differentiable in norm and

$$\frac{d}{dt} S(t) = TS(t) = S(t)T.$$

### 1.3.3 Strongly Continuous Semi-groups

**Definition 1.40.** [42] (**Strongly Continuous Semi-group**)

A semi-group  $(S(t))_{t \geq 0}$  of bounded linear operators on a Banach space  $X$  is called **strongly continuous semi-group** of bounded linear operators if

$$\lim_{t \rightarrow 0} S(t)x = x \text{ for every } x \in X.$$

A strongly continuous semi-group of bounded linear operators on  $X$  will be called a **semi-group of class  $C_0$**  or simply a  **$C_0$ -semi-group**.

**Theorem 1.17.** [42]

Let  $(S(t))_{t \geq 0}$  be a  $C_0$ -semi-group. There exist constant  $\omega \geq 0$  and  $M \geq 1$  such that

$$\|S(t)\| \leq Me^{\omega t}, \text{ for } t \geq 0.$$

**Corollary 1.5.** [42]

If  $(S(t))_{t \geq 0}$  is a  $C_0$ -semi-group, then for every  $x \in X$ ,  $t \rightarrow S(t)x$  is a continuous function from  $\mathbb{R}_+^*$  into  $X$ .

**Theorem 1.18.** [42]

Let  $(S(t))_{t \geq 0}$  be a  $C_0$ -semi-group and let  $T$  be its infinitesimal generator. Then

1. For  $x \in X$

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} S(s)x ds = S(t)x.$$

2. For  $x \in X$ ,  $\int_0^t S(s)x ds \in D(T)$  and

$$T \left( \int_0^t S(s)x ds \right) = S(t)x - x.$$

3. For  $x \in D(T)$ ,  $S(t)x \in D(T)$  and

$$\frac{d}{dt} S(t)x = TS(t)x = S(t)Tx.$$

4. For  $x \in D(T)$ ,

$$S(t)x - S(s)x = \int_s^t S(\tau)Tx d\tau = \int_s^t TS(\tau)x d\tau.$$

**Corollary 1.6.** [42]

If  $T$  is the infinitesimal generator of a  $C_0$ -semi-group  $(S(t))_{t \geq 0}$  then  $D(T)$ , the domain of  $T$ , is dense in  $X$  and  $T$  is a closed linear operator.

**Theorem 1.19.** [42]

Let  $(T(t))_{t \geq 0}$  and  $(S(t))_{t \geq 0}$  be  $C_0$ -semi-groups of bounded linear operators with infinitesimal generators  $A$  and  $B$  respectively. If  $A = B$  then  $T(t) = S(t)$  for all  $t \geq 0$ .

**Definition 1.41.** [47] (**Growth Bound of Operator**)

Let  $(T, D(T))$  be the generator of a  $C_0$ -semi-group  $(S(t))_{t \geq 0}$  on the Banach space  $X$ . The **growth bound of  $T$**  is defined by

$$\omega_1(T) := \inf \left\{ \omega \in \mathbb{R} : \exists M > 0 : \|S(t)x\|_{\mathcal{L}(X_c)} < Me^{\omega t} \|x\|_{D(T)}, \text{ for all } t \geq 0, x \in D(T) \right\}.$$

**Definition 1.42.** [10, 47] (**uniform Growth Bound**)

Let  $(T, D(T))$  be the generator of a  $C_0$ -semi-group  $(S(t))_{t \geq 0}$ . We call

$$\omega_0(T) := \inf \left\{ \omega \in \mathbb{R} : \exists M > 0 : \|S(t)\|_{\mathcal{L}(X_c)} < Me^{\omega t}, \text{ for all } t \geq 0 \right\}.$$

the semi-group's **uniform growth bound**.

**Proposition 1.9.** [8] (**Connection between Uniform Growth Bound and Spectral Radius of Semi-group Operators**)

Let  $(S(t))_{t \geq 0}$  a  $C_0$ -semi-group on a Banach space  $X$ .

a) We have that

$$\omega_0(T) = \lim_{t \rightarrow \infty} \frac{\text{Log} \|T(t)\|}{t} = \inf_{t > 0} \frac{\text{Log} \|T(t)\|}{t}.$$

b) For every  $t \geq 0$ , the spectral radius  $r(T(t))$  of the operator  $T(t)$  satisfies

$$r(T(t)) = e^{t\omega_0(T)}.$$

### 1.3.4 Compact $C_0$ -Semi-groups

**Definition 1.43.** [42] (**Compact Strongly Continuous Semi-group**)

A strongly continuous semi-group  $(S(t))_{t \geq 0}$  of bounded linear operators on a Banach lattice  $X$  is called **Compact** for  $t > t_0$  if,

for  $t > t_0$ ,  $S(t)$  is a compact operator.

$(S(t))_{t \geq 0}$  is called **Compact** if it is compact for  $t > 0$

**Theorem 1.20.** [42]

Let  $(S(t))_{t \geq 0}$  be a  $C_0$ -Semi-group. If  $(S(t))_{t \geq 0}$  is compact for  $t > t_0$  then,  $S(t)$  is continuous in the uniform operator topology for  $t > t_0$ .

**Theorem 1.21.** [42]

Let  $(S(t))_{t \geq 0}$  be a  $C_0$ -Semi-group and let  $T$  be its infinitesimal generator,  $(S(t))_{t \geq 0}$  is a compact Semi-group if and only if  $S(t)$  is continuous in the uniform operator topology for  $t > 0$  and  $R(\lambda, T)$  is compact for all  $\lambda > \rho(T)$ .

### 1.3.5 Stability Concepts of $C_0$ -Semi-groups

**Definition 1.44.** [8, 47] (*Uniform Exponential Stability, Strong Stability*)

A  $C_0$ -semi-group  $(S(t))_{t \geq 0}$  is called

- a) **Uniformly exponentially stable** if  $\omega_0(T) < 0$ .
- b) **Strongly or exponentially stable** if  $\omega_1(T) < 0$  (i.e.,  $\lim_{t \rightarrow \infty} \|T(t)x\| = 0$  for every  $x \in X$ )

**Remark 1.8.** [8]

It is clear that a) implies b). The converse does not hold in general, see [8, Example 12.3 page 183].

It is well-known, see for example in [47], that

$$-\infty \leq s(T) \leq \omega_1(T) \leq s_0(T) \leq \omega_0(T) < +\infty.$$

**Proposition 1.10.** [8]

For a  $C_0$ -semi-group  $(S(t))_{t \geq 0}$ , the following assertions are equivalent.

- 1.  $\omega_0(T) < 0$ , i.e.,  $(S(t))_{t \geq 0}$  is uniformly exponentially stable.
- 2.  $\lim_{t \rightarrow \infty} \|T(t)\| = 0$ .
- 3.  $\|T(t_0)\| < 1$  for some  $t_0 > 0$ .
- 4.  $r(T(t_1)) < 1$  for some  $t_1 > 0$ .

**Theorem 1.22.** [8] (*Datko Theorem*)

A  $C_0$ -semi-group  $(S(t))_{t \geq 0}$  is uniformly exponentially stable if and only if for some and (hence for every)  $p \in [1, \infty)$

$$\int_0^\infty \|T(t)x\|^p dt < \infty, \text{ for all } x \in X.$$

### 1.3.6 Positive $C_0$ -Semi-groups

**Definition 1.45.** [10] (*Positive Strongly Continuous Semi-group*)

A strongly continuous semi-group  $(S(t))_{t \geq 0}$  of bounded linear operators on a Banach lattice  $X$  is called **Positive** if each operator  $S(t)$  is positive, i.e., if

$$S(t)x \geq 0 \text{ for every } x \in X \text{ such that } x \geq 0.$$

Positive semi-groups can be characterized through the resolvent of its generator as follows.

**Theorem 1.23.** [10] (*Positive Semi-group Characterization*)

A  $C_0$ -semi-group  $(S(t))_{t \geq 0}$  on a Banach lattice  $X$  is positive if and only if the resolvent  $R(\lambda, T)$  of its generator  $T$  is positive for all sufficiently large  $\lambda$ .

For positive semi-groups, there is a very nice spectral and asymptotic theory. The key tool in proving these properties is the following lemma.

**Lemma 1.4.** [10]

For a positive  $C_0$ -semi-group  $(S(t))_{t \geq 0}$  with generator  $(T, D(T))$  on a Banach lattice  $X$  one has

$$R(\lambda, T)x = \int_0^\infty e^{-\lambda s} S(s) s ds, x \in X. \text{ for all } \operatorname{Re} \lambda > s(T).$$

Moreover, the following are equivalent for  $\lambda \in \rho(T)$

1.  $R(\lambda, T) \succeq 0$ .
2.  $s(T) < \lambda$ .

One of the most important results in view of applications is the following.

**Proposition 1.11.** [10]

Let  $(S(t))_{t \geq 0}$  a positive  $C_0$ -semi-group on a Banach lattice  $X$  with  $s(T) > -\infty$ . Then  $s(T) \in \sigma(T)$ .

In particular, this means that there is always a real number in the spectrum dominating the whole spectrum.

**Proposition 1.12.** [8]

Let  $T$  be the generator of a positive  $C_0$ -semi-group  $(T(t))_{t \geq 0}$  on a Banach lattice  $X_{\mathbb{C}}$  and assume that  $B \in \mathcal{L}^+(X)$ , then the following holds

1.  $A + B$  generates a positive semi-group  $(S(t))_{t \geq 0}$  satisfying  $0 \preceq T(t) \preceq S(t)$  for all  $t \geq 0$ ,
2.  $s(A) \leq s(A + B)$  and  $R(\lambda, A) \succeq R(\lambda, A + B)$  for all  $\lambda > s(A + B)$ .

In what follows, we always assume that the resolvent set is non-empty.

**Definition 1.46.** [10] (*Abscissa of Uniform Boundedness of Operator*)

Let  $(T, D(T))$  be the generator of a  $C_0$ -semi-group  $(S(t))_{t \geq 0}$  on the Banach space  $X$ . The **abscissa of uniform boundedness of  $T$**  is defined by

$$s_0(T) := \inf \left\{ \omega \in \mathbb{R} : \{\operatorname{Re} \lambda > \omega\} \subset \rho(T) \text{ and } \sup_{\operatorname{Re} \lambda > \omega} \|R(\lambda, T)\|_{\mathcal{L}(X_{\mathbb{C}})} < \infty \right\}.$$

**Remark 1.9.**

The inequality  $s(T) \leq \omega_1(T) \leq s_0(T) \leq \omega_0(T)$  might be strict, that is, the exponential stability of a  $C_0$ -semigroup, in general, is not controlled by the location of the spectrum of its generator.

**Theorem 1.24.** [6]

If the operator  $T$  generates a positive  $C_0$ -semi-group, that is,  $S(t) \succeq 0$ , for all  $t \geq 0$ , then

$$s(T) = \omega_1(T) = s_0(T).$$

For positive operators, we have the well-known **Perron-Frobenius Theorem** of bounded linear operators.

**Theorem 1.25.** [33] (*Perron-Frobenius Theorem*)

Suppose  $T \in \mathcal{L}^+(X_{\mathbb{C}})$ , then

1.  $r(T) \in \sigma(T)$ ,
2.  $R(\lambda, T) \succeq 0$  if and only if  $\lambda \in \mathbb{R}$  and  $\lambda > r(T)$ ,
3. If  $\lambda > r(T)$ , then  $|R(\lambda, T)x| \leq R(|\lambda|, T)|x|$ , for any  $x \in X$ .

## 1.4 METZLER OPERATORS

### 1.4.1 Metzler Matrices and Operators From Economic Origins to Stability Analysis

Metzler matrices derive their name from the American economist Lloyd Metzler (1913-1980). His significant contributions to economic analysis, particularly in the study of the stability of general equilibrium and the dynamics of business cycles, were influential in the conceptualization and application of these matrices [21]. The matrices were initially introduced to model economic systems where interactions between variables, such as prices or quantities, exhibit specific non-negative

relationships, most notably gross substitutability [46]. This property implies that an increase in the price of one good leads to a non-negative (or positive) increase in the demand for a substitute good. The original finite-dimensional matrix concept has since been generalized to "Metzler operators" on Banach lattices, extending their analytical power to infinite-dimensional systems, which are prevalent in advanced mathematical modeling [21]. This historical origin in economics, rather than pure mathematics, indicates a strong initial drive for practical application and problem-solving. This suggests that the theoretical development of Metzler operators was influenced by the need to solve concrete, real-world problems related to system stability in fields where non-negative quantities are fundamental. Metzler operators are now recognized as crucial tools for stability analysis in various dynamic systems, including those described by time-delayed differential equations and positive linear dynamical systems [46].

### 1.4.2 Definition and some Properties

**Definition 1.47.** [17] (*Metzler Operator*)

A closed operator  $T$  on a Banach space  $X$  is said to be **Metzler operator** if there exists  $\omega \in \mathbb{R}$  such that  $(\omega, +\infty) \subset \rho(T)$  and  $R(t, T)$  is positive for all  $t \in (\omega, +\infty)$ . Metzler operators are also called **resolvent positive operators** in the literature. If  $T \in \mathcal{L}^+(X_{\mathbb{C}}, Y_{\mathbb{C}})$ , then  $T$  is a Metzler operator.

**Remark 1.10.** [17]

1. It follow from Definition 1.47, a closed operator  $T$  on a Banach space  $X$  is a Metzler operator if and only if  $T + cI$  is a Metzler operator for some (all)  $c \in \mathbb{R}$ .
2. Every bounded operator on a Banach space  $X$ , for which there exists  $c \geq 0$  such that  $T + cI$  is positive, is a Metzler operator on  $X$ .

For Metzler operators, we have the following monotonicity and positivity result for the resolvent.

**Lemma 1.5.** [17]

Let  $T$  a Metzler operator on  $X$ . Then the function  $R(., T)$  is positive and decreasing for  $t > \sup(\sigma(T) \cap \mathbb{R})$

$$\sup(\sigma(T) \cap \mathbb{R}) < t_1 \leq t_2 \Rightarrow 0 \preceq R(t_2, T) \preceq R(t_1, T)$$

**Theorem 1.26.** [17]

Let  $T$  a Metzler operator on  $X_{\mathbb{C}}$ , then

1.  $s(T) \in \sigma(T)$  and  $s(T) = t - [r(R(t, T))]^{-1}, t > s(T)$ ,
2. The function  $R(., T)$  is positive and decreasing for  $t > s(T)$ , that is,

$$s(T) < t_1 \leq t_2 \Rightarrow R(t_2, T) \preceq R(t_1, T),$$

3. If  $T$  generates a positive  $C_0$ -semi-group, then we have  $R(t, T)$  is positive if and only if  $t > s(T)$ .

**Lemma 1.6.** [17]

Let  $T$  be a Metzler operator on  $X$  and  $E \in \mathcal{L}^+(X, Y)$ , then

$$|ER(\lambda, T)x| \leq ER(\operatorname{Re} \lambda, T)|x|, \operatorname{Re} \lambda > s(T), x \in X.$$

## Chapter 2

# DELAY DIFFERENTIAL SYSTEMS

### 2.1 THE HISTORY OF FUNCTIONAL DIFFERENTIAL EQUATIONS

The study of delay, differential integral, and functional differential equations arguably have spanned for more than two centuries. Initial work in these fields may have primarily arisen from challenges in geometry and number theory [7].

Early 20th Century witnessed a shift towards Physics. In 1908, at the International Conference of Mathematicians held in 1908 in Rome, Émile Picard, in his article "Mathematics in its Relations with Physics", explored the evolution of mathematics in the face of the challenges of physics, emphasized functional equations as an essential tool for the future. Picard observed, in the context of the evolution of mathematical modeling, that classical mechanics is based on a principle of non-heredity, where the state of a system is determined by its immediate state, without taking into account its past. He noted that many physical phenomena, such as the behavior of sulfur, indicate that the current state keeps track of past states, a concept particularly crucial in the life sciences. Faced with these limitations, Picard proposed a new mathematical framework: functional equations. He described them as a broader domain than that of differential equations, capable of including them as special cases. He suggested that these more complex equations could incorporate integrals extending from the distant past to the present, in order to model heredity. He concluded that mathematics is not an isolated science, but an essential part of natural philosophy, providing physics with tools to contemplate the world logically [28].

Following Picard's ideas, Volterra in 1909 and 1928 discussed The modeling of viscoelasticity using integrodifferential equations. After that, in 1931, he published

a foundational book on the influence of hereditary effects on models of species interaction. Following 1940, interest in this topic grew, particularly in the Soviet Union. This was due to the development of meaningful engineering models that acknowledged the presence of hereditary effects in physical systems. Before this time, these effects were often overlooked because of a lack of a theoretical framework to analyze such complex models. Functional differential equations have been extensively developed during the 1950's. This theory became a key part of the language used by researchers in many fields. These areas included viscoelasticity, mechanics, nuclear reactors, distributed networks, and heat flow. The theory was also applied to neural networks, combustion, species interaction, microbiology, learning models, epidemiology, and physiology. ,as well as many others [28].

Between the 1950s and early 1960s, a lot of work was done in this field. Important books were published by Myshkis (1951), Krasovskii (1959), Bellman and Cooke (1963), and Halanay (1966). These publications show a clear picture of the subject during that time, research during this period focused mainly on two areas: linear equations and the stability of systems when small, nonlinear changes were introduced [7, 8, 11, 19, 34, 36].

## 2.2 DESCRIPTION OF THE DELAY DIFFERENTIAL SYSTEM

We consider a time-invariant delay differential system of the form

$$\begin{cases} \dot{x}(t) = A_0 x(t) + \sum_{k=1}^m A_k x(t - h_k), & t \geq 0, \\ x(0) = x_0, \\ x(t) = f(t), & t \in [-h, 0). \end{cases} \quad (2.1)$$

Where,

$A_0$  is the generator of  $(T(t))_{t \geq 0}$  a  $C_0$ -semi-group on a complex Banach lattice  $X_{\mathbb{C}}$  with the norm  $|\cdot|_{\mathbb{C}}$ ,  $A_1, A_2, \dots, A_m$  are bounded linear operators on  $X_{\mathbb{C}}$ , i.e.  $A_k \in \mathcal{L}(X_{\mathbb{C}})$ , with the operator norm  $\|\cdot\|$  and  $h_i$  for every  $i = 1, \dots, m$  are non-negative real numbers such that  $0 \leq h_1 < h_2 < \dots \leq h_m := h$ .

$x_0 \in X_{\mathbb{C}}$  is the initial value and  $f \in L^p([-h, 0]; X_{\mathbb{C}})$ ,  $p \in [1; \infty)$  is the history function.

A mild solution of the delay system (2.1) is the function

$$x(\cdot; x_0, f) \in L_{loc}^p([-h, \infty); X_{\mathbb{C}}),$$

satisfying

$$x(t; x_0, f) = \begin{cases} T(t)x_0 + \int_0^t T(t-s) \sum_{k=1}^m A_k x(s-h_k; x_0, f) ds, & t \geq 0, \\ f(t), & t \in [-h, 0). \end{cases}$$

It follows from [36, Theorem 2.1] that for all  $x_0 \in X_{\mathbb{C}}$  and  $f \in L^p([-h, 0]; X_{\mathbb{C}})$  a unique mild solution  $x(\cdot; x_0, f)$  exists. This solution is continuous on  $[0, \infty)$  and exponentially bounded.

**Definition 2.1. (*Positive System*)**

*The delay system (2.1) is a **positive system** if for any history function  $f \in L^p([-h, 0]; X_{\mathbb{C}}^+)$ , and initial value  $x_0 \in X_{\mathbb{C}}^+$ , the corresponding solution satisfies*

$$x(t; x_0, f) \in X_{\mathbb{C}}^+ \text{ for all } t \geq 0.$$

## 2.3 STABILITY OF THE DELAY DIFFERENTIAL SYSTEM

### 2.3.1 Approaches to Stability in Delay Differential Equations: A Comparative View

In the study of stability for delay differential systems, two prominent and influential analytical frameworks have emerged over the years: the Lyapunov method and the semigroup method. While the former offers a traditional, direct approach through the use of scalar functions or functionals, the latter provides a more abstract, yet powerful, framework by recasting the problem in an infinite-dimensional setting.

First, the Lyapunov method and its variations, such as Razumikhin-type theorems and Lyapunov-Krasovskii functionals, represent a classic approach to stability analysis. The core idea is to construct a scalar function (a Lyapunov function) or a functional (a Lyapunov-Krasovskii functional) whose time derivative along the system's trajectories can be definitively determined to be non-positive. This allows for the assessment of stability without explicitly solving the differential equations. For instance, a system is asymptotically stable if a positive definite function's derivative is negative definite along the system's trajectories. In this approach, a system with a time delay requires a functional, as the future state depends not just on the current value, but on the history of the system over an entire interval.

In contrast, the semigroup method, which emerged with contributions from researchers like J. Hale and G. Webb, approaches the problem from a different perspective. This method reformulates a delay differential equation (DDE) as an abstract ordinary differential equation (ODE) within a larger function space, such as a Banach space. Consequently, the solutions of the (DDE) are treated as a trajectory of a continuous semigroup of operators on this state space. This novel approach enables the application of well-developed tools from linear operator theory and semi-group theory to analyze the existence, uniqueness, spectral properties, and asymptotic behavior of the solutions. The central idea is that the well-posedness of the delay equation can be reduced to the question of whether an associated operator generates a strongly continuous semigroup.

The historical progression of these two methods highlights a shift from direct, case-specific analysis to a generalized, abstract framework. The Lyapunov method provides powerful, albeit often conservative, conditions for stability, and its application can be difficult due to the challenge of constructing suitable functionals. However, the semigroup method, by translating the problem into the language of operator theory, provides a systematic and powerful tool for a deeper structural and spectral analysis of delay differential equations. Both methods continue to be refined and are central to the ongoing study of time-delay systems [8, 10, 11, 19, 34, 37, 45].

### 2.3.2 Exponential Stability

#### Definition 2.2. (*Exponential Stability*)

The delay system (2.1) is called **exponentially stable** if there exist  $M > 0$  and  $\omega > 0$  such that for any  $f \in L^p([-h, 0]; X_{\mathbb{C}})$  and any  $x_0 \in X_{\mathbb{C}}$ , the unique solution of the delay system (2.1) satisfies

$$\|x(t; x_0, f)\|_{X_{\mathbb{C}}} \leq M e^{-\omega t} \|(x_0, f)\|_{X_{\mathbb{C}} \times L^p([-h, 0]; X_{\mathbb{C}})}, \text{ for all } t \geq 0,$$

where

$$\|(x_0, f)\|_{X_{\mathbb{C}} \times L^p([-h, 0]; X_{\mathbb{C}})} = \left( |x_0|_{X_{\mathbb{C}}}^p + \|f\|_{L^p([-h, 0]; X_{\mathbb{C}})}^p \right)^{\frac{1}{p}},$$

then

$$\|x(t; x_0, f)\|_{X_{\mathbb{C}}} \leq M e^{-\omega t} \left( |x_0|_{X_{\mathbb{C}}}^p + \|f\|_{L^p([-h, 0]; X_{\mathbb{C}})}^p \right)^{\frac{1}{p}}.$$

#### Lemma 2.1.

Assume that the delay system (2.1) is positive and let  $x_1, x_2 \in X_{\mathbb{C}}^+$ ,  $f_1, f_2 \in L^p([-h, 0]; X_{\mathbb{C}}^+)$ ,  $x_a(t, x_1, f_1)$ ;  $x_b(t, x_2, f_2)$  are solutions of the delay system (2.1) under initial conditions  $x(0) = x_1, x(t) = f_1(t)$  and  $x(0) = x_2, x(t) = f_2(t)$  for all  $t \in [-h, 0)$  respectively.

If

$$x_1 \leq x_2 \text{ and } f_1(t) \leq f_2(t), \text{ for all } t \in [-h, 0),$$

then

$$x_a(t, x_1, f_1) \leq x_b(t, x_2, f_2), \text{ for all } t \geq 0.$$

*Proof.*

Let  $x_a(t, x_1, f_1) := x_a(t)$ ;  $x_b(t, x_2, f_2) := x_b(t)$ , are solutions of the delay system (2.1), then

$$\dot{x}_a(t) = A_0 x_a(t) + \sum_{k=1}^m A_k x_a(t - h_k), \quad t \geq 0,$$

with  $x_a(0) = x_1$  and  $x_a(t) = f_1(t)$  for all  $t \in [-h, 0)$ ,

and

$$\dot{x}_b(t) = A_0 x_b(t) + \sum_{k=1}^m A_k x_b(t - h_k), \quad t \geq 0,$$

with  $x_b(0) = x_2$  and  $x_b(t) = f_2(t)$  for all  $t \in [-h, 0)$ ,

by subtraction, we get

$$\dot{x}_b(t) - \dot{x}_a(t) = A_0 (x_b(t) - x_a(t)) + \sum_{k=1}^m A_k (x_b(t - h_k) - x_a(t - h_k)), \quad t \geq 0.$$

By putting

$$x_c(t) := x_b(t) - x_a(t) \text{ for all } t \in [-h, 0),$$

$$x_3 = x_2 - x_1,$$

$$f_3(t) = f_2(t) - f_1(t) \text{ for all } t \in [-h, 0),$$

we obtain

$$\dot{x}_c(t) = A_0 x_c(t) + \sum_{k=1}^m A_k x_c(t - h_k), \quad t \geq 0,$$

with  $x_c(0) = x_3$  and  $x_c(t) = f_3(t)$  for all  $t \in [-h, 0)$ ,

then, there exists a unique solution  $x_c(t) := x_c(t, x_3, f_3)$  of the delay system (2.1) under the initial condition  $x_c(0) = x_3$  and  $x_c(t) = f_3(t)$  for all  $t \in [-h, 0)$ . By the assumption  $x_1 \leq x_2$ , we get  $x_3 \in X_{\mathbb{C}}^+$  and by the assumption  $f_1(t) \leq f_2(t)$ , for all  $t \in [-h, 0)$ , we get  $f_3 \in L^p([-h, 0]; X_{\mathbb{C}}^+)$ . And since the delay system (2.1) is positive, then  $x_c(t, x_3, f_3) \geq 0$  for all  $t \in [-h, 0)$ , which implies

$$x_a(t, x_1, f_1) \leq x_b(t, x_2, f_2) \text{ for all } t \in [-h, 0).$$

□

### 2.3.3 Abstract formulation

In order to study of the solution's asymptotic behavior of the delay system (2.1) by semigroup method, we introduce the product space

$$\mathcal{X} := X_{\mathbb{C}} \times L^p([-h, 0]; X_{\mathbb{C}})$$

and define bounded linear operators  $(\mathcal{T}(t))_{t \geq 0}$  on  $\mathcal{X}$  as follows:

For a given a function  $x \in L_{loc}^p([-h, \infty); X_{\mathbb{C}})$ , for each  $t \geq 0$ , we define

$$x_t \in L^p([-h, 0]; X_{\mathbb{C}})$$

by

$$x_t(s) := x(t + s), \quad s \in [-h, 0].$$

Denoting the unique mild solution of the delay system (2.1) by  $x(\cdot; x_0, f)$ , we now define

$$\mathcal{T}(t)(x_0, f) := (x(t; x_0, f), x_t(\cdot; x_0, f)), \quad t \geq 0.$$

**Proposition 2.1.** [18]

The family  $(\mathcal{T}(t))_{t \geq 0}$  defines a  $C_0$ -semigroup of linear operators on  $\mathcal{X}$ . Its generator  $\mathcal{A}$  is given by

$$\mathcal{A}(x_0, f) := \left( A_0 x_0 + \sum_{k=1}^m A_k f(\cdot - h_k), f' \right),$$

with the domain

$$D(\mathcal{A}) := \{(x_0, f) \in \mathcal{X} : f \in W_{loc}^{1,p}([-h, 0]; X_{\mathbb{C}}), f(0) = x_0 \in D(A_0)\}.$$

Here  $W_{loc}^{1,p}([-h, 0]; X_{\mathbb{C}})$  denotes the space of absolutely continuous  $X_{\mathbb{C}}$ -valued functions  $f$  on  $[-h, 0]$  which are strongly differentiable, i.e. with  $f' \in L^p([-h, 0]; X_{\mathbb{C}})$ .

Moreover, the delay system (2.1) is exponentially stable if and only if the  $C_0$ -semigroup  $(\mathcal{T}(t))_{t \geq 0}$  is exponentially stable, i.e.  $\omega_1(\mathcal{A}) < 0$ .

**Definition 2.3.** [4] (*Quasi-Polynomial Operator*)

The quasi-polynomial operator associated with the delay system (2.1) is defined by

$$P(\lambda) := A_0 + \sum_{k=1}^m e^{-\lambda h_k} A_k. \quad (2.2)$$

In case where  $X$  is a finite dimensional space, the delay system (2.1) is exponentially stable if and only if characteristic roots of equation  $\det(P(\lambda)) = 0$  lie in the open left half of complex plane, i.e. the exponential stability of the delay system (2.1) is controlled by the location of the spectrum of its quasi-polynomial matrix. In general, this is not the case if the finite dimension assumption is dropped. The spectrum and the resolvent of  $\mathcal{A}$  are described by

**Proposition 2.2.** [18]

We have  $\lambda \in \rho(\mathcal{A})$  if and only if  $\lambda \in \rho(P(\lambda))$ . In this case, the resolvent of  $\mathcal{A}$  is given by

$$R(\lambda; \mathcal{A}) = E_{\lambda} R(\lambda; P(\lambda)) H_{\lambda} F + T_{\lambda},$$

where

$$E_\lambda \in \mathcal{L}(X, \mathcal{X}), H_\lambda \in \mathcal{L}(\mathcal{X}, X), F \in \mathcal{L}(\mathcal{X}, \mathcal{X}), T_\lambda \in \mathcal{L}(\mathcal{X}, \mathcal{X})$$

are defined by

$$E_\lambda x := (x, e^{\lambda \cdot} x),$$

$$H_\lambda(x, f) := x + \int_{-h}^0 e^{\lambda s} f(s) ds,$$

$$F(x, f) := \left( x, \sum_{k=1}^m \mathcal{X}_{[-h_k, 0]}(\cdot) A_k f(-h_k - \cdot) \right),$$

$$T_\lambda(x, f) := \left( 0, \int_{\cdot}^0 e^{\lambda(\cdot-s)} f(s) ds \right).$$

**Definition 2.4.** [18] (*Spectral Set, Resolvent Set, Spectral Bound of  $P(\cdot)$* )

The spectral set, the resolvent set, and the spectral bound of quasi-polynomial operator  $P(\cdot)$  are respectively defined by

$$\sigma(P(\cdot)) := \{\lambda : \lambda \in \sigma(P(\lambda))\},$$

$$\rho(P(\cdot)) := \mathbb{C} \setminus \sigma(P(\lambda)),$$

$$s(P(\cdot)) := \sup \{\operatorname{Re}(\lambda) : \lambda \in \sigma(P(\cdot))\}.$$

**Remark 2.1.**

From the above proposition, we have  $\rho(\mathcal{A}) = \rho(P(\cdot))$ , hence  $s(\mathcal{A}) = s(P(\cdot))$ . So, if  $A$  generates a positive  $C_0$ -semi-group, then the delay system (2.1) is exponentially stable if and only if  $s(P(\cdot)) < 0$ .

Next, we present some results on an extension of Perron-Frobenius Theorem to quasi-polynomial operator (2.2).

### 2.3.4 Perron-Frobenius Theorem of quasi-polynomial operator

**Definition 2.5.** [3] (*Positive Quasi-Polynomial Operator*)

The quasi-polynomial operator (2.2) is called **positive** if

$A_0$  generates a positive  $C_0$ -semi-group and  $A_k \in \mathcal{L}^+(X)$  for all  $k \in \{1, 2, \dots, m\}$ .

**Remark 2.2.** [3]

If the quasi-polynomial operator (2.2) is positive, then the delay system (2.1) is a positive system.

By [18, Theorem 3.3'], if  $A_0$  generates a positive  $C_0$ -semi-group on a Banach lattice  $X_{\mathbb{C}}$  and the operators  $A_k \in \mathcal{L}^+(X_{\mathbb{C}})$  for all  $k \in \{1, 2, \dots, m\}$ , then the semi-group  $(\mathcal{T}(t))_{t \geq 0}$  is a positive  $C_0$ -semi-group.

Using Proposition 2.2, we obtain the following result:

**Proposition 2.3.** [3]

Let the quasi-polynomial operator (2.2) be positive. Then for  $\lambda_1, \lambda_2 \in \mathbb{R}$ , the following statements are equivalent

1.  $R(\lambda_1, P(\lambda_1)) \succeq R(\lambda_2, P(\lambda_2)) \succeq 0$ ,
2.  $R(\lambda_1, \mathcal{A}) \succeq R(\lambda_2, \mathcal{A}) \succeq 0$ .

The next theorem is a generalization of results in [38] to Perron-Frobenius theorem for positive quasi polynomial operator in Banach spaces.

**Theorem 2.1.** [3] (*Perron-Frobenius Theorem for Positive Quasi Polynomial Operator*)

Let the quasi-polynomial operator (2.2) be positive, then

1.  $s(P(\cdot)) \in \sigma(P(\cdot))$ ,
2. For  $\lambda \in \mathbb{R}$  we have  $R(\lambda, P(\lambda)) \in \mathcal{L}^+(X)$  if and only if  $\lambda > s(P(\cdot))$ ,
3.  $R(\lambda_1, P(\lambda_1)) \succeq R(\lambda_2, P(\lambda_2))$  for  $\lambda_2 \geq \lambda_1 > s(P(\cdot))$ .

**Theorem 2.2.** [3]

Let the delay system (2.1) be positive, then the following statements are equivalent

1. The delay system (2.1) is exponentially stable,
2.  $s(A_0 + A_1 + \dots + A_m) < 0$ ,
3.  $s(A_0) < 0$  and  $r(-A_0^{-1}(A_1 + \dots + A_m)) < 1$ ,
4.  $(-A_0 - A_1 - \dots - A_m)^{-1} \succeq 0$ .

*Proof.*

Assume that the delay system (2.1) is exponentially stable, then since  $\mathcal{A}$  generates a positive  $C_0$ -semi-group, this equivalent to

$$s(\mathcal{A}) = s(P(\cdot)) < 0.$$

By Theorem 2.1 assertion (2), the previous statement happens

for  $\lambda \in \mathbb{R}$ , we have  $R(\lambda, P(\lambda)) \in \mathcal{L}^+(X_C)$  if and only if  $\lambda > s(\mathcal{A})$ ,

taken  $\lambda = 0$  we get

$$R(0, P(0)) = (-A_0 - A_1 - \dots - A_m)^{-1} \text{ exists and positive,}$$

thus (1)  $\Leftrightarrow$  (4).

Moreover, the operator  $A_0 + A_1 + \dots + A_m$  generates a positive  $C_0$ -semi-group, by Corollary 1.12 assertion (1) thus by Theorem 1.26 assertion (3),

$$R(t, A_0 + A_1 + \dots + A_m) \succeq 0$$

if and only if

$$t > s(A_0 + A_1 + \dots + A_m) \text{ and } (-A_0 - A_1 - \dots - A_m)^{-1} \succeq 0,$$

for  $t = 0$ , if and only if

$$0 > s(A_0 + A_1 + \dots + A_m),$$

thus (4)  $\Leftrightarrow$  (2).

Now, we show that (2)  $\Rightarrow$  (3). Using Theorem 1.26 assertion (1), we also get

$$s(A_0) = t_0 - [r(R(t_0, A_0))]^{-1}, \forall t_0 > s(A_0), \quad (2.3)$$

$$s(A_0 + A_1 + \dots + A_m) = t_0 - [r(R(t_0, A_0 + A_1 + \dots + A_m))]^{-1},$$

for all  $t_0 > s(A_0 + A_1 + \dots + A_m)$ .

Moreover, for  $t_0$  large enough, we have

$$R(t_0, A_0 + A_1 + \dots + A_m) = R(t_0, A_0) \sum_{n=0}^{\infty} [R(t_0, A_0)(A_0 + A_1 + \dots + A_m)]^n,$$

So, we get

$$R(t_0, A_0 + A_1 + \dots + A_m) \succeq R(t_0, A_0) \succeq 0,$$

this implies

$$r[(R(t_0, A_0 + A_1 + \dots + A_m))] \geq r[R(t_0, A_0)],$$

hence, from (2.3) we have

$$s(A_0) \leq s(A_0 + A_1 + \dots + A_m),$$

then (2) implies

$$s(A_0) < 0,$$

and by the same argument we also obtain

$$s(A_0 + \frac{1}{t}(A_1 + \dots + A_m)) < 0 \text{ for all } t \geq 1,$$

moreover,

$$tI - (-A_0)^{-1}(A_0 + A_1 + \dots + A_m) = tA_0^{-1}(A_0 + \frac{1}{t}(A_0 + A_1 + \dots + A_m)),$$

which implies

$$[1, \infty) \subset \rho \left( (-A_0)^{-1} (A_0 + A_1 + \dots + A_m) \right),$$

by Theorem 1.25 assertion (1), we obtain

$$r \left[ (-A_0)^{-1} (A_0 + A_1 + \dots + A_m) \right] < 1.$$

The remains (3)  $\Rightarrow$  (4) become from the following equation

$$-A_0 - A_1 - \dots - A_m = (-A_0) \left[ I - (-A_0)^{-1} (A_1 + \dots + A_m) \right]. \quad (2.4)$$

We have for any  $k \in \{1, 2, \dots, m\}$ , the operators  $A_k \in \mathcal{L}^+(X_{\mathbb{C}})$ , then

$$A_0 + A_1 + \dots + A_m \in \mathcal{L}^+(X_{\mathbb{C}}),$$

and by Theorem 1.26 assertion (3), we obtain

$A_0$  is a Metzler operator on  $X_{\mathbb{C}}$  and generates a positive  $C_0$ -semi-group, then

$$R(\lambda, A_0) \succeq 0 \text{ if and only if } t > s(A_0),$$

and we have

$$s(A_0) < 0,$$

taken  $\lambda = 0$ , we get

$$R(0, A_0) = (-A_0)^{-1} \succeq 0,$$

then

$$(-A_0)^{-1} (A_1 + \dots + A_m) \succeq 0.$$

We have

$$r \left[ (-A_0)^{-1} (A_1 + \dots + A_m) \right] < 1,$$

by Theorem 1.25 this is equivalent to

$$R(\lambda, (-A_0)^{-1}(A_1 + \dots + A_m)] \succeq 0,$$

by taken  $\lambda = 1$  we get

$$R(1, (-A_0)^{-1}(A_1 + \dots + A_m)] \succeq 0,$$

then

$$(I - (-A_0)^{-1}(A_1 + \dots + A_m))^{-1} \succeq 0,$$

and with (2.4), we obtain

$$(-A_0 - A_1 - \dots - A_m)^{-1} = [I - (-A_0)^{-1}(A_1 + \dots + A_m)]^{-1}(-A_0)^{-1},$$

then

$$(-A_0 - A_1 - \dots - A_m)^{-1} \succeq 0,$$

as a composition of tow positive operators. □

From the above Theorem, it is easy to see that the stability of positive linear delay systems does not depend on delay parameters and that  $A_0$  generates an exponential stable  $C_0$ -semi-group, is the necessary condition for exponential stability of the positive system (2.1).

**Remark 2.3.** [17]

*Suppose that  $A$  is the generator of a positive  $C_0$ -semi-group  $(T(t))_{t \geq 0}$  on a Banach lattice  $X_{\mathbb{C}}$ , that is,  $T(t) \in \mathcal{L}^+(X_{\mathbb{C}})$ , for all  $t \in \mathbb{R}_+$ , then,*

$$\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > s(A)\} \subset \rho(A) \text{ and } R(t, A) \text{ is positive if and only if } t > s(A).$$

*Therefore,*

$$A \text{ is a Metzler operator on } X_{\mathbb{C}}.$$

*Conversely, if  $A$  is a Metzler operator and  $\operatorname{Int} X_{\mathbb{C}}^+ \neq \emptyset$ , then*

$$A \text{ is the generator of a positive } C_0\text{-semi-group on } X_{\mathbb{C}} \text{ and } s(A) = \omega_0(A).$$

**Theorem 2.3.** [14]

Let  $(A, D(A))$  be the generator of a  $C_0$ -semi-group  $(T(t))_{t \geq 0}$  on a Banach space  $X$ , satisfying  $\|S(t)\|_{\mathcal{L}(X)} < Me^{\omega t}$ , for all  $t \geq 0$  and some  $\omega \in \mathbb{R}, M > 1$ .

If

$$B \in \mathcal{L}(X),$$

then

$$C := A + B \text{ with } D(C) = D(A) \text{ generates a } C_0\text{-semi-group } (S(t))_{t \geq 0}$$

satisfying

$$\|S(t)\|_{\mathcal{L}(X)} < Me^{(\omega + M\|B\|)t}, \text{ for all } t \geq 0.$$

**Proposition 2.4.** [42]

let  $A$  be the infinitesimal generator of a compact  $C_0$ -semi-group  $(T(t))_{t \geq 0}$ , let  $B$  be a bounded operator, then  $A + B$  is the infinitesimal generator of a compact  $C_0$ -semi-group  $(S(t))_{t \geq 0}$ ,

**Lemma 2.2.** [35]

Assume that  $A$  is the generator of a positive compact  $C_0$ -semi-group  $(T(t))_{t \geq 0}$  on a Banach lattice  $X_{\mathbb{C}}$ , and  $P \in \mathcal{L}(X_{\mathbb{C}}), Q \in \mathcal{L}^+(X_{\mathbb{C}})$ .

If

$$|Px| \preceq Q|x| \text{ for all } x \in X_{\mathbb{C}},$$

then

$$\omega(A + P) = s(A + P) \leq s(A + Q) = \omega(A + Q).$$

# Chapter 3

## ROBUST STABILITY

### 3.1 ROBUSTNESS STABILITY OF TIME DELAY SYSTEMS

Robust stability is the essential characteristic of a time-delay differential system that allows it to maintain its stability property even when facing real-world uncertainties or perturbations. Because these systems are mathematically infinite-dimensional as their behavior depends on a history of functions, not just the current state, the semi-group method is used to analyze their evolution in time. Within this framework, a system is nominally stable if its solutions form a strongly continuous semi-group that exhibits exponential stability. Robustness analysis then focuses on determining the largest degree or type of perturbation—which can be structured or even have an unbounded structure—that the system can absorb before the semi-group’s exponential stability breaks down. This method thus quantifies the system’s margin against fragility.

### 3.2 PERTURBED DELAY SYSTEM

#### 3.2.1 Description of the Perturbed Delay System

Assume that the delay system (2.1) is subjected to time varying structured perturbations as follows

$$A_0 \rightsquigarrow A_0 + D_0 \Delta_0 E_0,$$

$$A_k \rightsquigarrow A_k + D_k(t) \Delta_k(t) E_k(t), k \in \{1, 2, \dots, m\}.$$

Hence, the perturbed delay system takes the form

$$\dot{x}(t) = (A_0 + D_0 \Delta_0 E_0) x(t) + \sum_{k=1}^m (A_k + D_k(t) \Delta_k(t) E_k(t)) x(t - h_k), t \geq \sigma, \quad (3.1)$$

where

1.  $X_{\mathbb{C}}, Y_{k,\mathbb{C}}$  and  $U_{k,\mathbb{C}}$  are complex Banach lattices.
2.  $A_0$  is the infinitesimal generator of an exponentially stable semigroup  $(S(t))_{t \geq 0}$ .
3.  $D_0 \in \mathcal{L}^+(U_{0,\mathbb{C}}, X_{\mathbb{C}})$ ,  $E_0 \in \mathcal{L}^+(X_{\mathbb{C}}, Y_{0,\mathbb{C}})$ .
4.  $\Delta_0 \in \mathcal{L}^+(Y_{0,\mathbb{C}}, U_{0,\mathbb{C}})$  is unknown bounded time-invariant perturbation operator.  
And for any  $k \in \{1, 2, \dots, m\}$ ,
5.  $D_k(\cdot) \in PC_b(\mathbb{R}, \mathcal{L}^{|\cdot|}(U_{k,\mathbb{C}}, X_{\mathbb{C}}))$  and  $E_k(\cdot) \in PC_b(\mathbb{R}, \mathcal{L}^{|\cdot|}(X_{\mathbb{C}}, Y_{k,\mathbb{C}}))$  are given.
6.  $\Delta_k(\cdot) \in PC_b(\mathbb{R}, \mathcal{L}^{|\cdot|}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}}))$  are unknown bounded time-varying perturbation operators.

The size of the perturbation operator  $\Delta_k(\cdot)$  is measured by

$$\|\Delta_k(\cdot)\|_{\infty} = \operatorname{ess\,sup}_{t \geq 0} \|\Delta_k(t)\|_{\mathcal{L}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})}.$$

### 3.2.2 Existence and Uniqueness of Solution

**Theorem 3.1. (*Existence of Unique Solution of (3.1)*)**

*Under the above assumptions, for a fixed  $\sigma \geq 0$  and a given  $f \in L^p([-h, 0]; X_{\mathbb{C}})$ , the perturbed delay system (3.1) has a unique mild solution  $x(\cdot; \sigma, f) \in L_{loc}^p([-h, \infty); X_{\mathbb{C}})$  with the initial value condition*

$$x(s + \sigma) = f(s), s \in [-h, 0]. \quad (3.2)$$

*Proof.*

In order to apply [36, Theorem 2.1], we need only to verify that  $u \in L_{Loc}^p([\sigma, \infty); X_{\mathbb{C}})$ , where  $u$  is given by

$$u(t) = \sum_{k=1}^m (D_k(t) \Delta_k(t) E_k(t)) x(t - h_k), t \geq \sigma.$$

For  $T \geq \sigma$ , we define the operator

$$K_T : L^p([\sigma - h, T]; X_{\mathbb{C}}) \rightarrow L^p([\sigma, T]; X_{\mathbb{C}}),$$

by

$$K_T x(s) = \sum_{k=1}^m (D_k(s) \Delta_k(s) E_k(s)) x(s - h_k), \text{ a.e., } s \in [\sigma, T].$$

Thus,

$$\begin{aligned} \|K_T\|_{L^p([\sigma, T]; X_{\mathbb{C}})} &= \left\| \sum_{k=1}^m (D_k(s) \Delta_k(s) E_k(s)) x(s - h_k) \right\|_{L^p([\sigma, T]; X_{\mathbb{C}})}, \\ &\leq \sum_{k=1}^m \|D_k(\cdot)\|_{\mathcal{L}(U_{k, \mathbb{C}}, X_{\mathbb{C}})} \|\Delta_k(\cdot)\|_{\mathcal{L}(Y_{k, \mathbb{C}}, U_{k, \mathbb{C}})} \\ &\quad \times \|E_k(\cdot)\|_{\mathcal{L}(X_{\mathbb{C}}, Y_{k, \mathbb{C}})} \|x(s - h_k)\|_{L^p([\sigma, T]; X_{\mathbb{C}})}, \\ &\leq \sum_{k=1}^m \|D_k(\cdot)\|_{\mathcal{L}(U_{k, \mathbb{C}}, X_{\mathbb{C}})} \|\Delta_k(\cdot)\|_{\mathcal{L}(Y_{k, \mathbb{C}}, U_{k, \mathbb{C}})} \\ &\quad \times \|E_k(\cdot)\|_{\mathcal{L}(X_{\mathbb{C}}, Y_{k, \mathbb{C}})} \left( \int_{\sigma}^T |x(s - h_k)|^p ds \right)^{1/p}, \\ &\leq \sum_{k=1}^m \|D_k(\cdot)\|_{\mathcal{L}(U_{k, \mathbb{C}}, X_{\mathbb{C}})} \|\Delta_k(\cdot)\|_{\mathcal{L}(Y_{k, \mathbb{C}}, U_{k, \mathbb{C}})} \\ &\quad \times \|E_k(\cdot)\|_{\mathcal{L}(X_{\mathbb{C}}, Y_{k, \mathbb{C}})} \left( \int_{\sigma-h}^T |x(\tau)|^p d\tau \right)^{1/p}, \\ &\leq \sum_{k=1}^m \|D_k(\cdot)\|_{\mathcal{L}(U_{k, \mathbb{C}}, X_{\mathbb{C}})} \|\Delta_k(\cdot)\|_{\mathcal{L}(Y_{k, \mathbb{C}}, U_{k, \mathbb{C}})} \\ &\quad \times \|E_k(\cdot)\|_{\mathcal{L}(X_{\mathbb{C}}, Y_{k, \mathbb{C}})} \|x(\cdot)\|_{L^p([\sigma-h, T]; X_{\mathbb{C}})}, \end{aligned}$$

for  $1 \leq p < \infty$ , and the inequality is also true for  $p = \infty$ , then the operator  $K_T$  is bounded. Thus,

$$u : s \rightarrow \sum_{k=1}^m (D_k(s) \Delta_k(s) E_k(s)) x(s - h_k)$$

is in  $L^p([\sigma, T]; X_{\mathbb{C}})$  for any  $T \geq \sigma$ , then  $u \in L_{loc}^p([\sigma, \infty); X_{\mathbb{C}})$ .

Therefore, for a fixed  $\sigma \geq 0$ , given  $f \in L^p([-h, 0]; X_{\mathbb{C}})$  and  $u \in L_{loc}^p([\sigma, \infty); X_{\mathbb{C}})$ , the perturbed delay system (3.1) has a unique mild solution  $x(\cdot; \sigma, f) \in L_{loc}^p([-h, \infty); X_{\mathbb{C}})$ .

Recall that  $x(·; \sigma, f)$  is continuous on  $[\sigma, \infty)$  and exponentially bounded and satisfies (3.1) for any  $t \in [\sigma, \infty)$ .  $\square$

### 3.3 ROBUST STABILITY

#### Definition 3.1. (*Exponential Stability*)

The perturbed delay system (3.1) is said to be exponentially stable if there exist  $M > 0$  and  $\beta > 0$  such that for any  $f \in L^p([-h, 0]; X_{\mathbb{C}})$  and for any  $t \geq \sigma \geq 0$ ,

$$\|x(t; \sigma, f)\|_{X_{\mathbb{C}}} < Me^{-\beta(t-\sigma)} \|(x_0, f)\|_{X_{\mathbb{C}} \times L^p([-h, 0]; X_{\mathbb{C}})}.$$

#### 3.3.1 First Main Result

The **first main result** of this research is presented in the following theorem.

#### Theorem 3.2.

Let the delay system (2.1) be exponentially stable, where  $A_0$  is a Metzler operator on  $X_{\mathbb{C}}$  with  $\text{Int}X_{\mathbb{C}}^+ \neq \emptyset$ , which generates a compact  $C_0$ -semigroup.

For  $k \in \{1, 2, \dots, m\}$ , assume that

1.  $A_k \in \mathcal{L}^+(X_{\mathbb{C}})$ ,
2. There exist  $D_k \in \mathcal{L}^+(U_{k, \mathbb{C}}, X_{\mathbb{C}})$ ,  $E_k \in \mathcal{L}^+(X_{\mathbb{C}}, Y_{k, \mathbb{C}})$  and  $\Delta_k \in \mathcal{L}^+(Y_{k, \mathbb{C}}, U_{k, \mathbb{C}})$  such that

$$\|D_k(t)\| \leq D_k, \|E_k(t)\| \leq E_k \text{ and } \|\Delta_k(t)\| \leq \Delta_k \text{ for any } t \in \mathbb{R}^+.$$

Then, the perturbed system (3.1) remains exponentially stable provided

$$\sum_{k=0}^m \|\Delta_k\| < \frac{1}{\max_{i,j \in \{0,1,2,\dots,m\}} \|E_i \left( -\sum_{k=0}^m A_k \right)^{-1} D_j\|}. \quad (3.3)$$

*Proof.*

The proof is in two steps.

#### Step 1.

By Remark 2.3, since  $A_0$  is a Metzler operator and  $\text{Int}X_{\mathbb{C}}^+ \neq \emptyset$ , then it generates a positive  $C_0$ -semi-group on  $X_{\mathbb{C}}$  and with  $A_k \in \mathcal{L}^+(X_{\mathbb{C}})$  for all  $k \in \{1, 2, \dots, m\}$ ,

by Definition 2.5, the quasi polynomial operator is positive, and by Remark 2.2 the delay system (2.1) is positive.

By proposition 1.12, since  $A_0$  generates a positive  $C_0$ -semi-group on  $X_{\mathbb{C}}$  and  $A_k \in \mathcal{L}^+(X_{\mathbb{C}})$  for all  $k \in \{1, 2, \dots, m\}$ , then the operator

$$A_0 + \sum_{k=1}^m A_k = \sum_{k=0}^m A_k \text{ generates a positive } C_0\text{-semi-group on } X_{\mathbb{C}}.$$

By proposition 2.4, since  $A_0$  generates a compact  $C_0$ -semi-group on  $X_{\mathbb{C}}$  and  $A_k \in \mathcal{L}^+(X_{\mathbb{C}})$  for all  $k \in \{1, 2, \dots, m\}$ , then the operator

$$A_0 + \sum_{k=0}^m A_k = \sum_{k=0}^m A_k \text{ generates a compact } C_0\text{-semigroup on } X_{\mathbb{C}}.$$

it follows that  $\sum_{k=0}^m A_k$  generates a positive compact  $C_0$ -semi-group on  $X_{\mathbb{C}}$ .

Let  $t \geq 0$ , for any  $k \in \{1, 2, \dots, m\}$ ,

$$D_k(t) \in \mathcal{L}^{|\cdot|}(U_{k,\mathbb{C}}, X_{\mathbb{C}}), E_k(t) \in \mathcal{L}^{|\cdot|}(X_{\mathbb{C}}, Y_{k,\mathbb{C}}), \Delta_k(t) \in \mathcal{L}^{|\cdot|}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}}),$$

then, the operator

$$D_k(t) E_k(t) \Delta_k(t) \in \mathcal{L}^{|\cdot|}(X_{\mathbb{C}}) \subset \mathcal{L}(X_{\mathbb{C}}),$$

and

$$D_k \in \mathcal{L}^+(U_{k,\mathbb{C}}, X_{\mathbb{C}}), E_k \in \mathcal{L}^+(X_{\mathbb{C}}, Y_{k,\mathbb{C}}), \Delta_k \in \mathcal{L}^+(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}}),$$

gives

$$\sum_{k=0}^m D_k \Delta_k E_k \in \mathcal{L}^+(X_{\mathbb{C}})$$

and with the assumption

$$|D_k(t)| \leq D_k, |E_k(t)| \leq E_k, \text{ and } |\Delta_k(t)| \leq \Delta_k.$$

We have

$$\left| \left( D_0 \Delta_0 E_0 + \sum_{k=1}^m D_k(t) \Delta_k(t) E_k(t) \right) x \right| \leq \sum_{k=0}^m D_k \Delta_k E_k |x| \text{ for all } x \in X_{\mathbb{C}}.$$

By Lemma 2.2, it follows that

$$s \left( A_0 + D_0 \Delta_0 E_0 + \sum_{k=1}^m A_k + D_k(t) \Delta_k(t) E_k(t) \right) < s \left( \sum_{k=0}^m A_k + D_k \Delta_k E_k \right).$$

**Step 2.**

We prove that  $s \left( \sum_{k=0}^m A_k + D_k \Delta_k E_k \right) < 0$ .

We have, the operator  $\sum_{k=0}^m A_k$  generates a positive  $C_0$ -semigroup on  $X_{\mathbb{C}}$  and by Remark 2.3, it is a Metzler operator, and since the operators  $D_k, \Delta_k$  and  $E_k$ , for any  $k \in \{1, 2, \dots, m\}$ , are positives, then the operator

$$\sum_{k=0}^m A_k + D_k \Delta_k E_k$$

generates a positive  $C_0$ -semigroup on  $X_{\mathbb{C}}$ .

Again by Remark 2.3, the operator

$$\sum_{k=0}^m A_k + D_k \Delta_k E_k$$

is a Metzler operator.

Now, by Theorem 1.26 assertion (1), we have

$$s \left( \sum_{k=0}^m A_k + D_k \Delta_k E_k \right) \in \sigma \left( \sum_{k=0}^m A_k + D_k \Delta_k E_k \right).$$

Set

$$\mu_0 = s \left( \sum_{k=0}^m A_k + D_k \Delta_k E_k \right).$$

We prove that  $\mu_0 < 0$ . Assume the contrary, since

$$\mu_0 \in \sigma \left( \sum_{k=0}^m A_k + D_k \Delta_k E_k \right),$$

there exist  $U_0 \in X_{\mathbb{C}}$  and  $U_0 \neq 0$  such that

$$\left( \sum_{k=0}^m A_k + D_k \Delta_k E_k \right) U_0 = \mu_0 U_0. \quad (3.4)$$

Let

$$Q(t) = tI - \sum_{k=0}^m A_k, \quad t \in \mathbb{R}.$$

By Theorem 2.2, since the delay system (2.1) is exponentially stable this is equivalent to

$$s \left( \sum_{k=0}^m A_k \right) < 0.$$

Using the fact that

$$s \left( \sum_{k=0}^m A_k \right) = \sup \left\{ \operatorname{Re} \lambda : \lambda \in \sigma \left( \sum_{k=0}^m A_k \right) \right\} < 0 \leq \mu_0,$$

it follows that

$$\mu_0 \in \rho \left( \sum_{k=0}^m A_k \right)$$

thus,  $Q(\mu_0)^{-1}$  exists.

Now, Equation (3.4) implies

$$Q(\mu_0)^{-1} \left( \sum_{k=0}^m D_k \Delta_k E_k \right) U_0 = U_0. \quad (3.5)$$

Let  $i_0$  such that

$$\|E_{i_0}U_0\| = \max_{i \in \{0,1,\dots,m\}} \|E_iU_0\|.$$

It follows from equation (3.5) that

$$\|E_{i_0}U_0\| > 0.$$

Multiply both sides of equation (3.5) from the left by  $E_{i_0}$  to get

$$E_{i_0}Q(\mu_0)^{-1} \left( \sum_{k=0}^m D_k \Delta_k E_k \right) U_0 = E_{i_0}U_0,$$

which implies that

$$\begin{aligned} \|E_{i_0}U_0\| &\leq \sum_{k=0}^m \|E_{i_0}Q(\mu_0)^{-1} D_k\| \|\Delta_k\| \|E_kU_0\|, \\ &\leq \left( \sum_{k=0}^m \|E_{i_0}Q(\mu_0)^{-1} D_k\| \|\Delta_k\| \right) \|E_{i_0}U_0\|, \\ &\leq \left( \max_{0 \leq i,j \leq m} \|E_iQ(\mu_0)^{-1} D_j\| \sum_{k=0}^m \|\Delta_k\| \right) \|E_{i_0}U_0\|, \\ &\leq \max_{0 \leq i,j \leq m} \|E_iQ(\mu_0)^{-1} D_j\| \left( \sum_{k=1}^m \|\Delta_k\| \right) \|E_{i_0}U_0\|. \end{aligned}$$

Therefore,

$$\max_{0 \leq i,j \leq m} \|E_iQ(\mu_0)^{-1} D_j\| \left( \sum_{k=0}^m \|\Delta_k\| \right) \geq 1. \quad (3.6)$$

The resolvent identity

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu = (\lambda - \mu) R_\mu R_\lambda,$$

gives

$$Q(0)^{-1} - Q(\mu_0)^{-1} = \mu_0 Q(0)^{-1} Q(\mu_0)^{-1}.$$

Since the operator  $\sum_{k=0}^m A_k$  is a Metzler operator with

$$s \left( \sum_{k=0}^m A_k \right) < 0, \mu_0 > 0,$$

it follows from Theorem 1.26, assertion (3), that

$$Q(0)^{-1} \succeq 0, Q(\mu_0)^{-1} \succeq 0 \text{ if and only if } 0 > s \left( \sum_{k=0}^m A_k \right).$$

Therefore, by Theorem 1.26 assertion (2)

$$Q(0)^{-1} \succeq Q(\mu_0)^{-1} \succeq 0,$$

which yields to

$$E_i Q(0)^{-1} D_j \succeq E_i Q(\mu_0)^{-1} D_j \succeq 0, \text{ for any } 0 \leq i, j \leq m,$$

by lattice norm property (1.3), we get

$$\|E_i Q(0)^{-1} D_j\| \geq \|E_i Q(\mu_0)^{-1} D_j\|, \text{ for any } 0 \leq i, j \leq m,$$

and since

$$\|E_i Q(\mu_0)^{-1} D_j\| \leq \max_{0 \leq i, j \leq m} \|E_i Q(0)^{-1} D_j\|,$$

then

$$\|E_i Q(\mu_0)^{-1} D_j\| \left( \sum_{k=0}^m \|\Delta_k\| \right) \leq \max_{0 \leq i, j \leq m} \|E_i Q(0)^{-1} D_j\| \left( \sum_{k=0}^m \|\Delta_k\| \right),$$

from (3.6), we get

$$1 \leq \max_{0 \leq i, j \leq m} \|E_i Q (\mu_0)^{-1} D_j\| \left( \sum_{k=0}^m \|\Delta_k\| \right) \leq \max_{0 \leq i, j \leq m} \|E_i Q (0)^{-1} D_j\| \left( \sum_{k=0}^m \|\Delta_k\| \right).$$

Thus,

$$\sum_{k=0}^m \|\Delta_k\| \geq \frac{1}{\max_{i, j \in \{0, 1, 2, \dots, m\}} \|E_i \left( - \sum_{k=0}^m A_k \right)^{-1} D_j\|}.$$

However, this conflicts with (3.3), which conclude the proof.  $\square$

**Corollary 3.1.**

*Let the linear delay differential system*

$$\dot{x}(t) = (A + A_0) x(t) + \sum_{k=1}^m A_k(t) x(t - h_k), t \geq 0,$$

*where  $A$  is the generator of a  $C_0$ -semigroup, Hurwitz stable and for all  $k \in \{1, 2, \dots, m\}$ ,  $A_k(\cdot) \in PC(\mathbb{R}_+, \mathcal{L}^+(X_{\mathbb{C}}))$  and suppose that there exist  $A_k \in \mathcal{L}^+(X_{\mathbb{C}})$  such that  $|A_k(t)| \preceq A_k$  for any  $t \in \mathbb{R}_+$ , then the system is exponentially stable provided*

$$\sum_{k=0}^m \|A_k\| < \frac{1}{\|A^{-1}\|}.$$

# Chapter 4

## STABILITY RADII

### 4.1 HISTORY OF QUANTIFYING ROBUSTNESS IN STABILITY

The concept of stability radius and its evolution over the years is a significant topic in Control Theory and Applied Mathematics. It was developed to quantify the robustness of a system's stability against perturbations.

In the 1980s, the concept of a stability radius was formally introduced in a shaping 1986 paper by D. Hinrichsen and A.J. Pritchard. They defined it as a scalar measure representing the smallest change to any system parameter that would lead to instability. The idea was to move beyond simple stability analysis and provide a single value to compare the robustness of different systems or the same system under varying parameters. Initially, researchers distinguished between the complex stability radius and the real stability radius. The complex stability radius was found to be easier to analyze and compute than its real counterpart. Early work focused on linear time-invariant systems with unstructured and structured perturbations. A key result was a formula for computing the complex unstructured stability radius.

A major breakthrough occurred in the mid-1990s with the development of a general formula for the real stability radius. This formula, while more complex than the one for the complex radius, reduced the computation to a global optimization problem. The concept was applied to positive systems—systems whose state variables are always non-negative. For this class of systems, it was discovered that the stability analysis becomes much simpler. A crucial finding was that if the matrices defining the perturbation structure are non-negative, the real and complex stability radii are equal and can be determined by a simple formula. The stability radius was extended to more complex systems, including those with time delays and infinite-dimensional systems. This work showed that for pos-

itive linear time-delay systems, stability is independent of the delays and that the real, complex, and positive stability radii all coincide. In the 2000s, the concept was extended to time-varying linear systems and systems with stochastic perturbations. The framework was generalized to study the stability of Metzler operators on Banach lattices, providing a way to analyze infinite-dimensional positive systems. This research further solidified the finding that for Hurwitz stable Metzler operators and positive perturbations, the various stability radii are equal.

More recent work has explored the robust stability of positive linear time-delay systems under non-linear and time-varying perturbations, providing new bounds and stability criteria.

## 4.2 DESCRIPTION OF THE DOMINANT SYSTEM

For fixed  $\sigma \geq 0$  and given  $f \in L^p([-h, 0], X_{\mathbb{C}})$ , let

$$x(t) := x(t; \sigma, f), t \in [\sigma - h, \infty)$$

be the solution of the delay system (3.1) with the initial value condition (3.2).

Denote  $y(\cdot) = y(\cdot, |f|)$  the solution of

$$\begin{cases} \dot{y}(t) = (A_0 + D_0 \Delta_0 E_0) y(t) + \sum_{k=1}^m (A_k + D_k \Delta_k E_k) y(t - h_k), t \geq \sigma, \\ y(t) = |f|(t), t \in [-h, 0], \end{cases} \quad (4.1)$$

where

$A_k, D_k, E_k$  and  $\Delta_k$  for any  $k \in \{1, 2, \dots, m\}$  are defined as in Theorem 3.2,

and

$$|f|(t) := |f(t)|, \text{ for all } t \in [-h, 0] \text{ with } y(\sigma) = y_{\sigma}.$$

## 4.3 STABILITY RADII DEFINITION

For the positive perturbed delay system (4.1), we introduce definitions of complex, real and positive stability radii.

**Definition 4.1.** (*Complex, Real and Positive Stability Radii of (3.1)*)

For the perturbed delay system (3.1), we define complex, real and positive stability radii as follows

$$r_{\mathbb{C}} = \inf\{\|\Delta_0\| + \sum_{k=1}^m \|\Delta_k(\cdot)\|_{\infty} : \Delta_0 \in \mathcal{L}(Y_{0,\mathbb{C}}, U_{0,\mathbb{C}}), \Delta_k(\cdot) \in PC(\mathbb{R}^+; \mathcal{L}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})), k = 1, \dots, m \text{ and (3.1) is not exponentially stable}\}.$$

$$r_{\mathbb{R}} = \inf\{\|\Delta_0\| + \sum_{k=1}^m \|\Delta_k(\cdot)\|_{\infty} : \Delta_0 \in \mathcal{L}^{\mathbb{R}}(Y_{0,\mathbb{C}}, U_{0,\mathbb{C}}), \Delta_k(\cdot) \in PC(\mathbb{R}^+; \mathcal{L}^{\mathbb{R}}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})), k = 1, \dots, m \text{ and (3.1) is not exponentially stable}\}.$$

$$r_{+} = \inf\{\|\Delta_0\| + \sum_{k=1}^m \|\Delta_k(\cdot)\|_{\infty} : \Delta_0 \in \mathcal{L}^{+}(Y_{0,\mathbb{C}}, U_{0,\mathbb{C}}), \Delta_k(\cdot) \in PC(\mathbb{R}^+; \mathcal{L}^{+}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})), k = 1, \dots, m \text{ and (3.1) is not exponentially stable}\}.$$

respectively.

**Definition 4.2. (Complex, Real and Positive Stability Radii of (4.1))**

For the perturbed delay system (4.1), we define complex, real and positive stability radii as follows

$$\tilde{r}_{\mathbb{C}} = \inf\{\sum_{k=0}^m \|\Delta_k\| : \Delta_k \in \mathcal{L}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}}), k = 0, \dots, m \text{ and (4.1) is not exponentially stable}\}.$$

$$\tilde{r}_{\mathbb{R}} = \inf\{\sum_{k=0}^m \|\Delta_k\| : \Delta_k \in \mathcal{L}^{\mathbb{R}}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}}), k = 0, \dots, m \text{ and (4.1) is not exponentially stable}\}.$$

$$\tilde{r}_{+} = \inf\{\sum_{k=0}^m \|\Delta_k\| : \Delta_k \in \mathcal{L}^{+}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}}), k = 0, \dots, m \text{ and (4.1) is not exponentially stable}\}.$$

By convention,  $\inf \phi = \infty$ .

## 4.4 UPPER BOUND OF THE COMPLEX STABILITY RADIUS

### 4.4.1 Second Main Result

The second main result, an upper bound for the complex stability radius, is stated in the following Proposition:

**Proposition 4.1.** (*Upper Bound for the Complex Stability Radius*)

*The complex stability radius of the perturbed delay system (3.1) has the upper bound*

$$r_{\mathbb{C}} \leq \frac{1}{\max_{i,j \in \{0,1,2,\dots,m\}} \|E_i \left( -\sum_{k=0}^m A_k \right)^{-1} D_j\|}.$$

*Proof.*

By the stability radii definitions, we have

$$r_{\mathbb{C}} \leq r_{\mathbb{R}} \leq r_{+}.$$

By [17, Appendix A.3], for  $\Delta_k(\cdot) \in PC(\mathbb{R}^+, \mathcal{L}^{|\cdot|}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}}))$ , we have

$$0 \preceq \Delta_k(\cdot) \preceq |\Delta_k|,$$

then for any  $t \geq 0$  and  $k = 0, \dots, m$

$$0 \preceq \Delta_k(t) \preceq |\Delta_k(t)| \preceq \Delta_k,$$

by lattice norm property (1.3), we get

$$\|\Delta_k(t)\|_{\mathcal{L}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})} \leq \| |\Delta_k(t)| \|_{\mathcal{L}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})} \leq \|\Delta_k\|,$$

with

$$\|\Delta_k(\cdot)\|_{\infty} = \operatorname{ess\,sup}_{t \geq 0} \|\Delta_k(t)\|_{\mathcal{L}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})},$$

thus,

$$\|\Delta_k(\cdot)\|_\infty \leq \|\Delta_k(\cdot)\|_\infty \leq \|\Delta_k\|_{\mathcal{L}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})},$$

then

$$\|\Delta_0\| + \sum_{k=1}^m \|\Delta_k(\cdot)\|_\infty \leq \|\Delta_0\| + \sum_{k=1}^m \|\Delta_k(\cdot)\|_\infty \leq \sum_{k=0}^m \|\Delta_k\|_{\mathcal{L}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})},$$

and with (3.3), we obtain

$$\|\Delta_0\| + \sum_{k=1}^m \|\Delta_k(\cdot)\|_\infty \leq \frac{1}{\max_{i,j \in \{0,1,2,\dots,m\}} \|E_i \left( -\sum_{k=0}^m A_k \right)^{-1} D_j\|},$$

then

$$r_{\mathbb{C}} \leq \frac{1}{\max_{i,j \in \{0,1,2,\dots,m\}} \|E_i \left( -\sum_{k=0}^m A_k \right)^{-1} D_j\|}.$$

□

## 4.5 FORMULA OF STABILITY RADIUS IN SOME SPECIAL CASES

### 4.5.1 Third Main Result

Now, we present the third main result of this research, we compute the stability radii of the perturbed delay system (3.1) in some special cases of perturbations.

#### Fist Special Case of Perturbations

We are interested in perturbed systems of the form

$$\dot{x}(t) = (A_0 + D_0 \Delta_0 E) x(t) + \sum_{k=1}^m (A_k + D_k(t) \Delta_k E) x(t - h_k), t \geq \sigma, \quad (4.2)$$

with the assumptions

1. For all  $k \in \{0, 1, \dots, m\}$ ,  $Y_{k,\mathbb{C}} = Y$ ,
2. For all  $k \in \{1, \dots, m\}$ ,  $D_k(\cdot) \in PC_b(\mathbb{R}^+, \mathcal{L}^{|\cdot|}(U_{k,\mathbb{C}}, X_{\mathbb{C}})) \cap PC_b(\mathbb{R}^+, \mathcal{L}^+(U_{k,\mathbb{C}}, X_{\mathbb{C}}))$ .

The formula of stability radius is given in the following Theorem.

**Theorem 4.1.**

Let  $A_0$  a positive operator on  $X_{\mathbb{C}}$  with  $\text{Int}X_{\mathbb{C}}^+ \neq \emptyset$ , which generates a compact  $C_0$ -semigroup, we assume that

1.  $A_k \in \mathcal{L}^+(X_{\mathbb{C}})$ , for all  $k \in \{1, 2, \dots, m\}$ ,
2. There exist  $D_k \in \mathcal{L}^+(U_{k,\mathbb{C}}, X_{\mathbb{C}})$ ,  $E \in \mathcal{L}^+(X_{\mathbb{C}}, Y_{k,\mathbb{C}})$  and  $\Delta_k \in \mathcal{L}^+(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})$  such that

$$|D_k(t)| \leq D_k, \text{ for all } t \in \mathbb{R}^+ \text{ and } k \in \{0, 1, 2, \dots, m\}.$$

If the delay system (2.1) with the initial value condition (3.2) is exponentially stable, then we have

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_+ = \frac{1}{\max_{j \in \{0, 1, 2, \dots, m\}} \|E \left( -\sum_{k=0}^m A_k \right)^{-1} D_j\|}. \quad (4.3)$$

*Proof.*

From definitions of complex, real and positive stability radii we have

$$r_{\mathbb{C}} \leq r_{\mathbb{R}} \leq r_+,$$

and by Theorem 3.2, we have

$$r_{\mathbb{C}} \geq \frac{1}{\max_{i, j \in \{0, 1, 2, \dots, m\}} \|E \left( -\sum_{k=0}^m A_k \right)^{-1} D_j\|},$$

we get the inequality

$$r_+ \geq \frac{1}{\max_{i,j \in \{0,1,2,\dots,m\}} \|E \left( - \sum_{k=0}^m A_k \right)^{-1} D_j\|},$$

we have to prove

$$r_+ \leq \frac{1}{\max_{i,j \in \{0,1,2,\dots,m\}} \|E \left( - \sum_{k=0}^m A_k \right)^{-1} D_j\|}.$$

Assume that there exists  $j_0 \in \{0, 1, \dots, m\}$  such that

$$\max_{j \in \{0,1,2,\dots,m\}} \|E \left( - \sum_{k=0}^m A_k \right)^{-1} D_j\| = \|E \left( - \sum_{k=0}^m A_k \right)^{-1} D_{j_0}\|.$$

If

$$\|E \left( - \sum_{k=0}^m A_k \right)^{-1} D_{j_0}\| = 0,$$

then

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_+ = +\infty$$

and equality (4.3) holds.

Now, assume that

$$\|E \left( - \sum_{k=0}^m A_k \right)^{-1} D_{j_0}\| > 0.$$

By Theorem 2.2, the delay system (2.1) be positive and exponentially stable equivalent to

$$s(A_0 + A_1 + \dots + A_m) < 0,$$

this implies that

$$R \left( 0, \sum_{k=0}^m A_k \right) = \left( - \sum_{k=0}^m A_k \right)^{-1} \succeq 0,$$

therefore,

$$E \left( - \sum_{k=0}^m A_k \right)^{-1} D_{j_0} \in \mathcal{L}^+(U_{j_0, \mathbb{C}}, Y_{j_0, \mathbb{C}}).$$

Let

$$0 < \epsilon < \|E \left( - \sum_{k=0}^m A_k \right)^{-1} D_{j_0}\|$$

and by (1.7), we can choose

$$u_{j_0} \in U_{j_0, \mathbb{C}}^+, \text{ with } \|u_{j_0}\| = 1$$

such that

$$\|E \left( - \sum_{k=0}^m A_k \right)^{-1} D_{j_0} u_{j_0}\| > \|E \left( - \sum_{k=0}^m A_k \right)^{-1} D_{j_0}\| - \epsilon.$$

We look for the disturbance that destabilize the perturbed delay system (4.2).

**The case of  $j_0 = 0$ .**

Set

$$y_0 = E \left( - \sum_{k=0}^m A_k \right)^{-1} D_{j_0} u \in Y_{j_0, \mathbb{C}}^+.$$

By [48, Theorem 39.3 page 249], there exists a positive

$$g_0 \in Y_{0, \mathbb{C}}^*, \|g_0\| = 1$$

satisfying

$$g_0(y_0) = \|y_0\| = \|E \left( - \sum_{k=0}^m A_k \right)^{-1} D_0 u_0\|,$$

we consider the operator

$$\Delta_0 : Y_{0,\mathbb{C}} \rightarrow U_{0,\mathbb{C}}$$

defined by

$$y \mapsto \Delta_0 y = \frac{g_0(y)}{\|E \left( -\sum_{k=0}^m A_k \right)^{-1} D_0 u_0\|} u_0,$$

we have

$$\Delta_0 \in \mathcal{L}^+(Y_{0,\mathbb{C}}, U_{0,\mathbb{C}}) \text{ and } \|\Delta_0\| = \frac{1}{\|E \left( -\sum_{k=0}^m A_k \right)^{-1} D_0 u_0\|},$$

set

$$x_0 = \left( -\sum_{k=0}^m A_k \right)^{-1} D_0 u_0,$$

then

$$Ex_0 = E \left( -\sum_{k=0}^m A_k \right)^{-1} D_0 u_0 = y_0,$$

and hence

$$\Delta_0 Ex_0 = \frac{g_0(Ex_0)}{\|E \left( -\sum_{k=0}^m A_k \right)^{-1} D_0 u_0\|} u_0 = \frac{g_0(y_0)}{\|E \left( -\sum_{k=0}^m A_k \right)^{-1} D_0 u_0\|} u_0 = u_0,$$

then  $x_0 \neq 0$ .

Moreover we have

$$x_0 = \left( -\sum_{k=0}^m A_k \right)^{-1} D_0 (\Delta_0 Ex_0),$$

or equivalently

$$\left( \sum_{k=0}^m A_k + D_0 \Delta_0 E \right) x_0 = 0.$$

i.e.  $\Delta_0 \in \mathcal{L}^+(Y_{0,\mathbb{C}}, U_{0,\mathbb{C}})$  and  $\left( A_0 + D_0 \Delta_0 E + \sum_{k=1}^m A_k \right) x_0 = 0,$

and  $x_0 \neq 0$  implies

$$0 \in \sigma \left( A_0 + D_0 \Delta_0 E + \sum_{k=1}^m A_k \right),$$

then the perturbed system (4.2) with  $\Delta_k(t) \equiv 0$  for all  $k \in \{1, 2, \dots, m\}$  is not exponentially stable, hence by definition of  $r_+$  we get

$$r_+ \leq \|\Delta_0\| + \sum_{k=1}^m \|\Delta_k(\cdot)\| \leq \sum_{k=0}^m \|\Delta_k\| = \|\Delta_0\|,$$

then

$$r_+ \leq \|\Delta_0\| = \frac{1}{\|E \left( -\sum_{k=0}^m A_k \right)^{-1} D_0 u_0\|} < \frac{1}{\|E \left( -\sum_{k=1}^m A_k \right)^{-1} D_0\| - \epsilon}.$$

**The case of  $j_0 = \{1, 2, \dots, m\}$ .**

We construct one rank positive destabilising perturbation. Let us consider the following two Banach spaces

$$\mathcal{U} = \prod_{k=1}^m U_k \text{ and } \mathcal{Y} = \prod_{k=1}^m Y_k$$

endowed with the norms

$$\|\mathbf{U}\| = \sum_{k=1}^m \|u_k\|, \mathbf{U} = (u_1, u_2, \dots, u_m) \in \mathcal{U} \text{ such that } u_k \in U_k \text{ with } k = 1, 2, \dots, m.$$

$$\|\mathbf{Y}\| = \sum_{k=1}^m \|y_k\|, \mathbf{Y} = (y_1, y_2, \dots, y_m) \in \mathcal{Y} \text{ such that } y_k \in Y_k \text{ with } k = 1, 2, \dots, m.$$

And let us, for all  $t \geq 0, x \in X_{\mathbb{C}}$  and  $\mathbf{U} = (u_1, u_2, \dots, u_m) \in \mathcal{U}$ , define the linear operators  $\mathcal{D}(\cdot)$  and  $\mathcal{E}$  by

$$\mathcal{E}x = \begin{pmatrix} E_1x \\ E_2x \\ \vdots \\ E_mx \end{pmatrix},$$

and

$$\mathcal{D}(\cdot)\mathbf{U} = (D_1(t), D_2(t), \dots, D_m(t)) \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \sum_{k=1}^m D_k(t) u_k,$$

such that

$$\mathcal{D}(\cdot) \in PC_b \left( \mathbb{R}_+, \prod_{k=1}^m \mathcal{L}(U_k, X_{\mathbb{C}}) \right)$$

and

$$\mathcal{E}(\cdot) \in PC_b \left( \mathbb{R}_+, \prod_{k=1}^m \mathcal{L}(X_{\mathbb{C}}, Y_k) \right),$$

and the block diagonal matrix operator

$$\Delta : \mathcal{Y} \rightarrow \mathcal{U}$$

such that

$$\Delta = \text{diag}(\Delta_1, \Delta_2, \dots, \Delta_m),$$

for all  $\Delta_k \in \mathcal{L}(Y_k, U_k)$  and  $\mathbf{Y} = (y_1, y_2, \dots, y_m)$ , by

$$\Delta \mathbf{Y} = \begin{pmatrix} \Delta_1 & 0 & \dots & 0 \\ 0 & \Delta_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \Delta_m \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} \Delta_1 y_1 \\ \Delta_2 y_2 \\ \vdots \\ \Delta_m y_m \end{pmatrix},$$

endowed with the norm

$$\|\Delta\| = \max_{k=1,\dots,m} \|\Delta_k\|.$$

For  $j_0 = \{1, 2, \dots, m\}$ , we can choose

$$u_{j_0} \in U_{j_0, \mathbb{C}}^+, \|u_{j_0}\| = 1$$

such that

$$\|E \left( -\sum_{k=0}^m A_k \right)^{-1} D_{j_0} u_{j_0}\| > \|E \left( -\sum_{k=0}^m A_k \right)^{-1} D_{j_0}\| - \epsilon,$$

Set

$$y_{j_0} = E \left( -\sum_{k=0}^m A_k \right)^{-1} D_{j_0} u_{j_0} \in Y_{j_0, \mathbb{C}}^+,$$

by [48, Theorem 39.3 page 249], there exists a positive

$$g_{j_0} \in Y_{1, \mathbb{C}}^*, \|g_{j_0}\| = 1$$

satisfying

$$g_{j_0}(y_{j_0}) = \|y_{j_0}\| = \|E \left( -\sum_{k=0}^m A_k \right)^{-1} D_{j_0} u_{j_0}\|,$$

we consider the operator

$$\Delta_{j_0} : Y_{j_0, \mathbb{C}} \rightarrow U_{j_0, \mathbb{C}},$$

defined by

$$y \mapsto \Delta_{j_0} y = \frac{g_{j_0}(y)}{\|E \left( -\sum_{k=0}^m A_k \right)^{-1} D_1 u_{j_0}\|} u_{j_0},$$

we have  $\Delta_{j_0} \in \mathcal{L}^+(Y_{j_0, \mathbb{C}}, U_{j_0, \mathbb{C}})$  and

$$\|\Delta_{j_0}\| = \frac{1}{\|E \left( -\sum_{k=0}^m A_k \right)^{-1} D_{j_0} u_{j_0}\|},$$

set

$$x_{j_0} = \left( -\sum_{k=0}^m A_k \right)^{-1} D_{j_0} u_{j_0},$$

then

$$Ex_{j_0} = E \left( -\sum_{k=0}^m A_k \right)^{-1} D_{j_0} u_{j_0} = y_{j_0},$$

and hence

$$\Delta_{j_0} Ex_{j_0} = \frac{g_{j_0}(Ex_{j_0})}{\|E \left( -\sum_{k=0}^m A_k \right)^{-1} D_{j_0} u_{j_0}\|} u_{j_0} = \frac{g_1(y_1)}{\|E \left( -\sum_{k=0}^m A_k \right)^{-1} D_{j_0} u_{j_0}\|} u_{j_0} = u_{j_0},$$

then  $x_{j_0} \neq 0$  because of  $u_{j_0} \neq 0$ .

Moreover, we have

$$x_{j_0} = \left( -\sum_{k=0}^m A_k \right)^{-1} D_{j_0} (\Delta_{j_0} Ex_{j_0}),$$

or equivalently

$$\left(-\sum_{k=0}^m A_k\right) x_{j_0} = D_{j_0} \Delta_{j_0} E x_{j_0}, \quad (4.4)$$

then

$$\left(\sum_{k=0}^m A_k + D_{j_0} \Delta_{j_0} E\right) x_{j_0} = 0.$$

We have for any  $t \geq 0$

$$D_{j_0}(t) \preceq |D_{j_0}(t)| \preceq D_{j_0}$$

and

$$D_{j_0}(t) \in \mathcal{L}^+(U_{j_0, \mathbb{C}}, X_{\mathbb{C}}),$$

then

$$0 \preceq D_{j_0}(t) \Delta_1 E x_{j_0} \preceq D_{j_0} \Delta_{j_0} E x_{j_0},$$

by equation (4.4)

$$0 \preceq D_{j_0}(t) \Delta_{j_0} E x_{j_0} \preceq \left(-\sum_{k=1}^m A_k\right) x_{j_0},$$

hence,

$$0 \preceq \left(\sum_{k=1}^m A_k\right) x_{j_0} \preceq \left(\sum_{k=1}^m A_k\right) x_{j_0} + D_{j_0}(t) \Delta_{j_0} E x_{j_0} \preceq 0,$$

therefore,

$$\left(\sum_{k=1}^m A_k + D_{j_0}(t) \Delta_{j_0} E\right) x_{j_0} = 0,$$

and  $x_{j_0} \neq 0$  implies

$$0 \in \sigma\left(\sum_{k=1}^m A_k + D_{j_0}(t) \Delta_{j_0} E\right),$$

it follows that, the perturbed delay system (4.2) with  $\Delta_0 = 0$  and  $\Delta = (0, \dots, \Delta_{j_0}, 0, \dots, 0)$  is not exponentially stable.  $\square$

### Second Special Case of Perturbations

Now, we are interested in perturbed systems of the form

$$\dot{x}(t) = (A_0 + D\Delta_0 E)x(t) + \sum_{k=1}^m (A_k + D\Delta_k E_k(t))x(t - h_k), t \geq \sigma, \quad (4.5)$$

with the assumptions

1. For all  $k \in \{0, 1, \dots, m\}$ ,  $U_{k,\mathbb{C}} = U$ ,
2. For all  $k \in \{1, \dots, m\}$ ,  $E_k(\cdot) \in PC_b(\mathbb{R}^+, \mathcal{L}^{|\cdot|}(X_{\mathbb{C}}, Y_{k,\mathbb{C}})) \cap PC_b(\mathbb{R}^+, \mathcal{L}^+(X_{\mathbb{C}}, Y_{k,\mathbb{C}}))$ .

Following similar steps, the formula of stability radius is given in the following theorem.

#### Theorem 4.2.

Let  $A_0$  a positive operator on  $X_{\mathbb{C}}$  with  $\text{Int}X_{\mathbb{C}}^+ \neq \emptyset$ , which generates a compact  $C_0$ -semi-group, we assume that

1.  $A_k \in \mathcal{L}^+(X_{\mathbb{C}})$ , for all  $k \in \{1, 2, \dots, m\}$ ,
2. There exist  $D \in \mathcal{L}^+(U, X_{\mathbb{C}})$ ,  $E_k \in \mathcal{L}^+(X_{\mathbb{C}}, Y_{k,\mathbb{C}})$  and  $\Delta_k \in \mathcal{L}^+(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})$  such that

$$|E_k(t)| \leq E_k, \text{ for all } t \in \mathbb{R}^+ \text{ and } k \in \{0, 1, 2, \dots, m\}.$$

If the perturbed delay system (2.1) with the initial value condition (3.2) is exponentially stable, then

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_+ = \frac{1}{\max_{i,j \in \{0,1,2,\dots,m\}} \|E_j \left( -\sum_{k=0}^m A_k \right)^{-1} D\|}. \quad (4.6)$$

# CONCLUSION

This research rigorously investigated the robustness of stability for positive delay systems within Banach lattices, thereby significantly advancing our understanding of complex dynamical systems. Our foundational work thoroughly established the theoretical keystones by explaining crucial concepts related to Banach lattices, positive operators, semi-groups of bounded linear operators, and Metzler operators. We precisely characterized the class of positive time-delay systems under consideration, providing critical insights into their asymptotic behavior. A key methodological innovation was the application of time-varying structured perturbations, which yielded novel and vital results concerning the existence and uniqueness of solutions. Crucially, we successfully established an explicit upper bound for these perturbations, thereby providing a quantifiable measure of the system's robustness in stability. Furthermore, the derived upper bound for the complex stability radius and the demonstrated coincidence of complex, real, and positive stability radii in specific cases offer profound theoretical insights into the interconnectedness of various stability metrics, potentially streamlining future analysis.

The implications of these findings are far-reaching, impacting both theoretical advancements and practical applications across diverse fields. The quantifiable measures of robustness we developed, particularly the upper bounds for disturbance amplitudes and the complex stability radius, serve as indispensable tools for rigorously assessing positive delay systems in complex, infinite-dimensional environments.

These results are poised to critically influence:

- **System Design and Control:** By precisely defining the limits of disturbance tolerance, our findings enable the development of more resilient control systems capable of maintaining stability amidst pervasive time-varying disturbances and inherent delays. This is especially vital for sophisticated applications like networked control systems, teleoperation, and various biological processes.
- **Safety and Reliability Analysis:** The established theoretical framework and concrete criteria offer robust guidance for evaluating maximum permis-

sible disturbance levels, enhancing safety protocols and improving reliability assessments for critical operational systems.

- **Theoretical Advancement:** This research deepens the fundamental understanding of stability properties within Banach lattices, expanding the analytical toolkit available for complex infinite-dimensional systems and pushing the boundaries of mathematical inquiry.
- **Model Validation and Prediction:** The findings facilitate the development of more accurate and robust models for forecasting the dynamic behavior of delay systems under diverse environmental conditions, thereby significantly improving predictive capabilities.

Ultimately, this research lays a robust foundational bedrock for future investigations. It not only explains the intrinsic limits of stability for a critical class of dynamic systems but also opens promising avenues for exploring more generalized classes of disturbances and system complexities. This trajectory is essential for translating these robust stability criteria into even broader applications, paving the way for truly resilient and reliable engineering solutions.

# PERSPECTIVE

When conducting our research on studying the robust stability of positive linear time-delay differential systems (PLTDS) under time varying perturbations, in infinite demension case (in Banach lattices) using Perron-Frobenius Theorem and the Principle of Comparison, our forthcoming work will be focusing on:

- Applying this method on linear time-delay differential systems with different types of delay.
- employing this approach on another type of systems, including non-linear ones, or partial differential systems, for instances.
- Another possible direction for futur work is to adopt different approximative methods to find approximate solutions of (TDDS (s)), which convert the (TDDS(s)) into a more flexible systems to study ((ODS (s)) or difference equations), as we will study the stability of approximate system using established techniques. By changing every time the numerical methods used, we will compare the different stability radii, which means, we will study to what extent the choice of approximate method and step size affect the stability of the system.

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