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Fixed Point Theorems in R-metric Spaces with Applications

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Dedication - *DJERMOUNE Asma*

I dedicate my thesis work to my family and many friends. A special feeling of gratitude to my loving parents whose words of encouragement and push for tenacity ring in my ears. my sisters and brother have never left my side and are very special.

I also dedicate this thesis to my many friends who have supported me throughout the process. I will always appreciate all they have done.

Dedication - *BELAHSSINI Yousra*

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Asma - Yousra.

ملخّص

الهدف من هذا العمل هو تقديم فضاء جديد و هو (الفضاء ع ـ متري)، و الذي تم فيه تطوير بعض نظريات النقطة الثابتة. كما طبقنا النتائج المتحصل عليها في ايجاد الحل الوحيد لمعادلات مصفوفات خطية، معادلات مصفوفات غير خطية و معادلات تكاملية كسرية.

الكلمات المفتاحية :
النقطة الثابتة، الفضاء المتري، الفضاء ع ـ متري، معادلات مصفوفات، معادلات تكاملية كسرية .

Abstact

The objective of this work is to introduce a new space " R-metric space ", on which few fixed point theorems have been extended, also, we applied our results on linear matrix equations, non-linear matrix equations, and fractional integral equations in order to find the unique solution.

Key words :
Fixed point, Metric spaces, R-metric spaces, Matrices equations, Fractional integral equations.

Résumé

L'objectif de ce travail consiste à l'introduction d'un nouveau espace " l'espace R-métrique" sur lequel quelques théorèmes de point fixe ont été développés. De plus, nous appliquons nos résultats pour obtenir la solution unique des equations matricielles linéaires, des equations matricielles non-linéaires et des équations fractionnaires intégrales .

Mots clés :
Point fixe, Espace métrique, Espace R-métrique, Equations matricielles, Equations intégrales fractionnaires .

Contents

Introduction

We all know what the line (Δ) " defined by the equation $y = x$ " represent, or the points that remain stationary, by an application or a transformation. All these concepts are oriented towards the fixed point theory which is a very important branch of mathematics, and is one of the most powerful tools of modern mathematics. It is a combination of analysis, topology and geometry. In the last decades, fixed point theory has been revealed as a very powerful and important tool in the study of non-linear phenomena. we can see it through a lot of phenomena as a reproducible invariant temperature like boiling point or freezing point, a triple point of a substance such as the liquid used to calibrate a thermometer.

For this reason we have found numerous research about the existence of fixed point. asserts that under certain conditions, a mapping S of a space X into itself admits one or more fixed points. There are plenty of results in fixed point theory.
L. Brouwer (1881-1966) is the father of this theory which states that if a continuous function f satisfies certain properties, then there is a fixed point. But the most widely theorem used is **Banach 1922**, which is the product of previous works in particularly those of **E. Picard**. Around 1922, **Banach** recognized the fundamental role of metric completeness, a property shared by all spaces commonly used in analysis. For many years, activity in fixed-point theory was limited to **Banach's** mirror extensions of contraction principle and its multiple applications. Much of the acquired theory attributes to new results of **Browder's** pioneering work in the mid-sixties, as well as the development of non-linear functional analysis as an active and vital branch of mathematics, central to this evolution were **Browder's** theorems. The quality as well as the amount of research in fixed point theory in metric space increased greatly in the 1970's. Descriptions of important developments in this period prove the existence of fixed point theorems using more generalized contractive applications than previous contractive applications.

In this work, we present three theorem of fixed point in R-metric spaces, By these selected contractive properties, we applied to solve a linear matrix equation, a non-linear matrix equation and fractional integral equation .

This thesis is organized as follows:

Chapter 1 contains the essential notions regarding fixed point, metric spaces, R-metric spaces, and matrices. As well as some needed theorems and lemmas in the following sections.

Chapter 2 is devoted to develop three important theorems about fixed point:

fixed point theorem in partially ordered sets, Banach's and Brouwer's theorems in R-metric spaces.

Finally, in the **last chapter** we apply the previous fixed point theorems to solve different problems such as linear and non-linear matrix equations, and fractional integral equations.

Chapter 1

Preliminaries

In this chapter we explore the most important notions regarding fixed point, R-metric spaces and matrices. As well as some needed theorems and lemmas in our results.

1.1 Fixed point and metric spaces

fixed point a fixed point of a function is an element of the function's domain that is mapped to itself by the function. As an illustration, c is a fixed point of the function f if $f(c) = c$. This means $f^n(c) = c$, an important terminating consideration when recursively computing f.
A set of fixed points is sometimes called a fixed set.

For example, if f is defined on the real numbers by

$$f(x) = x^2 - 3x + 4,$$

then 2 is a fixed point of f, because $f(2) = 2$.

Definition 1.1. *A metric space is an ordered pair* (M, d) *where* M *is a set and* d *is a metric on* M*, i.e. a funciton* $d : M \times M \longrightarrow \mathbb{R}$
such that for any $x, y, z \in M$ *, the following holds:*
1. $d(x, y) = 0 \Longleftrightarrow x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Example 1.1.1. The real numbers with the distance function $d(x, y) = |y - x|$ given by the absolute difference , and more generally,Euclidean $n-$space with the Euclidean distance are complete metric spaces. The rational numbers with the same distance function also form a metric space, but not a complete one.

Banach's fixed point theorem

Theorem 1.1. *(Banach 1922). Let* (X, d) *be a complete metric space (or a Banach space if X has a norm) and* $f : X \to X$ *a contraction. Then f admits a unique fixed point in X, (i.e)* $\exists! u \in X$ *such that* $fu = u$*. In addition, this point can be obtained as limit of any sequence generated by the iteration*

$$x_{n+1} = fx_n, n \in X \textit{ with } x_0 \textit{ arbitrary element of } X$$

Proof.
let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence generated by the above iteration. then the contraction condition of f gives

$$d(x_1,x_2)=d(fx_0,fx_1)\le \theta d(x_0,x_1)$$
$$d(x_2,x_3)=d(fx_1,fx_2)\le \theta^2 d(x_0,x_1)$$

thus, for any n we get

$$d(fx_n,fx_{n+1})\le \theta^n d(x_0,x_1)$$

moreover, repeated applications of the triangular inequality give for arbitrary integers $n,p\in\mathbb{N}^*$

$$d(fx_n,fx_{n+p})\le d(fx_n,fx_{n+1})+d(fx_{n+1},fx_{n+2})+...+d(fx_{n+p-1},fx_{n+p})$$

with all the previous inequalities, this one gives us

$$d(x_n,x_{n+p})\le \tfrac{\theta^n(1-\theta^p)}{1-\theta}d(x_0,x_1)<\tfrac{\theta^n}{1-\theta}d(x_0,x_1)\to 0 \text{ ,if } n\to\infty$$

Obviously the strict inequality and the convergence to 0 follow from the fact that $\theta\in(0,1)$. Therefore $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and as X is complete, it converges to a limit u. Clearly, u is a fixed point of f since it is continuous. To prove the uniqueness, let us suppose that we have two different fixed points u, v such

$$0<d(u,v)=d(fu,fv)\le \theta d(u,v)$$

which implies
$$(1-\theta)d(u,v)\le 0$$

with $\theta\in(0,1)$. Or this is obscure. □

Brouwer's fixed-point theorem

Theorem 1.2. *(Brouwer 1910). Given that set $K\subset\mathbb{R}^n$ is compact and convex, and that function $f:K\to K$ is continuous, then there exists some $c\in K$ such that $f(c)=c$; that is c is a fixed point.*

Nieto and al. in [4] presented a new extension of Banach contractive mapping to partially ordered sets.

Theorem 1.3. *Let $(X,\le)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space Let $f:X\to X$ be a continuous and non-decreasing mapping such that there exists $k\in[0,1)$ with*

$$d(f(x),f(y))\le kd(x,y),\quad \forall x\ge y.$$

If there exists $x_0\in X$ with $x_0\le f(x_0)$, then f has a fixed point.

1.2 R-Metric space with its topology

Definition 1.2. *Suppose $(X,\ d)$ is a metric space and R is a relation on X. Then the triple $(X,\ d,\ R)$ or in brief X is called R-metric space.*

Example 1.2.1. Let $X = [0, \infty)$ equipped with Euclidean metric. Define xRy if $xy \leq (x \vee y)$ where $x \vee y = x$ or y. Then (X, d, R) is an R-metric space.

We note that for a given specified metric space (X, d) and any relation R on X we can consider an R-metric space (X, d, R) .

Definition 1.3. *A sequence $\{x_n\}$ is an R-metric space X is called an R-sequence if $x_n R x_{n+k}$ for each $n, k \in \mathbb{N}$.*

Definition 1.4. *An R-sequence $\{x_n\}$ is said to converge to $x \in X$ if for every $\epsilon > 0$ there is an integer N such that $d(x_n, x) < \epsilon$ if $n \geq N$.*

In this case, we write $x_n \xrightarrow{R} x$.

Example 1.2.2. Suppose $X = [1, 2]$ with Euclidean metric be given . let $R :=\geq$ and $x_n = 1 + \dfrac{1}{n}$ for each $n \in \mathbb{N}$. Clearly $\{x_n\}$ is an R-sequence and $x_n \xrightarrow{R} 1$. Note that $x_n = 2 - \dfrac{1}{n}$ is note an R-sequence.

Example 1.2.3. Let $X = \mathbb{R}$. Consider $P(\mathbb{R})$ with metric $d(A, B) = \inf\{|a - b|, \quad a \in A \quad and \quad b \in B\}$, let $R :=\subseteq$. Define $A_n = [1 + \frac{1}{n}, 4 - \frac{1}{n}]$, clearly $\{A_n\}$ is an R-sequence and $A_n \xrightarrow{R} (1, 4)$.

Remark 1.2.1. Every subsequence of an R-sequence in an R-sequence too.
In th followings it is supposed that X is an R-metric space.

Definition 1.5. *Let $E \subseteq X.x \in X$ is called an R-limit point of E if there exists an R-sequence $\{x_n\}$ in E such that $x_n \neq x$ for all $n \in \mathbb{N}$ and $x_n \xrightarrow{R} x$.*

Definition 1.6. *The set of all $R-$limit points of E in denoted by ${E'}^R$.*

Definition 1.7. *$E \subseteq X$ is called R-closed, if ${E'}^R \subseteq E$.*

Definition 1.8. *$E \subseteq X$ is called R-open, if E^c is R-closed.*

Theorem 1.4. *$E \subseteq X$ is R-open if and only if for any $x \in E$ and any R-sequence $\{x_n\}$ which $x_n \xrightarrow{R} x$, there exists $N \in \mathbb{N}$ such that $x_n \in E$ for all $n \geq N$.*

Proof. Let $x \in E$. Suppose there exists an R-sequence $\{x_n\}$ which $x_n \xrightarrow{R} x$ but for every $N \in \mathbb{N}$, there exists a natural number $n \geq N$ such that $x_n \notin E$. Hence, we obtain an R-subsequence $\{x_{n_N}\}$ of $\{x_n\}$, which $x_{n_N} \xrightarrow{R} x$, and $x_{n_N} \notin E$. Therefore $x \in E^{cC}$ This contradicts $x \in E$.

Conversely, if $x \in (E^c)'^R$, then $x \notin E$. Hence $(E^C)'^R \subset E^C$, so E^C is R-closed. Thus E is R-open. □

Definition 1.9. *Let $E \subseteq X$. The R-closure of E is the set $\overline{E}^R = E \cup E'^R$*

Theorem 1.5. *If $E \subset X$, then*

(a)$\overline{E}^R$ is R-closed.

(b) $E = \overline{E}^R$ if and only if E is R-closed.

Proof. (a) If $x \in (\overline{E}^R)^C$, then x is neither a point of E nor an R-limit point of E. Let $\{x_n\}$ be an R- sequence converging to x, then there exists $N \in \mathbb{N}$ such that $x_n \in (\overline{E}^R)^C$ for $n \geq N$. Thus $(\overline{E}^R)^C$ is R-open so that $\overline{E}^R$ is R-closed.
(b) If $E = \overline{E}^R$, (a) implies that E is R-close. If E is R-closed, then $E'^R \subset E$ [by **Definitions 1.7 and 1.9**], hence $\overline{E}^R = E$. □

Example 1.2.4. Suppose $X = \mathbb{R}$ equipped with standard topology.
Let $R := \leq$, let $E = (0, 1]$, then $\overline{E}^R = (0, 1]$. Hence E is R-closed but it is not closed.

Theorem 1.6. *The set $\{G \subseteq X | G$ is R-open $\}$ is a topology on X which is called an R-topology and is denoted by τ_R*

Proof. Trivially φ and X are R-open. Let $\{U_j\}_{j \in J}$ be a family of R-open sets.
Put $U = \bigcup_{j \in J} U_j$, let x be an R-limit point of $U^c = \bigcap_{j \in J} U_j^c$, hence there exists an R-sequence $\{x_n\}$ in $U^c \backslash \{x\}$ such that $x_n \xrightarrow{R} x$. Therefore for each $j \in J, x$ is an R-limit point of U_j^c. Since U_j^c is R-closed $x \in U_j^c$, so that $x \in U^c$, hence U is R-open.
Let $U_1, \ldots, U_n$ be R-open sets. Put $W = \bigcap_{j=1}^{n} U_j$
and let x be an R-limit point of $W^c = \bigcup_{j=1}^{n} U_j^c$, hence there exists an R-sequence $\{x_n\}$ in $W^c \backslash \{x\}$ such that $x_n \xrightarrow{R} x.$. It is easy to show that there exists $1 \leq j \leq n$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ in $U_j^c \backslash \{x\}$ which $x_{n_k} \xrightarrow{R} x.$. Since U_j^c is R-closed $x \in U_j^c$, thus $x \in W^c$ It follows that W is R-open. □

Corollary 1.7. *Let $X = \mathbb{R}$ equipped with standard topology. Let $R := \leq$, then:*
(a) The open intervals $(a,\ b)(a = -\infty$ or $b = +\infty)$ form a basis for the standard topology on $\mathbb{R}$,
(b) The intervals $(a,\ b)(a = -\infty$ or $b = +\infty)$ and $(a,\ b]$ form a basis for topology τ_R.

Theorem 1.8. *Let τ_d be a metric topology on X, then $\tau_d \subseteq \tau_R$.*

Proof. Let $G \in \tau_d$. Let x be an R-limit point of G^c There exists an R-sequence $\{x_n\}$ in $G^c \backslash \{x\}$ such that $x_n \xrightarrow{R} x$.. Since each R-sequence is a sequence hence x is a limit point of $G^C.G^C$ is closed, thus $x \in G^C$ Therefore $G \in \tau_R$. □

Lemma 1.9. *Let $R := X \times X$, then $\tau_R = \tau_d$.*

Proof. By **Theorem 1.8**, $\tau_d \subseteq \tau_R$. Let G is an R-open set and x is a limit point of G^C There exists a sequence $\{x_n\}$ in $G^c \backslash \{x\}$ such that $x_n \xrightarrow{R} x$.. Clearly $\{x_n\}$ is an R-sequence, hence $x_n \xrightarrow{R} x$. It follows that $x \in G^c$ Thus $\tau_R \subseteq \tau_d$. □

Remark 1.2.2. *Suppose (X, d) be a metric space and $R := X \times X$, then R-metric space $(X,\ d,\ R)$ is equivalent to metric space $(X,\ d)$.*

Definition 1.10. *Let R be a relation R^k. $E \subseteq \mathrm{R}^k$ is called R-convex if $\lambda x + (1 - \lambda)y \in E$, whenever $x \in E, y \in E, xRy$, and $0 < \lambda < 1$.*

Lemma 1.10. *Let E is a convex set, then E is R-convex.*

Proof. Let $x \in E, y \in E, xRy$, and $0 < \lambda < 1$. By the definition of convex set
$\lambda x + (1 - \lambda)y \in E$. □

The converse is not true.

Example 1.2.5. Let $E = (0, 1] \cup (2, 4]$. Define xRy if $x, y \in (0, 1]$ or $x, y \in (2, 4]$. Clearly E is R-convex but it is not convex.

Definition 1.11. *$K \subseteq X$ is called R-compact if every R-sequence $\{x_n\}$ in K has a convergent subsequence in K.*

Example 1.2.6. Suppose the Euclidean metric space $X = \mathbb{R}$ be given. Let $K = (0, 1)$ and let $R := \geq$. Then K is R-compact.

Lemma 1.11. . *Suppose $K \subseteq X$ is compact, then K is R-compact.*

Proof. Let $\{x_n\}$ be an R-sequence in K. It is clear that $\{x_n\}$ is a sequence too.So, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ in K. The theorem follows. □

The converse is not true.

Example 1.2.7. Suppose the Euclidean metric space $X = \mathrm{R}$ be given. Let $k = [0, 1)$, and let R defined on X by

$$xRy \Longleftrightarrow \begin{cases} x \leq y \leq \frac{1}{3} \\ or \\ x = 0 \qquad if\ y > \frac{1}{3} \end{cases}$$

Since K is not closed so it is not compact. Let $\{x_n\}$ be an R-sequence in K. Then:
i) For all $n \in \mathbb{N}$,$x_n = 0$, hence $\{x_n\}$ converges to 0.
ii) $\{x_n\}$ is increasing and bounded above to $\dfrac{1}{3}$, therefore it is convergent.

It follows that K is R-compact.

Example 1.2.8. Suppose $X = \mathrm{R}$ equipped with the Euclidean metric. Let $K = (0, 1]$. Let $R :=\leq$, and let $\{x_n\}$ be an R-sequence in K. It is clear that $\{x_n\}$ is increasing and bounded above, hence it is convergent. Therefore K is R-compact but it is not compact.

Theorem 1.12. *R-compact subsets of R-metric spaces are R-closed.*

Proof. Let K be an R-compact subset of X. Let $x \in K'^R$ Then there exists an R-sequence $\{x_n\}$ in K such that $x_n \xrightarrow{R} x.$. By **Definition 1.11**, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ in K , hence $x_{n_k} \xrightarrow{R} x.$, so that $x \in K$. The theorem follows. □

Theorem 1.13. *An R-closed subset of an R-compact set , is R-compact.*

Proof.
Suppose $F \subseteq K \subseteq X$. F is R-closed (relative to X) , and K is R-compact. Let $\{x_n\}$ be an R-sequence in F, hence $\{x_n\}$ is an R-sequence in K too. Since K in R-compact.there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{R} x$. Since F is R-closed $x \in F$. The theorem follows. □

Corollary 1.14. *Let F be R-closed and K be R-compact. Then $F \cap K$ is R-compact.*

Proof.
By **Theorems 1.6** and **1.12** $F \cap K$ s R-closed; since $F \cap K \subset K$,**Theorem 1.13** shows that $F \cap K$ is R-compact. □

Definition 1.12. *A set $K \subseteq X$ is called strong R-compact, if each R-sequence $\{x_n\}$ in K, that has a subsequence $\{x_n\}$ which converges to $x^* \in K$,i.e $x_{n_k} \xrightarrow{R} x^*$, then $x_{n_{k+1}} \xrightarrow{R} x^*$.*

Example 1.2.9. Suppose $X = \mathbb{R}$ with standard topology be given, let $R := \leq$and let $K = (0, 10]$. Suppose $\{x_n\}$ is an R-sequence, then each subsequence $\{x_{n_k}\}$ of $\{x_n\}$ in increasing bounded above, so there exists $x^* \in k$, such that $x_{n_k} \xrightarrow{R} x^*$ and $x_{n_{k+1}} \xrightarrow{R} x^*$. Therefore K is a strong R-compact set

Definition 1.13. *An R-sequence $\{x_n\}$ in X is said to be an R-Cauchy sequence, if for very $\epsilon > 0$ there exists an integer N such that $d(x_n, x_m) < \epsilon$ if $n \geq N$ if $m \geq N$. It is clear that $x_n R x_m$ or $x_m R x_n$.*

Lemma 1.15.

(a) Every convergent R-sequence in X is an R-Cauchy sequence.

(b) Suppose K be an R-compact set and $\{x_n\}$ be a R-Cauchy sequence in K. Then $\{x_n\}$ converges to $x \in K$

Definition 1.14. *X is said to be r-complete if every R-Cauchy sequence in X converges to a point in X.*

Corollary 1.16. *Every R-compact space in R-complete, but the converse in not true.*

Example 1.2.10. Suppose $X = \mathbb{R}$ with standard topology be given and let $R := \leq$. It is easy to show that X is R-complete but it is not R-compact. Since the R-sequence $\{n\}$ has no convergent subsequence.

Definition 1.15. *Let $f : X \to X$ be a mapping. f is said to be R-continuous at $x \in X$ if for every R-sequence $\{x_n\}$ in X with $x_n \xrightarrow{R} x.$, we have $f(x_n) \xrightarrow{R} f(x)$. Also, f is said to be R-continuous on X if f is R-continuous in each $x \in X$.*

Lemma 1.17. *Every continuous mapping $f : X \to X$, is R-continuous.*

Proof.
Since each R-sequence is a sequence.
The converse is not true. □

Example 1.2.11. Suppose $X = [0, 1]$ equipped with standard topology and let $f : X \to X$ be a Dirichlet mapping, i.e,

$$f(x) = \begin{cases} 1 & if \quad x \in Q \cap [0,1] \\ 0 & if \quad x \in Q^c \cap [0,1] \end{cases}$$

Let R be an equality relation on X, then f is discontinuous at each point of X but f is R-continuous on X.

Example 1.2.12. Suppose the Euclidean metric space $X = \mathbb{R}$ be given. Define xRy if $x, y \in (n + \frac{2}{3},\ n + \frac{4}{5})$ for some $n \in \mathbb{Z}$ or $x = 0$. Define $f : X \to X$ by $f(x) = [x]$. Let $x \in X$ and $\{x_n\}$ be an arbitrary R-sequence in X such that converges to x, then the following cases are satisfied:

Case 1: If $x_n = 0$ for all n, then $x = 0$, and $f(x_n) = 0 = f(x)$.

Case 2: If $x_n \neq 0$ for some n, then there exists $m \in \mathbb{Z}$ such that $x \in [m + \frac{2}{3},\ m + \frac{4}{5}]$, and $f(x_n) = m = f(x)$.

Therefore f is R-continuous on X, but it is not continuous on X.

Definition 1.16. *A mapping $f : X \to X$ is said to be an R-contraction with Lipschutz constant $0 < \lambda < 1$ if for all $x, y \in X$ such that xRy, we have*

$$d(f(x),\ f(y)) \leq \lambda d(x, y) \ .$$

It is easy to show that every contraction is an R-contraction but the converse is not true. See the next example.

Example 1.2.13. Let $X = [0, 0.99)$, and X equipped with Euclidean metric. Let xRy if $xy \in \{x, y\}$ for all $x, y \in X$. Let $f : X \to X$ be a mapping defined by

$$f(x) = \begin{cases} x^2 & if\ x \in Q \cap X \\ 0 & if\ x \in Q^c \cap X \end{cases}$$

Suppose $x = 0.9, y = \frac{\sqrt{2}}{2}$ and $0 < \lambda < 1$, then

$$|f(x) - f(y)| = 0.81 \not\leq \lambda |0.9 - \frac{\sqrt{2}}{2}|.$$

Hence f is not a contraction. Now, let xRy, therefore $x = 0$ or $y = 0$. Suppose $x = 0$, thus

$$|f(x) - f(y)| = \begin{cases} y^2 & if\ y \in Q \cap X \\ 0 & if\ y \in Q^c \cap X \end{cases}$$

Hence by choosing $\lambda = 0.99$ it follows that

$$|f(x) - f(y)| \leq y^2 \leq \lambda|0 - y| = \lambda y.$$

So that f is R-contraction.

Definition 1.17. *Let $f : X \to X$ be a mapping f is called R-preserving if xRy, then $f(x)Rf(y)$ for all $x, y \in X$.*

Example 1.2.14. Suppose $X = \mathbb{R}$ with standard topology be given and let $R :=\geq$. Let $f : X \to X$ be a mapping defined by $f(x) = x^3$. Let $x_1 \geq x_2$, then $f(x_1) = x_1^3 \geq x_2^3 = f(x_2)$. Hence f is R-preserving.

1.3 Recall about matrices

Hermitian matrix

A Hermitian matrix (or self-adjoint matrix) is a complex square matrix that is equal to its own conjugate transpose, that is, the element in the i-th row and j-th column is equal to the complex conjugate of the element in the j-th row and i-th column, for all indices i and j:

$$A \text{ Hermitian} \Leftrightarrow a_{ij} = \overline{a}_{ji}.$$

or in matrix form:

$$A \text{ Hermitian} \Leftrightarrow A = \overline{A^T}$$

Hermitian matrices can be understood as the complex extension of real symmetric matrices. If the conjugate transpose of a matrix A is denoted by A^H, then the Hermitian property can be written concisely as:

$$A \text{ Hermitian} \Leftrightarrow A = A^H$$

Example 1.3.1. In this section, the conjugate transpose of matrix A is denoted as A^H, the transpose of matrix A is denoted as A^T and conjugate of matrix A is denoted as $\overline{A}$.

See the following example:

$$\begin{bmatrix} 0 & a-ib & c-id \\ a+ib & 1 & m-in \\ c+id & m+in & 2 \end{bmatrix}$$

The diagonal elements must be real, as they must be their own complex conjugate.

Positive definite matrix

In the following definitions, x^T is the transpose of x, x^* is the conjugate transpose of x and 0 denotes the n-dimensional zero-vector.
An $n \times n$ symmetric real matrix A is said to be positive-definite if $x^TAx > 0$ for all non-zero x in $\mathbb{R}^n$. Formally,

$$A \text{ positive definite} \Leftrightarrow x^TAx > 0 \qquad \forall x \in \mathbb{R}^n \setminus \{0\}$$

An $n \times n$ symmetric real matrix A is said to be positive semi-definite or non-negative-definite if $x^TAx \geq 0$ for all non-zero x in $\mathbb{R}^n$. Formally,

$$A \text{ positive semi-definite} \Leftrightarrow x^TAx \geq 0 \qquad \forall x \in \mathbb{R}^n$$

Chapter 2

Fixed point theorems in R-metric spaces

In this section we state three important theorems about fixed point:
fixed point theorem in partially ordered sets, Banach's and Brouwer's theorems in R-metric spaces

2.1 Fixed point theorem in partially ordered sets

In this section we proved the following fixed point theorem, which, to the best of our knowledge, is new.

Theorem 2.1. *Let T be a partially ordered set such that every pair $x, y \in T$ has a lower bound and an upper bound. Furthermore, let d be a metric on T such that (T, d) is a complete metric space. If $\mathcal{F}$ is a continuous, monotone (i.e., either order-preserving or order-reversing) map from T into T such that*

$$\exists 0 < c < 1 : d(\mathcal{F}(x),\ \mathcal{F}(y)) \leq cd(x,\ y)\ ,\ \forall x \geq y, \tag{2.1}$$

$$\exists x_0 \in T : x_0 \leq \mathcal{F}(x_0)\ \text{or}\ x_0 \geq \mathcal{F}(x_0)\ , \tag{2.2}$$

then $\mathcal{F}$ has a unique fixed point $\overline{x}$. Moreover, for every $x \in T$,

$$\lim_{n\to\infty} \mathcal{F}^n(x) = \overline{x}.$$

Proof.

Let $x_0 \in T$ be such that $x_0 \leq F(x_0)$ or $x_0 \geq F(x_0)$. The monotonicity of $\mathcal{F}$ implies that either $\mathcal{F}^n(x_0) \leq \mathcal{F}^{n+1}(x_0)$ or $\mathbb{F}^n(x_0) \geq \mathcal{F}^{n+1}(x_0)$ for $n = 0, 1, 2, ...$ So from (2.1) it follows that

$$d(\mathcal{F}^{n+1}(x_0),\ \mathcal{F}^n(x_0)) \leq cd(\mathcal{F}^n(x_0),\ \mathcal{F}^{n-1}(x_0))\ .$$

Hence, induction gives

$$d(\mathcal{F}^{n+1}(x_0),\ \mathcal{F}^n(x_0)) \leq c^n d(\mathcal{F}(x_0),\ x_0)\ .$$

At this point we follow the proof of Banach's fixed point theorem. So now we will prove that $\{\mathcal{F}^n(x_0)\}_{n=0}^{\infty}$ is a Cauchy sequence. Let $n < m$. Then

$$d(\mathcal{F}^n(x_0),\ \mathcal{F}^m(x_0)) \leq \sum_{i=n+1}^{m} d(\mathcal{F}^i(x_0),\ \mathcal{F}^{i-1}(x_0))$$

$$\leq (c^n + c^{n+1} + \cdots + c^{m-n-1})d(\mathcal{F}(x_0),\ x_0)$$

$$= c^n \tfrac{1-c^{m-n-1}}{1-c} d(\mathcal{F}(x_0),\ x_0)\ .$$

So $\{\mathcal{F}^n(x_0)\}_{n=0}^{\infty}$ is indeed a Cauchy sequence. Since T is complete, it follows that

$$\lim_{n\to\infty} \mathcal{F}^n(x_0) = \overline{x}$$

for some $\overline{x} \in T$. Because $\mathcal{F}$ is continuous, $\overline{x}$ is a fixed point of $\mathcal{F}$.

It remains for us to show that $\overline{x}$ is the unique fixed point of $\mathcal{F}$. We will do this by showing that $\lim\limits_{n\to\infty} \mathcal{F}^n(x) = \overline{x}$ for every $x \in T$. Thus, for $x \le x_0$ and $x \ge x_0$ it is obvious, because in both cases $\mathcal{F}^n(x) \le \mathcal{F}^n(x_0)$ or $\mathcal{F}^n(x) \ge \mathcal{F}^n(x_0)$. Hence, with (2.1) we see that $d(\mathcal{F}^n(x),\ \mathcal{F}^n(x_0)) \le c^n d(x,\ x_0)$. Note that the right-hand side tends to 0 if $n \to \infty$. So

$$\lim_{n\to\infty} \mathcal{F}^n(x) = \lim_{n\to\infty} \mathcal{F}^n(x_0) = \overline{x}.$$

Finally, let $x \in T$ be arbitrary and let x_1, resp. x_2, be an upper bound, resp. a lower bound, of x and x_0. Then

$$x_1 \ge x \ge x_2 \text{ and } x_1 \ge x_0 \ge x_2. \tag{2.3}$$

From (2.3) it follows that

$$\mathcal{F}^n(x_1) \ge \mathcal{F}^n(x) \ge \mathcal{F}^n(x_2) \text{ or } \mathcal{F}^n(x_1) \le \mathcal{F}^n(x) \le \mathcal{F}^n(x_2) \tag{2.4}$$

and

$$\lim_{n\to\infty} \mathcal{F}^n(x_1) = \lim_{n\to\infty} \mathcal{F}^n(x_2) = \overline{x}. \tag{2.5}$$

By combining (2.4) and(2.5) we get $\lim\limits_{n\to\infty} \mathcal{F}^n(x) = \overline{x}$. This proves the theorem.

□

Recall that Banach's fixed point theorem states that $\mathcal{F}$ has a unique fixed point if $\mathcal{F}$ maps a complete metric space S into itself and satisfies

$$\exists 0 < c < 1\ :\ d(\mathcal{F}(x),\ \mathcal{F}(y)) \le cd(x,\ y)\ , \forall x,\ y \in S.$$

Observe that this condition is slightly stronger than condition (2.1) of Theorem 2.1.

2.2 Banach's and Brouwer's theorems in R-metric spaces

In this section, it is proved two main theorems. The first one is the real extension of one of the most important theorem in mathematics which is named Banach contraction principle theorem 1.1, and the second one is the version of Brouwer fixed point theorem.

Theorem 2.2. *Let X be an R-complete metric space(not necessarily complete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be R-continuous, R- contraction with Lipschutz constant λ and R- preserving. Suppose there exists $x_0 \in X$ such that $x_0 R y$ for all $y \in f(X)$. Then f has a unique fixed point x^*. Also, f is a Picard operator, that is, $\lim_{n\to\infty} f^n(x) = x^* for$ all $x \in X$.*

Proof.

Let $x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), \ldots, x_n = f(x_{n-1}) = f^n(x_0), \ldots$, for all $n \in \mathbb{N}$. let $n, m \in \mathbb{N}$, and $n < m$, put $k = m - n$. We have $x_0 R f^k(x_0)$ since f is R-preserving $(x_n = f^n(x_0))R(f^{n+k}(x_0) = x_m)$. Hence $\{x_n\}$ is an R-sequence. On the other hand, f is R-contraction, thus we have

$$d(x_n,\ x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq$$

$$\lambda d(x_{n-1},\ x_n) \leq \cdots \leq \lambda^n d(x_1,\ x_0)\ ,$$

for all $n \in \mathbb{N}$. If $m, n \in \mathbb{N}$, and $n \leq m$, then

$$\begin{aligned} d(x_n,\ x_m) &\leq d(x_n,\ x_{n+1}) + \cdots + d(x_{m-1},\ x_m) \\ &\leq \lambda^n d(x_0,\ x_1) + \cdots + \lambda^{m-1} d(x_0,\ x_1) \\ &\leq \tfrac{\lambda^n}{1-\lambda} d(x_1,\ x_0)\ . \end{aligned}$$

Hence $d(x_n,\ x_m) \to 0$ as $m, n \to \infty$. Therefore $\{x_n\}$ is an R-Cauchy sequence. Since X is R-complete, there exists $x^* \in X$ such that $x_n \xrightarrow{R} x^*$ On the other hand, f is R-continuous so $f(x_n) \xrightarrow{R} f(x^*)$, therefore $f(x^*) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x^*$ Thus x^* is a fixed point of f.

To prove the uniqueness, let $y^* \in X$ be a fixed point of f, then $x_0 R f(y^*) = y^*$. Hence $(x_n = f^n(x_0))R y^*$ for all $n \in \mathbb{N}$. Therefore, by triangle inequality, we have

$$\begin{aligned} d(x^*, y^*) = d(f^n(x^*),\ f^n(y^*)) &\leq d(f^n(x^*),\ f^n(x_0)) + d(f^n(x_0),\ f^n(y^*)) \\ &\leq \lambda^n d(x^*,\ x_0) + \lambda^n d(x_0, y^*) \to 0, \end{aligned}$$

as $n \to \infty$. Thus it follows that $x^* = y^*$

Finally, let x be an arbitrary element of X. We have $x_0Rf(x)$, hence $f^n(x_0)Rf^{n+1}(x)$ for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned} d(x^*, f^n(x)) = d(f^n(x^*),\ f^n(x)) &\leq d(f^{n-1}(x^*), f^{n-1}(x_0)) + d(f^{n-1}(x_0),\ f^{n-1}(f(x))) \\ &\leq \lambda^{n-1}d(x^*,\ x_0) + \lambda^{n-1}d(x_0, f(x)) \to 0, \end{aligned}$$

as $n \to \infty$. Hence $\lim_{n\to\infty} f^n(x) = x^*$

We now show that our theorem is an extension of Banach contraction principle. □

Corollary 2.3. *(Banach contraction principle)* *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping such that for some $\lambda \in (0, 1)$,*

$$d(f(x),\ f(y)) \leq \lambda d(x, y)$$

for all $x, y \in X$. Then f has a unique fixed point in X.

Proof. Suppose that $R := X \times X$. Fix $x_0 \in X$. Clearly, x_0Ry for all $y \in f(X)$. Since X is complete, it is R-complete. It is clear that f is R-preserving, R-contraction and R-continuous. Using Theorem 2.2, f has a unique fixed point in X.

□

The following example, shows that our theorem is a real extension of Banach contraction principle.

Example 2.2.1. Let $X = [O,\ 1)$ and let the metric on X be the Euclidean metric. Define xRy if $xy \in \{x, y\}$, that is, $x = 0$ or $y = 0$. Let $f : X \to X$ be a mapping defined by

$$f(x) = \begin{cases} \frac{x^2}{3} & if\ x \leq \frac{1}{3} \\ 0 & if\ x > \frac{1}{3} \end{cases}$$

Let $\{x_n\}$ be an R-sequence in X. It is obvious that there exists $N \in \mathbb{N}$ such that $x_n = 0$ for all $n \geq N, n \in \mathbb{N}$. Hence $x_n \xrightarrow{R} 0$. Therefore X is R-complete but not complete. f is R-continuous but not continuous. Suppose xRy, thus $x = 0$ or $y = 0$. Let $x = 0$, then

$$|f(x) - f(y)| = \begin{cases} \frac{y^2}{3} & if\ y \leq \frac{1}{3} \\ 0 & if\ y > \frac{1}{3} \end{cases}$$

Hence by choosing $\lambda = 0.99$ it follows that

$$|f(x) - f(y)| \leq \frac{y^2}{3} \leq \lambda|0 - y| = \lambda y.$$

So f is R-contraction. Let xRy, then $x = 0$ or $y = 0$, hence $f(x) = 0$ or $f(y) = 0$, so $f(x)Rf(y)$, that is, f is R-preserving. Let $y \in f(X)$, then $0Ry$. Therefore using Theorem 2.2, f has a unique fixed point in X. However, by Banach contraction principle , we can not find any fixed point of f in X.

As, we know in Brouwer theorem a continuous mapping $f : E \to E$ from a convex and compact, set $E(\subseteq \mathbb{R}^n)$ into E has a fixed point without mentioning how to find the fixed point.

In our theorem, we omit the convexity and substitute the compactness with strong R-compactness to show how the fixed point can be constructed.

Theorem 2.4. *Suppose $X =]\mathbb{R}^n$ equipped with standard topology and let $E \subseteq X$ be a strong R- compact set. Let $f : E \to E$ is R- continuous and R- preserving. Suppose $x_0 \in E$ be given such that x_0Ry for all $y \in f(E)$. Then f has a fixed point.*

Proof.

Let $x_1 = f(x_0), x_2 = f(x_1) = f(f(x_0)) = f^2(x_0), \ldots, x_n = f^n(x_0), \ldots$, for all $n \in \mathbb{N}$. Let $n, m \in \mathbb{N}$ where $n < m$, put $k = m - n.x_0Rf^k(x_0)$, hence $f^n(x_0)Rf^{n+k}(x_0) = f^m(x_0)$, i.e, x_nRx_m, therefore, the sequence $\{x_n\}$ is an R-sequence. Since E is R-compact, there exists a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \xrightarrow{R} x^*$ Thus $f(x_{n_k}) \xrightarrow{R} f(x^*)$, therefore $f(x^*) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} x_{n_k+1} = x^*$ □

Example 2.2.2. Suppose $X =]\mathbb{R}$ with standard topology be given. Let $R :=\leq$ and $E = (0, 1]$. Let $f : E \to E$ be a mapping defined by $f(x) = \frac{x+1}{2}$. Let $x_0 = \frac{1}{2}$, then $x_1 = f(x_0) = \frac{3}{4}, x_2 = f(x_1) = \frac{7}{8}, x_3 = f(x_2) = \frac{15}{16}, x_4 = f(x_3) = \frac{31}{32}, \ldots$. Hence we obtain an $R-$sequence $\{x_n\}$ such that $x_n \to 1 = x^*$ Thus $f(x^*) = f(1) = \frac{1+1}{2} = 1 = x^*$

Example 2.2.3. Suppose $X = \mathbb{R}$ with standard topology be given. Let $R :=\leq$ and let $E = (-1, 2] \cup (3, 4]$. Let $f : E \to E$ be a mapping defined by $f(x) = \frac{x+1}{4}$. Put $x_0 = 0$, then $x_1 = f(x_0) = f(0) = \frac{1}{4}, x_2 = f(x_1) = f(\frac{1}{4}) = \frac{5}{4^2}, x_3 = f(x_2) = f(\frac{5}{4^2}) = \frac{21}{4^3}, x_4 = f(x_3) = f(\frac{21}{4^3}) = \frac{85}{4^4}, \ldots$. It is easy to show that $x_n \xrightarrow{R} \frac{1}{3} = x^*$ Hence the fixed point is $\frac{1}{3}$

Chapter 3

Applications

In this chapter we apply the previous fixed point theorems to solve different problems such as linear and non-linear matrix equations, and fractional integral equations.

3.1 Application to linear matrix equations

We shall apply the fixed point theorem to linear matrix equations of the type

$$X - A_1^* X A_1 - \cdots - A_m^* X A_m = Q \tag{3.1}$$

and

$$X + A_1^* X A_1 + \cdots + A_m^* X A_m = Q, \tag{3.2}$$

where Q is a positive definite matrix and $A_1, ..., A_m$ are arbitrary $n \times n$ matrices. We are particularly interested in (unique) positive definite solutions. The results we shall derive below were obtained there by two methods which are totally different from the method we will use in this paper. The first method is based on the fact that (3.1) is equivalent to the matrix equation $Gx = q$, where $G = \sum_{j=1}^{m} A_i^T \otimes A_i^*$ and where q is a vector constructed from the elements of Q. The second method involves results from the theory of operators mapping a positive cone into itself.

We shall use the following notation: $\mathcal{M}(n)$ denotes the set of all $n \times n$ matrices, $\mathcal{H}(n) \subset \mathcal{M}(n)$ the set of all $n \times n$ Hermitian matrices and $\mathcal{P}(n) \subset \mathcal{H}(n)$ is the set of all $n \times n$ positive definite matrices. Instead of $X \in \mathcal{P}(n)$ we will also write $X > 0$. Furthermore, $X \geq 0$ means that X is positive semi-definite. As a different notation

for $X - Y \geq 0 (X - Y > 0)$ we will use $X \geq Y (X > Y)$. If $X, Y \in \mathcal{H}(n)$ such that $X \leq Y$, then $[X,\ Y]$ will be the set of all $Z \in \mathcal{H}(n)$ satisfying $X \leq Z \leq Y$.

We denote by $\|.\|$ the spectral norm, i.e., $\|A\| = \sqrt{\lambda^+(A^*A)}$ where $\lambda^+(A^*A)$ is the largest eigenvalue of A^*A. The $n \times n$ identity matrix will be written as I_n.

Define maps $\mathcal{G}$ and $\mathcal{K}$ on $\mathcal{H}(n)$ by

$$\mathcal{G}(X) = Q + \sum_{j=1}^{m} A_j^* X A_j,\ \mathcal{K}(X) = Q - \sum_{j=1}^{m} A_j^* X A_j.$$

Note that the fixed points of $\mathcal{G}$ are the solutions of (3.1), while the fixed points of $\mathcal{K}$ are the solutions of (3.2).

Most of the conditions of Theorem 2.1 are satisfied for $\mathcal{G}$ and $\mathcal{K}$. Indeed, $\mathcal{G}$ and $\mathcal{K}$ both are monotone and continuous. The set $\mathcal{H}(n)$ is partially ordered and for every $X, Y \in \mathcal{H}(n)$ there is a greatest lower bound and a least upper bound. Condition (2.2) is satisfied for

$x_0 = 0$, since $\mathcal{G}(0) = \mathcal{K}(0) = Q > 0$. So we only have to derive a condition on $\mathcal{G}$ and $\mathcal{K}$ such that (2.1) is fulfilled. For this we also have to specify which metric we shall use on $\mathcal{H}(n)$. It turns out that it is convenient here to use the metric induced by the trace norm $\|.\|_1$. Recall that this norm is given by $\|A\|_1 = tr(A)$. In fact, we shall use a slight modification of this norm. For $Q \in \mathcal{P}(n)$ we define $\|A\|_{1,Q} = \|Q^{\frac{1}{2}} A Q^{\frac{1}{2}}\|_1$. This again defines a norm, and $\mathcal{H}(n)$ equipped with the metric induced by $\|\cdot\|_{1,Q}$ is a complete metric space for any positive definite Q.

The following lemma is useful for our application.

Lemma 3.1. *Let $A \geq 0$ and $B \geq 0$ be $n \times n$ matrices. Then $0 \leq tr(AB) \leq \|A\| \cdot tr(B)$.*

Proof.

It is well known that the eigenvalues of the product of two positive semi-definite matrices are non-negative. In particular, $tr(AB) \geq 0$. Furthermore, since $A \leq \|A\| I_n$, we have

$$0 \leq tr((\|A\| - A)B) = tr(\|A\|B - AB) = \|A\|tr(B) - tr(AB) \ ,$$

which completes the proof.

□

Theorem 3.1. *Let $Q \in \mathcal{P}(n)$ and let $\mathcal{K}(\tilde{Q}) > 0$ for some $\tilde{Q} \in \mathcal{P}(n)$. Then*

$\mathcal{G}$ and $\mathcal{K}$ have a unique fixed point in $\mathcal{H}(n)$.

Proof.

We will only consider $\mathcal{G}$. The proof for $\mathcal{K}$ is the same. Let $X, Y \in \mathcal{H}(n)$ such that $X \leq Y$. Then $\mathcal{G}(X) \leq \mathcal{G}(Y)$. Therefore,

$$\|\mathcal{G}(Y) - \mathcal{G}(X)\|_{1,Q^-} = tr(\tilde{Q}^{\frac{1}{2}}(\mathcal{G}(Y) - \mathcal{G}(X))\tilde{Q}^{\frac{1}{2}})$$

$$= tr(\textstyle\sum_{j=1}^m \tilde{Q}^{\frac{1}{2}} A_j^*(Y - X)A_j\tilde{Q}^{\frac{1}{2}}) = \textstyle\sum_{j=1}^m tr(\tilde{Q}^{\frac{1}{2}} A_j^*(Y - X)A_j\tilde{Q}^{\frac{1}{2}})$$

$$= \textstyle\sum_{j=1}^m tr(A_j\tilde{Q}A_j^*(Y - X)) = \textstyle\sum_{j=1}^m tr(A_j\tilde{Q}A_j^*\tilde{Q}^{-\frac{1}{2}}\tilde{Q}^{\frac{1}{2}}(Y - X)\tilde{Q}^{\frac{1}{2}}\tilde{Q}^{-\frac{1}{2}})$$

$$= \sum_{j=1}^{m} tr(\tilde{Q}^{-\frac{1}{2}} A_j \tilde{Q} A_j^* \tilde{Q}^{-\frac{1}{2}} \tilde{Q}^{\frac{1}{2}} (Y - X) \tilde{Q}^{\frac{1}{2}})$$

$$= tr(\sum_{j=1}^{m} \tilde{Q}^{-\frac{1}{2}} A_j \tilde{Q} A_j^* \tilde{Q}^{-\frac{1}{2}} \tilde{Q}^{\frac{1}{2}} (Y - X) \tilde{Q}^{\frac{1}{2}})$$

$$= tr((\sum_{j=1}^{m} \tilde{Q}^{-\frac{1}{2}} A_j \tilde{Q} A_j^* \tilde{Q}^{-\frac{1}{2}})(\tilde{Q}^{\frac{1}{2}} (Y - X) \tilde{Q}^{\frac{1}{2}}))$$

$$\leq \| \sum_{j=1}^{m} \tilde{Q}^{-\frac{1}{2}} A_j \tilde{Q} A_j^* \tilde{Q}^{-\frac{1}{2}} \| \cdot \|Y - X\|_{1,\overline{Q}}.$$

The inequality follows from Lemma 3.1 Because of the assumption that $0 \leq \sum\limits_{j=1}^{m} A_j \tilde{Q} A_j^* < \tilde{Q}$, condition (2.1) of Theorem 2.1 is satisfied with

$$c = \| \sum_{j=1}^{m} \tilde{Q}^{-\frac{1}{2}} A_j \tilde{Q} A_j^* \tilde{Q}^{-\frac{1}{2}} \|.$$

□

3.2 Application to non-linear matrix equations

In this section we study the following class of non-linear matrix equations:

$$X = Q \pm \sum_{j=1}^{m} A_j^* \mathcal{F}(X) A_j. \qquad (3.3)$$

Here, again Q is positive definite and the A_j are arbitrary $n \times n$ matrices, and $\mathcal{F}$ is a continuous map, mapping $\mathcal{P}(n)$ into $\mathcal{P}(n)$. We shall also assume that $\mathcal{F}$ is either order-preserving or order-reversing. The map $X \to Q \pm \sum_{j=1}^{m} A_j^* \mathcal{F}(X) A_j$ is denoted by $\mathcal{G}$.

we consider the equation

$$X = Q + \sum_{j=1}^{m} A_j^* \mathcal{F}(X) A_j, \qquad (3.4)$$

with $\mathcal{F}$ being order-preserving. In that case $\mathcal{G}$ is well defined on $\mathcal{P}(n)$ and order- preserving as well. Since $\mathcal{G}(X) \geq Q$ for all $X \in \mathcal{P}(n)$, it is easily seen that the sequence $\{\mathcal{G}^j(Q)\}_{j=0}^{\infty}$ is an increasing sequence. In particular, $\mathcal{G}(Q) \geq Q$. So condition (2.2) in Theorem 2.1 is satisfied.

Our first result discusses existence of a solution.

Proposition 3.1 *Assume that there exists an X_0 such that $\mathcal{G}(X_0) \leq X_0$. Then $\mathcal{G}$ maps the set $[Q,\ X_0]$ into itself, the limit $X_- = \lim_{j\to\infty} \mathcal{G}^j(Q)$ exists and is the smallest solution to (3.4). Moreover, the sequence $\{\mathcal{G}^j(X_0)\}_{j=0}^{\infty}$ decreases to a solution X_+, which is the largest solution in the set $[Q,\ X_0]$.*

Proof.

Since $\mathcal{G}(X_0) \leq X_0$, we have $Q \leq \mathcal{G}(X_0) \leq X_0$. Now if $Q \leq X \leq X_0$, we easily see that $Q \leq \mathcal{G}(Q) \leq \mathcal{G}(X) \leq \mathcal{G}(X_0) \leq X_0$.

Applying $\mathcal{G}$ repeatedly, we see that the sequence $\{\mathcal{G}^j(Q)\}_{j=0}^{\infty}$ is an increasing sequence, bounded above by $\mathcal{G}^l(X_0)$ for any value of l. Also, the sequence $\{\mathcal{G}^j(X_0)\}_{j=0}^{\infty}$ is a decreasing sequence bounded below. Thus, both sequences con- verge.

Now let X be any solution to the equation (3.4). Then $Q \leq X = \mathcal{G}(X)$, and by repeatedly applying $\mathcal{G}$ we see that $X_- \leq X$. If $X \in [Q,\ X_0]$, i.e., if $X \leq X_0$, we see, again by applying $\mathcal{G}$ repeatedly, that $X \leq X_+$. □

To get uniqueness of the solution, we can follow the same line of argument as in the proof of Theorem to derive the following result. For the sake of completeness we shall provide the proof in this case.

Theorem 3.2. *Let $Q \in \mathcal{P}(n)$. Assume there is a positive number M for which $\sum_{j=1}^{m} A_j A_j^* < M \cdot I_n$ and such that for $X \leq Y$ we have*

$$|tr(\mathcal{F}(Y) - \mathcal{F}(X))| \leq \frac{1}{M}|tr(Y - X)|$$

Then equation (3.4) *has a unique solution in $\mathcal{P}(n)$.*

Proof.

Let $X, Y \in \mathcal{H}(n)$ such that $X \leq Y$. Then $\mathcal{G}(X) \leq \mathcal{G}(Y)$. Therefore,

$$\|\mathcal{G}(Y) - \mathcal{G}(X)\|_1 = tr(\mathcal{G}(Y) - \mathcal{G}(X)) = tr(\textstyle\sum_{j=1}^{m} A_j^*(\mathcal{F}(Y) - \mathcal{F}(X))A_j)$$

$$= \textstyle\sum_{j=1}^{m} tr(A_j^*(\mathcal{F}(Y) - \mathcal{F}(X))A_j) = \sum_{j=1}^{m} tr(A_j A_j^*(\mathcal{F}(Y) - \mathcal{F}(X)))$$

$$= tr(\textstyle\sum_{j=1}^{m} A_j A_j^*(\mathcal{F}(Y) - \mathcal{F}(X))) = tr((\sum_{j=1}^{m} A_j A_j^*)(\mathcal{F}(Y) - \mathcal{F}(X)))$$

$$\leq \| \textstyle\sum_{j=1}^{m} A_j A_j^*\| \|\mathcal{F}(Y) - \mathcal{F}(X)\|_1.$$

As before, the inequality follows from Lemma 3.1 Because of the assumptions in the theorem we see that condition (2.1) of Theorem 2.1 is satisfied. So we can apply that theorem to get the desired result. □

3.3 Application to fractional integral equation

Our aim here is to apply **Theorem 2.2** to prove the existence and uniqueness of solution for the following fractional integral equation of the type

$$\begin{cases} x(t) = \frac{a(t)}{\Gamma(\alpha)} \int_0^t b(u)(t-u)^{\alpha-1} x(u)du + f(t) & t \in I = [0,\ T] \\ f(t) \geq 1 & t \in I \end{cases} \tag{3.5}$$

where $\Gamma(\alpha) = \int_0^t e^{-w} w^{\alpha-1} dw$, $0 < \alpha < 1$ and $a, b, f : [0,\ T] \to]\mathbb{R}$ are continuous functions.

Theorem 3.3. *Under the above conditions, for all $T > 0$ the fractional integral equation* (3.5) *has a unique solution.*

Proof.

Let $X = \{u \in C(I,\ ]\mathbb{R}) : u(t) > 0, \forall t \in I\}$. We define the following relation R in X:

$$xRy \Longleftrightarrow x(t)y(t) \geq (x(t) \vee y(t)),\ \forall t \in I.$$

Define

$$\|x\|_r = \max_{t \in l} e^{-rt}|x(t)|, x \in X, \quad for \quad r > (\|a\| \cdot \|b\|)^{\frac{1}{\alpha}} \tag{3.6}$$

It is easy to show that (X, d) is an R-metric space. Suppose $x_0(t) \equiv 1$ for all $t \in I$, then $x_0 Rx$ for all $x \in X$.let $\{x_n\} \subseteq X$ be a Cauchy R-sequence. It is easy to show that $\{x_n\}$ is converging to an element x uniformly in $C(I, \mathbb{R})$. Fix $t \in I$, the definition of R implies $x_n(t)x_{n+k}(t) \geq (x_n(t) \vee x_{n+k}(t))$ for each $n, k \in \mathbb{N}$. Since $x_n(t) > 0$ for all $n \in \mathbb{N}$, there exists an R-subsequence $\{x_{n_i}\}$ of $\{x_n$ such that $x_{n_i}(t) \geq 1$ for all $i \in \mathbb{N}$. Clearly $\{x_{n_i}\}$ also converges to x, therefore $x \geq 1$, hence $x \in X$.

Define $F : X \to X$ by

$$F.x(t) = \frac{a(t)}{\Gamma(\alpha)} \int_0^t b(u)(t-u)^{\alpha-1} x(u)du + f(t)\ .$$

Clearly the fixed points of F are the solutions of the fractional integral equation (3.5).

It is enough to prove the following three steps:

Step(1)

F is R-preserving.

Proof

Let xRy and $t \in I, F.wx(t) = \frac{a(t)}{\Gamma(\alpha)} \int_0^t b(u)(t-u)^{\alpha-1}x(u)du + f(t) \geq 1$ which implies that $F_x(t)F_y(t) \geq F_y(t)$. Thus $FxRFy$.

Step(2)

Proof

F is R-contraction.

Let xRy and $t \in I$,

$$
\begin{aligned}
e^{-rt}|Fx(t)-Fy(t)| &\leq e^{-rt}\tfrac{a(t)}{\Gamma(\alpha)}\textstyle\int_0^t b(u)(t-u)^{\alpha-1}|x(u)-y(u)|du \\
&\leq e^{-rt}\tfrac{a(t)}{\Gamma(\alpha)}\textstyle\int_0^t b(u)(t-u)^{\alpha-1}e^{r\iota t}e^{-r\iota t}|x(u)-y(u)|du \\
&\leq e^{-rt}\tfrac{a(t)}{\Gamma(\alpha)}\|x-y\|_r\textstyle\int_0^t b(u)(t-u)^{\alpha-1}e^{r\iota}du
\end{aligned}
$$

We have $a(t) \leq \|a\|$

$$\leq \tfrac{1}{\Gamma(\alpha)}\|a\|\|b\|\|x-y\|_r\tfrac{1}{r^{\alpha-1}}\textstyle\int_0^t r^{\alpha-1}(t-u)^{\alpha-1}e^{rl4}e^{-rt}du.$$

Put $w = r(t-u)$, then $0 \leq w \leq rt$, we get

$$
\begin{aligned}
e^{-rt}|Fx(t)-Fy(t)| &\leq \tfrac{1}{\Gamma(\alpha)}\|a\|\|b\|\|x-y\|_r\tfrac{1}{r^{\alpha}}\textstyle\int_0^{rt} w^{\alpha-1}e^{-w}dw \\
&\leq (\|a\|\|b\|\tfrac{1}{r^{\alpha}})\|x-y\|_r.
\end{aligned}
$$

Since t is an arbitrary element of I, by (3.6) it follows that $0 \leq \lambda = (\|a\|\|b\|\frac{1}{r^{\alpha}}) < 1$, hence $\|Fx-Fy\|_r \leq \lambda\|x-y\|_r$.

- ***Step (3)***

F is R-continuous.

Proof

Let $\{x_n\}$ be an R-sequence converging to $x \in X$. Using the first part of the proof $x(t) \geq 1$ for all $t \in I$, hence $x_n(t)x(t) \geq x_n(t)$ for all $n \in \mathbb{N}$ and all $t \in I$, therefore x_nRx. By step (2), we have

$$e^{-rt}|Fx_n(t)-Fx(t)| \leq \lambda\|x_n-x\|_r.$$

Thus

$$\|Fx_n-Fx\|_r \leq \lambda\|x_n-x\|_r.$$

Therefore $Fx_n \xrightarrow{R} Fx$.

Now applying theorem 2.2 it follows that the fractional integral equation (3.5) has a unique solution.

$\square$

Conclusion

Fixed-point theory is one of the important fields in mathematics that contribute to the solution of many problems, especially non-linear.

In this work we attempted to present the theory of fixed points in R-metric spaces ,with selected contractive properties and applied in to solve some linear matrix equations, non-linear matrix equations, and fractional integral equations. Several works have been carried out previously in this field, the differentiation between these works can be found: either in the properties of the applications, or in the conditions of contraction, or in the spaces used.

Fixed point theory remains in need of further development and search for more important and diverse fields of application in mathematics and other sciences.

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