University ABBAS Laghrour
Faculty of Sciences & Technology
Department of Mechanical Engineering

Elasticity
Licence S5 Construction Mécanique
Academic year 2025/2026

FINAL EXAM

Exercise N°1 (4 pts)

Given an orthonormal basis $(\vec{\imath},\vec{\jmath},\vec{k})$ and vectors: $\vec{V}_1 = a\vec{\imath} + \vec{\jmath} + b\vec{k}$; $\vec{V}_2 = b\vec{\imath} + \vec{\jmath} + c\vec{k}$

1) Write the tensor $[P] = \vec{V}_1 \otimes \vec{V}_2$
2) If $a = 2$
   - 2-1) Find b and c to get $[P]$ symmetric
   - 2-2) Calculate the contracted product $[P]\!:\!\vec{V}_1$
   - 2-3) Show that $\vec{V}_1$ is an Eigenvector for the representative matrix $[P]$

Exercise N°2 (8 pts)

Consider Cauchy stress tensor $[\sigma]$ defined at the point M(x,y,z):

$$\sigma_{ij} = \begin{cases} 5ij \Leftrightarrow i \neq j \\ 2i \Leftrightarrow i = j \end{cases} \quad (i = 1{,}3 \text{ and } j = 1{,}3) \quad \text{(MPa)}$$

1) Write the tensor $[\sigma]$
2) At the facet perpendicular to X direction and without any calculation, write:
   2-1) Stress vector
   2-2) Normal stress
   2-3) Shear vector
3) At the facet with normal $\vec{n} = \frac{1}{3}\begin{Bmatrix} 2 \\ 1 \\ 2 \end{Bmatrix}$, calculate:
   3-1) The normal stress
   3-2) The magnitude of the shear vector
4) Write the deviatoric tensor

Exercise N°3 (8 pts)

Given a structure, the material is supposed to be a homogeneous continuum. At a point M(x,y,z) is defined the following stress tensor:

$$[\sigma] = 100\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{(MPa)}$$

1) Determine principal stresses
2) Determine the principal directions
3) Given Young Modulus E=2.10^{5}MPa and Poisson's Ratio v=0.3, calculate the linear strain tensor
4) By using Strength Rankine Criterion, choose the suitable steel to ensure the strength.

Steel	**X001**	**X002**	**X003**	**X004**
Elastic Limit (MPa)	210	180	320	460

****************** MODEL ANSWER ****************************

Exercise N°1 (4pts)

1) $[P] = \vec{V}_1 \otimes \vec{V}_2 = (\mathbf{0.25}) \begin{Bmatrix} a \\ 1 \\ b \end{Bmatrix} \otimes \begin{Bmatrix} b \\ 1 \\ c \end{Bmatrix} = \begin{bmatrix} ab & a & ac \\ b & 1 & c \\ b^2 & b & bc \end{bmatrix}$ **(0.5)**

2) If *a*=2

2-1) **(0.25*3)** $\begin{cases} a = b \\ b^2 = ac \\ b = c \end{cases} \rightarrow a = b = c = 2$ **(0.5)**

2-2) $[P]:\vec{V}_1 = (\mathbf{0.25}) \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} : \begin{Bmatrix} 2 \\ 1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 18 \\ 9 \\ 18 \end{Bmatrix}$ **(0.5)**

2-3) **(0.5)** $\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} : \begin{Bmatrix} 2 \\ 1 \\ 2 \end{Bmatrix} = 2 \begin{Bmatrix} 2 \\ 1 \\ 2 \end{Bmatrix}$ 2= Eigenvalue **(0.75)**

Exercise N°2 (8pts)

$$\sigma_{ij} = \begin{cases} 5ij \Leftrightarrow i \neq j \\ 2ij \Leftrightarrow i = j \end{cases} \quad (i = 1{,}3 \text{ and } j = 1{,}3)$$

1) Stress tensor

$$[\sigma] = \begin{bmatrix} 2 & 10 & 15 \\ 10 & 8 & 30 \\ 15 & 30 & 18 \end{bmatrix} \quad \mathbf{(1)}$$

2) Stress vector at the facet perpendicular to X direction

$\vec{T}(M,\vec{\imath}) = \begin{Bmatrix} 2 \\ 10 \\ 15 \end{Bmatrix}$ **(0.5)**

The normal stress: $\sigma_i = 2$ **(0.5)**

Shear vector: $\vec{\tau} = 10\vec{\jmath} + 15\vec{k} = \begin{Bmatrix} 0 \\ 10 \\ 15 \end{Bmatrix}$ **(0.5)**

3) At the facet with normal $\vec{n} = \frac{1}{3}\begin{Bmatrix} 2 \\ 1 \\ 2 \end{Bmatrix}$, calculate:

3-1) The normal stress

$\vec{T}(M,\vec{n}) = \begin{bmatrix} 2 & 10 & 15 \\ 10 & 8 & 30 \\ 15 & 30 & 18 \end{bmatrix} \frac{1}{3}\begin{Bmatrix} 2 \\ 1 \\ 2 \end{Bmatrix} \; (\mathbf{0.5}) = \frac{1}{3}\begin{Bmatrix} 44 \\ 88 \\ 96 \end{Bmatrix}$ **(0.5)**

$$\sigma_n = \vec{T}(M,\vec{n}).\vec{n} = (\mathbf{0.5}) \frac{1}{3}\begin{Bmatrix} 44 \\ 88 \\ 96 \end{Bmatrix} \frac{1}{3}\begin{Bmatrix} 2 \\ 1 \\ 2 \end{Bmatrix} = \frac{370}{9} = 41.11MPa \quad \mathbf{(0.5)}$$

3-2) The magnitude of the shear vector

(0.5) $\vec{T}(M,\vec{n}) = \sigma_n\vec{n} + \vec{\tau} \rightarrow \vec{\tau} = \vec{T}(M,\vec{n}) - \sigma_n\vec{n} = \frac{1}{3}\begin{Bmatrix} 44 \\ 88 \\ 96 \end{Bmatrix} - 41.11\begin{Bmatrix} 2 \\ 1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} -67.55 \\ -11.78 \\ -50.22 \end{Bmatrix}$ **(0.5)**

(0.5) $|\vec{\tau}| = \sqrt{(-67.55)^2 + (-11.78)^2 + (-56.22)^2}$=88.67MPa **(0.5)**

4) Write the deviatoric tensor

$$[\sigma] = [\sigma]_S + [\sigma]_D \rightarrow [\sigma]_D = [\sigma] - [\sigma]_S \; \mathbf{(0.5)}$$

$$\mathbf{(0.5)}\; [\sigma]_D = \begin{bmatrix} 2 & 10 & 15 \\ 10 & 8 & 30 \\ 15 & 30 & 18 \end{bmatrix} - \begin{bmatrix} \frac{28}{3} & 0 & 0 \\ 0 & \frac{28}{3} & 0 \\ 0 & 0 & \frac{28}{3} \end{bmatrix} = \begin{bmatrix} -\frac{22}{3} & 10 & 15 \\ 10 & -\frac{4}{3} & 30 \\ 15 & 30 & \frac{8}{3} \end{bmatrix} \quad \mathbf{(0.5)}$$

Exercise N°3

$$[\sigma] = 100\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{(MPa)}$$

1) Determine principal stresses

(0.5) $[\sigma] = \begin{vmatrix} 200-\sigma_p & -100 & 0 \\ -100 & 100-\sigma_p & 0 \\ 0 & 0 & -100-\sigma_p \end{vmatrix} = 0$

$\rightarrow (100+\sigma_p)[(100-\sigma_p)(200-\sigma_p) - 10000] = 0$ **(0.5)**

$\sigma_p^2 - 300\sigma_p + 10000 = 0 \quad \rightarrow \Delta = 90000 - 40000 = 50000$ **(0.5)**

$\sigma_p = -100MPa; \sigma_p = \frac{300+\sqrt{50000}}{2} = 261.80MPa$ and $\sigma_p = \frac{300-\sqrt{50000}}{2} = 38.20MPa$

$\sigma_{p1} = 261.80MPa$ **(0.5)**; $\sigma_{p2} = 38.20MPa$ **(0.5)** and $\sigma_{p3} = -100MPa$ **(0.5)**

$[\sigma] = \begin{vmatrix} 261.80 & 0 & 0 \\ 0 & 38.20 & 0 \\ 0 & 0 & -100 \end{vmatrix}$ **(0.5)**

2) Determine the principal directions

$[\sigma] = \begin{vmatrix} 200 & -100 & 0 \\ -100 & 100 & 0 \\ 0 & 0 & -100 \end{vmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sigma_p \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$ **(0.25)**

(0.25) $\sigma_{p1} \rightarrow \begin{cases} 200x - 100y = 261.80x \\ -100x + 100y = 261.80\,y \\ -100z = 261.80z \end{cases} \rightarrow \begin{cases} 61.80x + 100y = 0 \\ 100x + 161.80y = 0 \\ -361.80z = 0 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \vec{V}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ **(0.25)**

(0.25) $\sigma_{p2} \rightarrow \begin{cases} 200x - 100y = 38.20x \\ -100x + 100y = 38.20\,y \\ -100z = 38.20z \end{cases} \rightarrow \begin{cases} 161.80x + 100y = 0 \\ -100x + 61.80y = 0 \\ -61.80z = 0 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \vec{V}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ **(0.25)**

(0.25) $\sigma_{p3} \rightarrow \begin{cases} 200x - 100y = -100x \\ -100x + 100y = -100\,y \\ -100z = -100z \end{cases} \rightarrow \begin{cases} 300x - 100y = 0 \\ -100x + 200y = 0 \\ z = z \end{cases} \rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = z \end{cases} \vec{V}_2 = z\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$ let be $\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$ **(0.25)**

3) Given Young Modulus E=2.10^{5}MPa and Poisson's Ratio v=0.3, calculate the linear strain tensor

$$\varepsilon_{ij} = \frac{1}{2G}\left(\sigma_{ij} - \frac{v}{1+v}\delta_{ij}\sigma_{kk}\right) \qquad G = \frac{E}{2(1+v)}$$

(0.25) $\varepsilon_{11} = \frac{1+v}{E}\left(\sigma_{11} - \frac{v}{1+v}\sigma_{kk}\right) = \frac{1.3}{210}\left(261.80 - \frac{0.3}{1.3}200\right) =$0,001401722 **(0.25)**

(0.25) $\varepsilon_{22} = \frac{1+v}{E}\left(\sigma_{22} - \frac{v}{1+v}\sigma_{kk}\right) = \frac{1.3}{210}\left(38.20 - \frac{0.3}{1.3}200\right) =$-5,17221E-05 **(0.25)**

(0.25) $\varepsilon_{33} = \frac{1+v}{E}\left(\sigma_{33} - \frac{v}{1+v}\sigma_{kk}\right) = \frac{1.3}{210}\left(-100 - \frac{0.3}{1.3}200\right) =$-0,00095 **(0.25)**

4) By using Strength Rankine Criterion, choose the suitable steel to ensure the strength.

(0.25) $\sigma_R = Max(|\sigma_{p1}|, |\sigma_{p2}|, |\sigma_{p3}| = 261.80MPa$ **(0.5)** $\rightarrow$

We choose Steel X003 (σ_e=320MPa) **(0.5)**